changed from (26) to

$$V_1=(R^2/\Delta)[R\omega_1'+Z^3\omega_2'-Z^2(\omega_3'+\sigma)]\tilde{\gamma},$$
$$V_2=(R^2/\Delta)[R\omega_2'+Z^1(\omega_3'+\sigma)-Z^3\omega_1']\tilde{\gamma},$$
$$V_3=(R^2/\Delta)[R(\omega_3'+\sigma)+Z^2\omega_1'-Z^1\omega_2']\tilde{\gamma}, \quad (30)$$
$$V_4=-c[1-R^2\sigma(\omega_3'+\sigma)/c^2]\tilde{\gamma},$$
$$\tilde{\gamma}=[1-(R^2/c^2)\{\omega_1'^2+\omega_2'^2+(\omega_3'+\sigma)^2\}]^{-\frac{1}{2}}.$$

Thus in this case the relative velocity σ does not occur in the contravariant components explicitly, but instead in the covariant expressions. Symmetry would be achieved by using $\boldsymbol{\omega}'-\frac{1}{2}\boldsymbol{\sigma}$ instead of $\boldsymbol{\omega}'-\boldsymbol{\sigma}$ in the expressions (24).

We denote once more S_z's rest mass of P by m, and find for the energy the quantity $-mcV_4$, but since as before $m=m_0\gamma_\sigma$ we have

$$E_z'=\frac{m_0c^2[1-R^2(\omega_3'+\sigma)\sigma/c^2]\gamma_\sigma}{[1-(R^2/c^2)\{\omega_1'^2+\omega_2'^2+(\omega_3'+\sigma)^2\}]^{\frac{1}{2}}}. \quad (31)$$

This is easily verified to be the Lorentz transform by (23b) of S_x's expression for the energy of P; hence agreement is again achieved.

The Clifford momentum-energy four-vector $m_0R\omega_\iota$, $m_0c\gamma_\omega=E_x/c$ could have been introduced for this demonstration, but we did not wish to multiply symbols any more than has been found necessary as it is.

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New Techniques in the Lamb Shift Calculation*

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Based on the analogy between the calculations of radiative corrections to electron scattering and the Lamb shift, a new procedure applicable to bound state self-energy problems is developed, wherein the electron propagator is expanded in powers of the external potential. In the resulting sequence, a change in the gauge of the virtual photon field conveniently removes from each term spurious lower order contributions and yields a new and considerably simpler sequence; the expectation value of the first two terms of the latter is shown to account for the major portion of the Bethe logarithm. A simple method is developed to sum all the $\alpha(Z\alpha)^4\mu$ dependence, and the result is the lowest order Lamb shift formula. The ease of the calculation, as well as that involved in obtaining the relativistic level shift correction of order $\alpha(Z\alpha)^5\mu$ (not given in the present paper), indicates that the method may find application in the calculation of further higher order effects.

1. INTRODUCTION

THERE is a close analogy between the calculations of radiative corrections to electron scattering[1] and those of the Lamb shift.[2–6] This is because both effects require the calculation of the vertex operator for the interaction of an electron with an external potential. In the approximation in which this vertex operator depends only on the momentum transfer and not explicitly on the electron's initial and final momenta, it may be replaced by an equivalent effective potential which may then be used to predict both effects. In practice this approximation breaks down because of the infrared divergence, which makes the vertex operator depend logarithmically on the deviation of the electron's four-momentum from its free value. In most of the earlier calculations[4] the infrared divergence difficulty was avoided by introducing a photon mass and by using the resulting scattering operator to obtain only the level shift contribution arising from the "high-energy" virtual photons. In the calculation of the contribution of the "low-energy" virtual photons, the intermediate electron states were treated correctly in the nonrelativistic dipole approximation; and the famous "Bethe logarithm"[3] was obtained. When the two contributions were added, using the connection formula of French,[4] the level shift was obtained correct to lowest order. The more recent calculation of Karplus, Klein, and Schwinger[5] avoids the infrared problem entirely by evaluating the mass operator in the appropriate atomic state. Baranger, Bethe, and Feynman[6] also evaluate the mass operator; but as a matter of convenience, to

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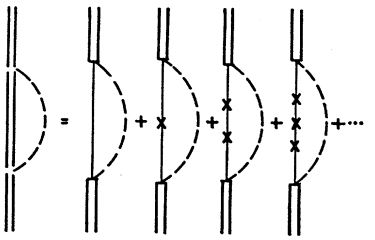

FIG. 1. The pictorial equality demonstrating the result of iterating the bound state propagator. The double-line notation is used to denote electron propagation in the Coulomb field of the nucleus.

make use of the techniques developed by Feynman,[7] they introduce a small photon mass. In the end, the photon mass cancels out; but during the course of the calculation it serves to prevent the separate contributions from becoming divergent. The latter two calculations were carried out to lowest order in the number of virtual photons, and to order $Z\alpha$ beyond the lowest order level shift. When all other known corrections are included, the theoretical results are in good agreement with experiment for hydrogen, but there exists a discrepancy in the case of singly ionized heilum.[8]

