Variational Methods for Scattering Problems*

MARIUS KOLSRUD
Institute for Theoretical Physics, University of Oslo, Blindern, Norway
(Received August 4, 1958)

Some new stationary expressions for the phase shifts and total amplitude by central force scattering are presented. As examples values of $\tan\delta_0$ are given for the exponential and Yukawa potential, and compared with the results by other methods and with the exact values.

THE phase shift for the lth partial wave is given by the integral[1]

$$\tan\delta_l = -k(u,v), \tag{1}$$

where v is given by the integral equation

$$v - Kv = u. \tag{2}$$

Our notation is the following:

$$v(r) = r[U(r)]^{\frac{1}{2}} R_l(r),$$

$$u(r) = r[U(r)]^{\frac{1}{2}} j_l(kr),$$

$$K(r,r') = rr'[U(r)U(r')]^{\frac{1}{2}} G_l(r,r'),$$

$$(u,v) = \int_0^\infty u(r)v(r)dr,$$

$$(u,Kv) = \int_0^\infty dr u(r) \int_0^\infty dr' K(r,r')v(r').$$

Schwinger's variational principle is obtained by writing

$$(u,v) = \frac{(u,v)^2}{(v,v)-(v,Kv)}. \tag{3}$$

We may, however, deduce infinitely many new variational principles by means of successive iterations of Eq. (2). Below we present the first two of these stationary expressions:

$$(u,v) = (u,u) + \frac{(u,Kv)^2}{(v,Kv)-(v,K^2v)}, \tag{4}$$

$$(u,v) = (u,u) + (u,Ku) + \frac{(u,K^2v)^2}{(v,K^2v)-(v,K^3v)}. \tag{5}$$

On the right-hand side, v may now be replaced by an un-normalized trial function. However, in order to simplify the integrations we may rather substitute a trial function for $Kv \equiv w$. Hence Eqs. (4) and (5) yield the following new stationary expressions:

$$(u,v) = (u,u) + \frac{(u,w)^2}{(w,K^{-1}w)-(w,w)}, \tag{6}$$

$$(u,v) = (u,u) + (u,Ku) + \frac{(u,Kw)^2}{(w,w)-(w,Kw)}. \tag{7}$$

Equation (6) can be shown to have the explicit form[2]

$$\tan\delta_l = -k\int_0^\infty U(r) j_l(kr)^2 r^2 dr - \frac{k\left[\int_0^\infty U(r) j_l(kr) S_l(r) r^2 dr\right]^2}{\int_0^\infty rS_l(r)\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + k^2 - U(r)\right] rS_l(r) dr}, \tag{6'}$$

where

$$R_l(r) = j_l(kr) + S_l(r),$$

corresponding to $v = u + w$.

Stationary expressions for the total scattering amplitude $f(\mathbf{k},\mathbf{k}_0)$ are deduced in a similar manner from the equations

$$4\pi f(\mathbf{k},\mathbf{k}_0) = -\int e^{-i\mathbf{k}\cdot\mathbf{r}} U(r)\psi(\mathbf{r},\mathbf{k}_0) d\mathbf{r},$$

$$\psi(\mathbf{r},\mathbf{k}_0) = e^{i\mathbf{k}_0\cdot\mathbf{r}} - \int G(\mathbf{r},\mathbf{r}')U(r')\psi(\mathbf{r}',\mathbf{k}_0)d\mathbf{r}' \equiv e^{i\mathbf{k}_0\cdot\mathbf{r}} + \phi(\mathbf{r},\mathbf{k}_0),$$

$$G(\mathbf{r},\mathbf{r}') = \exp(ik|\mathbf{r}-\mathbf{r}'|)/4\pi|\mathbf{r}-\mathbf{r}'|.$$

TABLE I. Values of $\tan\delta_0$ ($l=0$) for the potential $U=-3e^{-r}$ and $k=1$, calculated by different methods.

Method	Trial function	$\tan\delta_0$	Deviation from $(\tan\delta_0)_{\text{exact}}$ (%)
1. Born approx.		1.200	43
2. Born approx.		1.245	41
Schwinger	$rR=\sin\kappa r$	1.935	8.7
Eq. (6')	$rS=e^{-r}\cos\kappa r-\cos r$	2.0780	1.9
Eq. (4)	$rR=\sin\kappa r$	2.1069	0.55
Eq. (7)	$rS=e^{-r}\cos\kappa r-\cos r$	2.1159	0.12
Exact		2.1185	0

* Part of this work was presented at the Conference of the Norwegian Physical Society, Bergen, May, 1958 (unpublished).

[1] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), second edition, p. 181.

[2] *Note added in proof.*—The expressions (6') and (8) were found independently by M. Moe and D. S. Saxon, Phys. Rev. **111**, 950 (1958). Equation (8) was also found by H. E. Moses, 1956 (unpublished, see same reference).