There were two main reasons for undertaking the present work. The first of these is mainly pedagogical. It seemed interesting to carry out the level shift calculation by making a straightforward expansion of the bound state electron propagator in powers of the external potential. It is well known, of course, that such an expansion is not directly an expansion in powers of $Z\alpha$ because of the dependence of the atomic state on the external potential. However, it would be interesting to see from which order in the expansion the major contribution comes. In the past, such a procedure seemed doomed to failure because the contribution due to the Bethe logarithm would be difficult to reproduce. This contribution is nonrelativistic and depends on the electronic wave functions at relatively large radii (Bohr radius); this, together with the fact that the logarithm has $Z\alpha$ as a factor of its argument, suggests that in the intermediate states many scatterings of the electron by the Coulomb potential may be important. In fact, as will be seen below, the expansion can be carried out and the bulk of the contribution comes from the first two terms; however, in order to avoid spurious terms of *lower* than the desired order in the separate parts of the expansion, it is found necessary to make a special choice of the gauge of the quantized photon field. The second reason was that it was hoped that a more straightforward procedure might lead to a simplification of the calculation and permit the evaluation or estimation of some of the previously neglected contributions. Although the present agreement between theory and experiment for hydrogen is quite good, as a result of the recalculation of the electron's anomalous moment,[8] these neglected terms might be sufficiently large to destroy the agreement. A few years ago, when there was a discrepancy, it was commonly felt that the cause might be found in higher order corrections. Aside from this consideration, an estimate of the higher order contributions is necessary for the evaluation of level shifts in heavier elements and might resolve the discrepancy between theory and experiment for ionized helium.

In free scattering, the infrared divergence problem arises because an electron can emit and absorb soft photons without being displaced very far off the free-energy shell. In terms of the formalism of Feynman,[7] this results in electron propagators which have very small denominators; and in combination with the photon propagator, these lead to the infrared divergence. By giving the photon a small mass λ, the divergence is eliminated; the resulting matrix element then depends logarithmically on λ. In the Lamb shift calculation, on the other hand, no infrared divergence arises because the electron's four-momentum is off the free-energy shell. Thus, even for very small photon energies, the electron propagator has a nonvanishing denominator equal to $(p^2+\mu^2)$; the resulting vertex operator then depends logarithmically on this quantity. As will be demonstrated below, the expectation value of this logarithm yields an appreciable portion of the Bethe logarithm. These remarks may be summarized by saying that the quantity $\ln(\lambda/\mu)$ of the free-particle scattering is effectively replaced by a term of the form $\ln[(p^2+\mu^2)/\mu^2]$. Since $p^2+\mu^2$ is approximately $\mathbf{p}^2+2\mu\epsilon_a$, where ϵ_a is the (positive) binding energy in the atomic state a, the correspondence here is then

$$\lambda\rightarrow 2[\epsilon_a+(\mathbf{p}^2/2\mu)].$$

The expansion of the bound state electron propagator S_F^e leads in a natural way to the terminology of "zero-potential" (ZP), "one-potential" (OP), and "many-potential" (MP) terms, represented in the familiar way by the Feynman diagrams of Fig. 1. It should be pointed out that this classification differs from that of reference 6, because in that work the equation satisfied by the atomic state was used to rearrange the powers of the potential. After mass renormalization has been performed on the ZP term, there remains a spurious ultraviolet divergence of the charge renormalization type. As in the case of free electron scattering, this is cancelled by a corresponding divergence arising from the OP term, as a consequence

[7] See references 1 and 4, and also Schweber, Bethe, and de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, New York, 1955), Vol. I, and J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Cambridge, 1955).

[8] E. E. Salpeter, Phys. Rev. **87**, 328 (1952); **89**, 92 (1953); C. M. Sommerfield, Phys. Rev. **107**, 328 (1957). The most recent comparison between theory and experiment for hydrogen and helium is given in E. Lipworth and R. Novick, Phys. Rev. **108**, 1446 (1957). The difference between the experimental and theoretical values is -0.22 ± 0.23 Mc/sec in hydrogen and -16.6 ± 7.5 Mc/sec in helium.

of Ward's identity.[9] The usual scattering prescription[10] for the isolation of these divergent terms will not be followed, since this would have the immediate consequence of introducing spurious infrared divergences into the calculation.

One difficulty peculiar to this approach is the presence of terms in the "zero- and one-potential" (ZOP) contributions which are of order $\alpha(Z\alpha)^2\mu$ and $\alpha(Z\alpha)^2\mu \times\ln(Z\alpha)$. Such spurious terms must be cancelled by corresponding quantities in the remaining MP terms, and it is not difficult to see that contributions of this order are, in fact, present in the latter. The presence of these terms is related to the choice of gauge of the virtual photon. By changing the gauge from the usual one, a new sequence of ZOP and MP terms is obtained, each of which is free of such lower order contributions. In the new gauge, the operator whose matrix element is desired is much simpler than the original operator. The choice of a new gauge corresponds to using the equation satisfied by the atomic state to rearrange the powers of the potential, as in reference 6; but it is a different, and perhaps simpler, rearrangement.

In this paper the aforementioned analogy will be exploited to develop a new procedure for the Lamb shift calculation and a simple means for the estimation of the Bethe logarithm. Although the relativistic corrections of order $\alpha(Z\alpha)^5\mu$, obtained in references 5 and 6, may be found in a very straightforward manner using a slight variation of the methods presented below, only the lowest order Lamb shift formula will be derived here; it is planned to include the former along with a treatment of the as yet unknown $\alpha(Z\alpha)^6\mu$ corrections in a subsequent paper.

2. "ZERO- AND ONE-POTENTIAL" CONTRIBUTIONS

The expression for the energy level shift of the atomic state a, due to the virtual emission of a photon of four-momentum k_ν,[11] is given by[12]

$$\Delta E_a = -2\alpha \int d^3p_1 d^3p\ \bar{\varphi}_a(\mathbf{p}_1) \int_F \frac{d^4k}{k^2} \times \gamma_\mu S_F{}^e(\boldsymbol{p}_1-\boldsymbol{k}, \boldsymbol{p}-\boldsymbol{k})\gamma_\mu \varphi_a(\mathbf{p}). \quad (2.1)$$

The bound-state propagator $S_F{}^e$ satisfies the integral equation

$$S_F{}^e(\boldsymbol{p}_1-\boldsymbol{k}, \boldsymbol{p}-\boldsymbol{k}) = \frac{\delta(\mathbf{p}_1-\mathbf{p})}{\boldsymbol{p}_1-\boldsymbol{k}-i\mu} + \frac{1}{\boldsymbol{p}_1-\boldsymbol{k}-i\mu}\int d^3p' \times \boldsymbol{a}(\mathbf{p}_1-\mathbf{p}')S_F{}^e(\boldsymbol{p}'-\boldsymbol{k}, \boldsymbol{p}-\boldsymbol{k}), \quad (2.2)$$

where the quantity $\boldsymbol{a}(\mathbf{q})$ is the Fourier transform of the Coulomb field $V(r)$:

$$V(r) = -i\gamma_4(Z\alpha)/r,$$

$$\boldsymbol{a}(\mathbf{q}) = (2\pi)^{-3}\int d^3x\ V(r)e^{-i\mathbf{q}\cdot\mathbf{r}} = i\gamma_4 a_0(\mathbf{q}) = -i\gamma_4(Z\alpha)/2\pi^2q^2.$$

For the purpose of formal manipulation, it will often prove convenient to replace the propagator $S_F{}^e(\boldsymbol{p}_1-\boldsymbol{k}, \boldsymbol{p}-\boldsymbol{k})$ by an operator $S_F{}^e(\boldsymbol{p}-\boldsymbol{k})$, whose matrix elements between momentum eigenstates yield the propagator in the momentum representation. This operator is defined by the formal relation

$$S_F{}^e(\boldsymbol{p}-\boldsymbol{k}) = \frac{1}{\boldsymbol{p}-\boldsymbol{k}-\boldsymbol{V}-i\mu}, \quad (2.3)$$

where $\boldsymbol{V}$ in turn is defined by the statement that

$$\boldsymbol{V}f(\mathbf{p}) \equiv \int d^3p'\ \boldsymbol{a}(\mathbf{p}-\mathbf{p}')f(\mathbf{p}')$$

for all sufficiently well-behaved $f(\mathbf{p})$.[13]

Defining the function

$$I(\boldsymbol{p}_1,\boldsymbol{p}) = \int_F \frac{d^4k}{k^2}\gamma_\mu S_F{}^e(\boldsymbol{p}_1-\boldsymbol{k}, \boldsymbol{p}-\boldsymbol{k})\gamma_\mu, \quad (2.4)$$

one obtains, by the iteration of Eq. (2.2),

$$I(\boldsymbol{p}_1,\boldsymbol{p}) = \delta(\mathbf{p}_1-\mathbf{p})I_0(\boldsymbol{p}) + I_1(\boldsymbol{p}_1,\boldsymbol{p}) + I_2(\boldsymbol{p}_1,\boldsymbol{p}) + \cdots + I_n(\boldsymbol{p}_1,\boldsymbol{p}) + \cdots,$$

where

$$I_0(\boldsymbol{p}) = \int_F \frac{d^4k}{k^2}\gamma_\mu \frac{1}{\boldsymbol{p}-\boldsymbol{k}-i\mu}\gamma_\mu,$$

$$I_1(\boldsymbol{p}_1,\boldsymbol{p}) = \int_F \frac{d^4k}{k^2}\gamma_\mu \frac{1}{\boldsymbol{p}_1-\boldsymbol{k}-i\mu}\boldsymbol{a}(\mathbf{p}_1-\mathbf{p})\frac{1}{\boldsymbol{p}-\boldsymbol{k}-i\mu}\gamma_\mu,$$

$$I_n(\boldsymbol{p}_1,\boldsymbol{p}) = \int_F \frac{d^4k}{k^2}\gamma_\mu \frac{1}{\boldsymbol{p}_1-\boldsymbol{k}-i\mu}\Big\{\int d^3p_1' \cdots \times d^3p_{n-1}'\boldsymbol{a}(\mathbf{p}_1-\mathbf{p}_1')\frac{1}{\boldsymbol{p}_1'-\boldsymbol{k}-i\mu}\boldsymbol{a}(\mathbf{p}_1'-\mathbf{p}_2')\cdots \times \boldsymbol{a}(\mathbf{p}_{n-1}'-\mathbf{p})\Big\}\frac{1}{\boldsymbol{p}-\boldsymbol{k}-i\mu}\gamma_\mu.$$

The matrix element of the nth term in this sequence

[9] J. C. Ward, Phys. Rev. **77**, 293 (1950); **78**, 182 (1950).

[10] See, for example, Schweber, Bethe, and de Hoffmann, reference 7, Vol. 1, Chap. 24, or Jauch and Rohrlich, reference 7, Chap. 10.