Below we give the variational principles analogous to (6′) and (7):

$$4\pi f(\mathbf{k},\mathbf{k}_0)=-\int e^{i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{r}}U(r)d\mathbf{r}-\frac{\int e^{-i\mathbf{k}\cdot\mathbf{r}}U(r)\phi(\mathbf{r},\mathbf{k}_0)d\mathbf{r}\int e^{i\mathbf{k}_0\cdot\mathbf{r}}U(r)\phi(\mathbf{r},-\mathbf{k})d\mathbf{r}}{\int\phi(\mathbf{r},-\mathbf{k})[\nabla^2+k^2-U(r)]\phi(\mathbf{r},\mathbf{k}_0)d\mathbf{r}},\tag{8}$$

and[2]

$$4\pi f(\mathbf{k},\mathbf{k}_0)=-\int e^{i(\mathbf{k}_0-\mathbf{k})\cdot\mathbf{r}}U(r)d\mathbf{r}+\int\int G(\mathbf{r},\mathbf{r}')U(r)U(r')e^{-i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}_0\cdot\mathbf{r}'}d\mathbf{r}d\mathbf{r}'$$
$$+\frac{\int\int G(\mathbf{r},\mathbf{r}')U(r)U(r')e^{-i\mathbf{k}\cdot\mathbf{r}}\phi(\mathbf{r}',\mathbf{k}_0)d\mathbf{r}d\mathbf{r}'\int\int G(\mathbf{r},\mathbf{r}')U(r)U(r')e^{i\mathbf{k}_0\cdot\mathbf{r}}\phi(\mathbf{r}',-\mathbf{k})d\mathbf{r}d\mathbf{r}'}{\int\phi(\mathbf{r},\mathbf{k}_0)\phi(\mathbf{r},-\mathbf{k})U(r)d\mathbf{r}-\int\int G(\mathbf{r},\mathbf{r}')U(r)U(r')\phi(\mathbf{r},\mathbf{k}_0)\phi(\mathbf{r}',-\mathbf{k})d\mathbf{r}d\mathbf{r}'}.\tag{9}$$

To compare these methods we present the results for $\tan\delta_0$ ($l=0$) for $U(r)=-3e^{-r}$ and $k=1$ in Table I.

As a second example we consider the Yukawa potential

$$U(r)=-1.5e^{-r}/r \quad\text{and}\quad k=0.8.$$

With the trial function (λ=variation parameter):

$$rS_0=[1-e^{-r}+\lambda(e^{-r}-e^{-2r})]\cos kr,$$

the variation principle (6′) yields

$$\tan\delta_0=1.109086.$$

Numerical integration gives[3]

$$\tan\delta_0=1.109103.$$

[3] P.-O. Löwdin and A. Sjölander, Arkiv Fysik 3, 11, 155 (1951).

PHYSICAL REVIEW VOLUME 112, NUMBER 5 DECEMBER 1, 1958

Conductivity of Plasmas to Microwaves*

EDWARD A. DESLOGE, STEVEN W. MATTHYSSE, AND HENRY MARGENAU
Laboratory of Marine Physics, Yale University, New Haven, Connecticut
(Received August 4, 1958)

If one assumes that the effect of an alternating electric field on an electron in a slightly ionized gas is on the average an addition, in the direction of the field, of a velocity component, the magnitude of which is dependent on the speed of the electron, then it is possible to derive very simply the expression for conductivity obtained earlier by Margenau. In addition an alternative expression is proposed which is of interest because it avoids difficulties with negative conductivities encountered by Margenau's formula.

INTRODUCTION

PREVIOUSLY[1,2] an expression was derived for the conductivity of a plasma to microwaves in which the electron distribution function is not restricted to be Maxwellian, and the collision frequency is not necessarily constant. In certain cases this expression leads to negative conductivities. If one assumes that the effect of an alternating electric field on an electron in a slightly ionized gas is, on the average, an addition, in the direction of the field, of a velocity component, the magnitude of which is dependent on the speed of the electron, then it is possible to derive the conductivity formula very simply. In addition this method suggests an alternative expression which in most cases agrees closely with the earlier result, but which avoids the difficulties of negative conductivities. The present method differs from the former in two respects. First, instead of expanding the distribution function in spherical harmonics about the axis of the electric field, we allow the field to cause a change in the velocity component along its axis. Secondly, in the absence of the orthogonality properties of the spherical harmonics, we exploit the parity of different parts of the distribution

* Supported by the Office of Naval Research.
[1] H. Margenau, Phys. Rev. 69, 508 (1946).
[2] H. Margenau, Phys. Rev. 109, 6 (1958).