[11] The relativistic notation used is such that $a_\mu = (\mathbf{a}, ia_0)$, $(a\cdot b) = \mathbf{a}\cdot\mathbf{b} - a_0b_0$, $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$, $\gamma_\mu{}^+ = \gamma_\mu$, $a_\mu\gamma_\mu = \boldsymbol{a}$, $\bar{\psi} = \psi^+\gamma_4$, $d^4k = d^3k dk_0$. Natural units ($\hbar = c = 1$) are employed throughout.

[12] The equivalent equation is presented in reference 6, Eq. (10). The momentum space wave functions used here are defined by $\varphi_a(\mathbf{p}) = (2\pi)^{-3}\int d^3p\ \psi_a(\mathbf{r})e^{-i\mathbf{p}\cdot\mathbf{r}}$. The fourth components of the momenta p_1 and p are the same: $p_{10} = p_0 = \mu - \epsilon_a$; this will be true of all other momenta which appear upon iteration of the electron propagator. The normalization of the latter differs from that of reference 18 by a factor of $(2\pi)^4/2$.

[13] For example, using this operator, the Dirac equation in momentum space reads $(\boldsymbol{p}-i\mu)\varphi_a = \boldsymbol{V}\varphi_a$.

corresponds to that member of the right-hand-side graphs of Fig. 1 containing n electron-Coulomb interactions.

The only infinity occurring in Eq. (2.4) is one requiring a mass renormalization. This is carried out in the standard manner by subtracting from $I_0(\not{p})$ the quantity $I_0(i\mu)$, obtained by replacing $\not{p}$ by $i\mu$ and p^2 by $-\mu^2$ in the final expression for $I_0(\not{p})$. After mass renormalization there remain ultraviolet divergences in I_0 and I_1; but as a consequence of Ward's identity these must cancel each other. Since we desire to treat I_0 and I_1 separately, we employ the covariant regulator method by making the substitution

$$\frac{1}{k^2}\rightarrow\int_0^{\Lambda^2}\frac{dL}{(k^2+L)^2}.$$

The methods for evaluating such integrals are well known,[7] and only the results will be stated here. With $I_0'(\not{p})=I_0(\not{p})-I_0(i\mu)$, the mass renormalized result is found to be[14]

$$I_0'(\not{p})=2\pi^2 i\Big\{(\not{p}-i\mu)\Big[-\frac{1}{2}\ln\Big(\frac{\Lambda^2}{\mu^2}\Big)-\frac{5}{4}+\frac{\Delta(\Delta-2\mu^2)}{2(\mu^2-\Delta)^2}\ln\Big(\frac{\Delta}{\mu^2}\Big)-\frac{\Delta}{2(\mu^2-\Delta)}\Big]+i\mu\Big[\frac{(2\mu^2-3\Delta)}{2(\mu^2-\Delta)^2}\ln\Big(\frac{\Delta}{\mu^2}\Big)-\frac{1}{2(\mu^2-\Delta)}\Big]\Delta\Big\},\quad(2.5)$$

where $\Delta=p^2+\mu^2$ and $p^2=\mathbf{p}^2-(\mu-\epsilon_a)^2$.

The contribution of Eq. (2.5) to the level shift has the form

$$\int d^3p\,\bar{\varphi}_a(\mathbf{p})\Big\{(\not{p}-i\mu)f(\Delta)+\frac{\Delta}{\mu^2}g(\Delta)\Big\}\varphi_a(\mathbf{p}).\quad(2.6)$$

For more direct comparison with the terms to be obtained from I_1, this expression may be rearranged by noting that $\phi_a(\not{p})$ satisfies the Dirac equation in the momentum representation, and that $\Delta=(\not{p}-i\mu)(\not{p}+i\mu)$. An alternative expression is therefore

$$\int d^3p_1d^3p\,\bar{\varphi}_a(\mathbf{p}_1)\Big\{f(\Delta_1)+\frac{2i\mu}{\mu^2}g(\Delta_1)+\frac{1}{\mu^2}(\not{p}_1-i\mu)g(\Delta_1)\Big\}a(\mathbf{p}_1-\mathbf{p})\varphi_a(\mathbf{p}),\quad(2.7)$$

where $\Delta_1=p_1^2+\mu^2$ and $p_1^2=\mathbf{p}_1^2-(\mu-\epsilon_a)^2$. This possibility of rearranging the terms of Eq. (2.6) incidentally demonstrates the lack of a definite $Z\alpha$ order to be associated with each separate term of the expansion of S_F^e.

At this stage it is convenient to discuss the general principles by which an expansion in powers of $Z\alpha$ may be obtained. These general principles will be stated first, and then we shall give their justification. Typically we have to evaluate expressions of the form

$$\langle F\rangle=\int d^3p_1d^3p\;\bar{\varphi}_a(\mathbf{p}_1)F(\mathbf{p}_1,\mathbf{p})\varphi_a(\mathbf{p}).\quad(2.8)$$

This integral may be assigned a "nominal order" by using the fact that a typical momentum in a hydrogen-like atom is of order $Z\alpha\mu$, and by keeping only the leading term of F in the resulting power series in $Z\alpha$. If the resulting integral over $\mathbf{p}_1$ and $\mathbf{p}$ actually converges in the nonrelativistic approximation, the result will be the correct answer, to this nominal order. On the other hand, if the integral diverges, it is an indication of an improper expansion of F, and the true order is less than the nominal order. To obtain higher powers of $Z\alpha$, both F and ϕ_a must be expanded carefully. To justify these remarks, we shall use the nonrelativistic approximation to both the large and small components of ϕ_a; the effect of relativistic corrections to the wave functions need be considered only when one is interested in contributions to the Lamb shift of order $\alpha(Z\alpha)^6\mu$, or higher. If we make the substitution

$$\mathbf{p}=Z\alpha\mu\mathbf{t},$$

and write

$$\varphi_a(\mathbf{p})=(Z\alpha\mu)^{-\frac{3}{2}}\omega_a(\mathbf{t}),$$

then ω_a will be a dimensionless spinor whose first two components are of order unity (that is, independent of $Z\alpha$) and decrease as t^{-4} for $t\gg1$; the last two components will be of order $Z\alpha$ and will decrease as t^{-3} for large t. We may remark that in the relativistic case the exponents for large t will differ from integers by amounts of order $(Z\alpha)^2$ and there will be an additional normalizing factor which differs from unity by an amount of order $(Z\alpha)^2$; the first two components then also have a contribution of order $(Z\alpha)^2$ decreasing roughly as t^{-3}. Thus in terms of dimensionless variables $\mathbf{t}$ and $\mathbf{t}_1$, expressions like Eq. (2.8) will take the form

$$\langle F\rangle=(Z\alpha\mu)^3\int d^3t_1d^3t\;\bar{\omega}_a(\mathbf{t}_1)F(Z\alpha\mu\mathbf{t}_1,Z\alpha\mu\mathbf{t})\omega_a(\mathbf{t}).$$

We can now proceed to assign orders to the various terms occurring in Eq. (2.7). In terms of dimensionless variables, we have

$$a=-i\gamma_4(Z\alpha\mu^2)^{-1}/2\pi^2|\mathbf{t}_1-\mathbf{t}|^2,$$
$$\Delta=(Z\alpha\mu)^2[\mathbf{t}^2+(2\epsilon_a/Z^2\alpha^2\mu)-(\epsilon_a^2/Z^2\alpha^2\mu^2)].$$

Also, if we refer all contributions to matrix element between large components, it is clear that $(\not{p}-i\mu)$ is of relative order $(Z\alpha)^2\mu$. To obtain a final result correct to

[14] The right-hand side of Eq. (2.5) is not singular when $\Delta=\mu^2$; this may most easily be seen by examining the integral before the parametric integrations used in its evaluation are performed.

order $\alpha(Z\alpha)^4\mu$, it is thus permissible to make the substitution

$$(\not{p}_1-i\mu)\rightarrow\Delta_1/2i\mu \tag{2.9}$$

and take the resulting operator only between the large components of ϕ_a. All of the integrals in Eq. (2.7) are convergent; however, if one tries to expand the integrand in powers of $Z\alpha$, a point will be reached beyond which the integrals will no longer converge. For clarity, consider a specific example. A typical term of I_0' yields a contribution proportional to

$$\int d^3p_1d^3p\ \bar{\varphi}_a(\mathbf{p}_1)\left(\frac{\Delta_1}{\mu^2-\Delta_1}\right)\boldsymbol{a}\varphi_a(\mathbf{p})$$

$$\equiv\int d^3p_1d^3p\ \bar{\varphi}_a(\mathbf{p}_1)\left(\frac{\Delta_1}{\mu^2}\right)\boldsymbol{a}\varphi_a(\mathbf{p})$$

$$+\int d^3p_1d^3p\ \bar{\varphi}_a(\mathbf{p}_1)\left[\frac{\Delta_1^2}{\mu^2(\mu^2-\Delta_1)}\right]\boldsymbol{a}\varphi_a(\mathbf{p}).$$

The first term on the right-hand side of this equation gives a result of order $(Z\alpha)^4\mu$ between the large components of ϕ_a. The second term is of higher order, but it would become a divergent integral if we expanded the integrand in powers of $Z\alpha$ (the remark of reference 14 is pertinent here). The second term is actually of order $(Z\alpha)^5\mu$; this may be understood from the following considerations. Without the factor $(\mu^2-\Delta_1)$ in the denominator, the integral would diverge linearly with the upper limit in $\mathbf{p}_1$ space; the effect of the denominator is to cut off the integral when $p_1\sim\mu$. Thus relative to the situation when the integral converges, there is a factor of order $(\mu/Z\alpha\mu)$. In the case of the small components of ϕ_a, the left-hand side of the equation cannot be expanded, but by the same type of consideration, it is seen to be of order $(Z\alpha)^5\mu$. In the future, when the expression "expanding in powers of Δ_1/μ^2" is used, it refers to the treatment just discussed.

Carrying out the expansions in this way, one finds that to the order $\alpha(Z\alpha)^4\mu$ in the final result the following substitution may be made

$$I_0'(\mathbf{p})\rightarrow 2\pi^2i\boldsymbol{a}(\mathbf{p}_1-\mathbf{p})\left\{-\frac{1}{2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)-\frac{1}{4}\left(1-\frac{\Delta_1}{\mu^2}\right)\right.$$

$$\left.-2\left(1+\frac{3}{4}\frac{\Delta_1}{\mu^2}\right)\ln\left(\frac{\Delta_1}{\mu^2}\right)\right\}. \tag{2.10}$$

Using the Λ^2 regulator replacement and the definitions $\mathbf{q}=\mathbf{p}_1-\mathbf{p}$, $\Delta_y=y\Delta_1+(1-y)\Delta$, $I_1(\not{p}_1,\not{p})$ may be evaluated and the result "expanded" in powers of Δ_1/μ^2, Δ/μ^2, q^2/μ^2. One then finds

$$I_1(\not{p}_1,\not{p})\rightarrow 2\pi^2i\left\{\frac{i\mu}{2\mu^2}\boldsymbol{a}\boldsymbol{q}+\boldsymbol{a}\left[\frac{1}{2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)\right.\right.$$

$$\left.+\frac{1}{4}+2+\frac{1}{12}\left(\frac{\mathbf{q}^2}{\mu^2}\right)+\frac{(\Delta_1+\Delta)}{4\mu^2}\right]$$

$$+\boldsymbol{a}\int_0^1 dy\left[2+4\frac{\Delta_y}{\mu^2}-\frac{(\Delta_1+\Delta)}{2\mu^2}\right]\ln\left(\frac{\Delta_y}{\mu^2}\right)$$

$$\left.+\frac{\mathbf{q}^2}{\mu^2}\boldsymbol{a}\int_0^1 dy[1-2y(1-y)]\ln\left(\frac{\Delta_y}{\mu^2}\right)\right\}. \tag{2.11}$$

The quantity $\boldsymbol{a}[\frac{1}{2}\ln(\Lambda^2/\mu^2)+\frac{1}{4}]$ cancels with the corresponding term of I_0', and the resulting matrix element is then free of all divergences (none of the remaining I_n are divergent). As previously mentioned, the cancellation of such spurious "charge renormalization" infinities is required by Ward's identity. In the present case, if one writes $I_1(\not{p}_1,\not{p})=a_\nu(\not{p}_1-\not{p})I_\nu(\not{p}_1,\not{p})$, this takes the form of the readily verified expression

$$I_\nu(\not{p},\not{p})=-\frac{\partial}{\partial p_\nu}I_0(\not{p})=-\frac{\partial}{\partial p_\nu}I_0'(\not{p}).$$

[Incidentally, this provides a convenient check on the calculation of I_0' and I_1, with I_0' in its original form, Eq. (2.5).] The cancellation of the $\ln(\Lambda^2/\mu^2)$ terms between I_0' and I_1 must therefore occur.

As is well known, one can derive Ward's identity using a simple argument concerning invariance with respect to gauge transformations of the external potential. If the latter is shifted by the constant amount Λ,

$$V(x)=i\gamma_4V_0(x)\rightarrow i\gamma_4[V_0(x)+\Lambda],$$

then the fourth component of $\not{p}$, appearing in the formal relation for S_F^e, Eq. (2.3), will in turn be displaced by the amount $i\Lambda$,

$$i\gamma_4E\rightarrow i\gamma_4(E+\Lambda),$$

which is just what is required to leave S_F^e unchanged. Thus, if one considers the expansion of S_F^e in powers of V before this gauge transformation is performed, one has

$$S_F^e=S(E)+S(E)VS(E)+S(E)VS(E)VS(E)+\cdots,$$

where we have written $S(E)=S_F^0(\not{p}-\not{k})=(\not{p}-\not{k}-i\mu)^{-1}$. This is to be compared with the identical expression obtained after making the gauge transformation

$$S_F^e=S(E+\Lambda)+S(E+\Lambda)[V+i\gamma_4\Lambda]S(E+\Lambda)$$
$$+S(E+\Lambda)[V+i\gamma_4\Lambda]S(E+\Lambda)[V+i\gamma_4\Lambda]S(E+\Lambda)+\cdots.$$

To first order in Λ, one then finds

$$\left[\frac{\partial S}{\partial E}+iS\gamma_4S\right]+\left[\frac{\partial}{\partial E}(SVS)+i(S\gamma_4SVS+SVS\gamma_4S)\right]+\cdots=0.$$

Since each order of V must vanish separately, each square bracket is identically zero; the vanishing of each such bracket then yields the Ward conditions relating the matrix elements I_{n+1} with no momentum transfer at any one of its vertices to the derivative of I_n with respect to E. (For example, the vanishing of the first square bracket is equivalent to the previous demonstration proving the cancellation of the spurious charge renormalization terms between I_0' and I_1.)

Since the entire matrix element is gauge-invariant, we can expect that all non-gauge-invariant terms, such as the one proportional to $(\not{p}_1-i\mu)\not{a}$ which appears in Eq. (2.15), will exactly cancel. That this is in fact true can easily be shown by the following argument. Suppose that correct to the order $\alpha(Z\alpha)^4\mu$, I_2 contains the term $T_2=\xi\not{a}\not{a}$, where ξ is some numerical constant. Then, by the vanishing of the second square bracket, there must be a term in I_1, say T_1, which is related to T_2 by

$$\partial T_1/\partial E+i\xi[\gamma_4\not{a}+\not{a}\gamma_4]=0.$$

Therefore

$$T_1=-\xi[(\not{p}_1-i\mu)\not{a}+\not{a}(\not{p}-i\mu)]+C(\mathbf{p}_1,\mathbf{p}),$$

where C is some spinor function of $\mathbf{p}_1$ and $\mathbf{p}$ which does not depend on E (e.g., $C\sim\not{a}q^2$). This can be the only such term of this form in I_1; otherwise, by reversing the argument just presented, one would have to infer that the value of ξ is different from that originally assumed. Now using the vanishing of the first square bracket, and the fact that I_1 contains T_1, there must be a term in I_0, say T_0, which is related to T_1 by

$$\partial T_0/\partial E-i\xi[(\not{p}-i\mu)\gamma_4+\gamma_4(\not{p}-i\mu)]+iC(\mathbf{p},\mathbf{p})=0,$$

or

$$T_0=\xi(\not{p}-i\mu)(\not{p}-i\mu)-iEC(\mathbf{p},\mathbf{p})+C'(\mathbf{p}).$$

The non-gauge-invariant $\alpha(Z\alpha)^4\mu$ contributions of $I_0+I_1+I_2$ are then given by the expectation value of T_2 and the first terms of both T_1 and T_0

$$\xi\langle a|(\not{p}-i\mu)(\not{p}-i\mu)|a\rangle-\xi\langle a|(\not{p}_1-i\mu)\not{a}+\not{a}(\not{p}-i\mu)|a\rangle+\xi\langle a|\not{a}\not{a}|a\rangle=0,$$

using the equation satisfied by the atomic state. Thus these terms exactly cancel, as they must.

The quantities I_0' and I_1 contain terms which yield contributions of order $\alpha(Z\alpha)^2\mu$ and $\alpha(Z\alpha)^2\mu\ln(Z\alpha)$. One might have expected such spurious terms to cancel as a consequence of Ward's identity, since they do not appear in the free-particle scattering calculation, but this is not the case. In fact, by using Ward's identity it is not difficult to show that their failure to cancel is due to the presence of terms proportional to $\ln(\Delta/\mu^2)$ in I_0'. Had we used a photon mass and treated the electron momenta as free, these terms would not have arisen. In the sum of I_0' and I_1, the $\alpha(Z\alpha)^2\mu\ln(Z\alpha)$ terms cancel, but there is left a peculiar quantity of the form

$$\not{a}\int_0^1 dy\ln(\Delta_y/\Delta_1),$$

which is of order $\alpha(Z\alpha)^2\mu$ (note that $\ln(\Delta_y/\Delta_1)$ vanishes for $\mathbf{p}_1=\mathbf{p}$, as required by Ward's identity). This $\alpha(Z\alpha)^2\mu$ contribution must be canceled by corresponding terms arising from the remaining I_n, since a level shift of the order $\alpha(Z\alpha)^4\mu$ permits no terms of order less than $\alpha(Z\alpha)^4\mu$ for the sum of the I_n. Unless this cancellation can be explicitly demonstrated, the method of expanding $S_F{}^e$ in powers of V is in serious difficulty. For example, one cannot merely neglect such terms in a calculation of the difference of two energy-level shifts since the $\alpha(Z\alpha)^2\mu$ contributions will in general have different values for each state; this has been explicitly verified for the $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ levels.

This difficulty may be overcome by proper choice of the gauge of the quantized electromagnetic field. Feynman[1] showed, in the free-particle scattering case, that the replacement of any polarization vector e_ν by the photon momentum k_ν yields a zero result upon adding all the terms from the contributing graphs. For the bound-state case, a corresponding gauge statement would be

$$\langle a|\{\not{k}S_F{}^e(\not{p}-\not{k})\not{e}+\not{e}S_F{}^e(\not{p}+\not{k})\not{k}\}|a\rangle=0, \quad (2.12)$$

which is easy to verify by the use of Eq. (2.3) and the equation satisfied by the state $|a\rangle$; essentially, the effect of $\not{k}S_F{}^e(\not{p}-\not{k})$ acting on $\langle a|$ is to produce $(-1)\langle a|$. This latter property is now utilized as follows:

$$\langle a|\not{k}S_F{}^e(\not{p}-\not{k})\not{k}|a\rangle=-\langle a|\not{k}|a\rangle,$$

and

$$\int_F\frac{d^4k}{k^2}F(k^2)\langle a|\not{k}|a\rangle=0,$$

when the integration is performed over a symmetrical k space.

A gauge transformation is now given by making the following substitution in Eq. (2.1):

$$\gamma_\mu\rightarrow\gamma_\mu+\lambda k_\mu\not{k}/k^2.$$

It might be expected that the proper choice of λ would be (-1) since this leads to gauge-invariant emission and absorption operators and makes each term of the expansion gauge invariant by itself, in contrast to the situation in the original form where various terms must be combined. However, for reasons which are not understood physically, this is not the case; there exists another choice which leads to simpler results. The effect of such a substitution is to add to I an operator J given by

$$J=\zeta\int_F\frac{d^4k}{k^2k^2}\not{k}S_F{}^e(\not{p}-\not{k})\not{k}, \quad (2.13)$$

where ζ is a constant to be determined. This operator has zero expectation value in the given state, and consequently the infinite set of expansion terms

$$J(\not{p}_1,\not{p})=\delta(\mathbf{p}_1-\mathbf{p})J_0(\not{p})+J_1(\not{p}_1,\not{p})+J_2(\not{p}_1,\not{p})+\cdots+J_n(\not{p}_1,\not{p})+\cdots,$$

$$J_0(\not{p})=\zeta\int_F\frac{d^4k}{k^2k^2}\not{k}\frac{1}{\not{p}-\not{k}-i\mu}\not{k},$$

$$J_1(\not{p}_1,\not{p})=\zeta\int_F\frac{d^4k}{k^2k^2}\not{k}\frac{1}{\not{p}_1-\not{k}-i\mu}\not{a}(\mathbf{p}_1-\mathbf{p})\frac{1}{\not{p}-\not{k}-i\mu}\not{k},$$

$$\cdots$$

$$J_n(\not{p}_1,\not{p})=\zeta\int_F\frac{d^4k}{k^2k^2}\not{k}\frac{1}{\not{p}_1-\not{k}-i\mu}\Big\{\int d^3p_1'\cdots\times d^3p_{n-1}'\not{a}(\mathbf{p}_1-\mathbf{p}_1')\frac{1}{\not{p}_1'-\not{k}-i\mu}\not{a}(\mathbf{p}_1'-\mathbf{p}_2')\cdots\times\not{a}(\not{p}_{n-1}'-\not{p})\Big\}\frac{1}{\not{p}-\not{k}-i\mu}\not{k},$$

may be added to the expansion of $I(\mathbf{p}_1,\mathbf{p})$ without altering the level shift. J_0 and J_1 have ultraviolet divergences of the charge renormalization type only; these are calculated using the same regulator as used for I_0 and I_1, and cancel in the same manner. To order $(Z\alpha)^4\mu$, the results are

$$J_0\rightarrow\zeta\pi^2 i\boldsymbol{a}\Big\{-\ln\Big(\frac{\Lambda^2}{\mu^2}\Big)+1+\frac{1}{2}\Big(\frac{\Delta_1}{\mu^2}\Big)+\Big(2+\frac{3}{2}\frac{\Delta_1}{\mu^2}\Big)\ln\Big(\frac{\Delta_1}{\mu^2}\Big)\Big\},$$

$$J_1\rightarrow\zeta\pi^2 i\Big\{\boldsymbol{a}\Big[\ln\Big(\frac{\Lambda^2}{\mu^2}\Big)-3-\frac{1}{6}\Big(\frac{q^2}{\mu^2}\Big)-\frac{(\Delta+\Delta_1)}{\mu^2}\Big]+\boldsymbol{a}\int_0^1 dy\Big[\frac{\Delta_1+\Delta}{2\mu^2}-4\frac{\Delta_y}{\mu^2}-2\Big]\ln\Big(\frac{\Delta_y}{\mu^2}\Big)+\frac{q^2}{\mu^2}\boldsymbol{a}\int_0^1 dy[6y(1-y)-1]\ln\Big(\frac{\Delta_y}{\mu^2}\Big)\Big\}. \quad (2.14)$$

Comparing Eq. (2.14) with (2.10) and (2.11), it is evident that if $\zeta=2$ the combination $(I_0'+J_0)+(I_1+J_1)$ is free of all $(Z\alpha)^2\mu$ dependence; with $M_0=I_0'+J_0$, $M_1=I_1+J_1$, one obtains the result

$$M_0+M_1\rightarrow 2\pi^2 i\Big\{-\frac{3i\mu}{2\mu^2}(\mathbf{p}_1-i\mu)\boldsymbol{a}+\frac{q^2}{\mu^2}\boldsymbol{a}\Big[-\frac{1}{12}+4\int_0^1 dyy(1-y)\ln\Big(\frac{\Delta_y}{\mu^2}\Big)\Big]+\frac{i}{2\mu}\boldsymbol{aq}\Big\}. \quad (2.15)$$

Note that the magnetic moment portion of I_1 is unchanged [the last term of Eq. (2.15) or the first of (2.11)], and that the term

$$\frac{q^2}{\mu^2}\boldsymbol{a}\int_0^1 dy[6y(1-y)-1]\ln\Big(\frac{\Delta_y}{\mu^2}\Big)$$

arising from J_1 is proportional to $(Z\alpha)^4\mu$ rather than to $(Z\alpha)^4\mu\ln(Z\alpha)$. This then represents an additive constant to the logarithm due to that portion of I_1,

$$\frac{q^2}{\mu^2}\boldsymbol{a}\int_0^1 dy[1-2y(1-y)]\ln\Big(\frac{\Delta_y}{\mu^2}\Big),$$

which is the only term giving a contribution of the form $(Z\alpha)^4\mu\ln(Z\alpha)$. Upon writing down the $Z\alpha$ expansion of the nth term of the combination $M=I+J$, it is not difficult to show that the result is free of all $(Z\alpha)^2\mu$ and $(Z\alpha)^2\mu\ln(Z\alpha)$ dependence; this will be demonstrated in the following section.

The simple form of Eq. (2.15) in comparison with its separate components suggests that the gauge used has a special physical significance. It is related to the fact that in this gauge the "spurious charge renormalization" has no infrared divergence, while with the other choice of gauge mentioned earlier it would have had no ultraviolet divergence. The spurious charge renormalization is given by the vertex operator evaluated on the energy shell with zero momentum transfer. For the special gauge used here, this operator contains the term

$$\boldsymbol{a}\int_F d^4k\,\frac{1}{k^2(k^2-2k\cdot p)^2}\Big[4p^2+\frac{8(p\cdot k)^2}{k^2}\Big]$$

in addition to other contributions which are not infrared divergent. Integrating over k_0, we find that only the poles in the photon propagator contribute to the infrared divergence; the first term has simple poles and the second double poles. The result is

$$-\pi i\boldsymbol{a}\int\frac{dk}{k}\,d\Omega\Big[\frac{m^2}{(E-p\cos\theta)^2}-1\Big]=0.$$

This gives a simple, explicit proof of the cancellation of the infrared contribution to the spurious charge renormalization. Since soft photons are essentially independent of each other[15] the result is true for any number of soft photons. As a direct consequence of this proof, we see that terms of the form

$$\boldsymbol{a}\int_0^1 dy\ln\Big(\frac{\Delta_y}{\mu^2}\Big)$$

cannot occur in M_1, and hence there can be no residual contributions of order $\alpha(Z\alpha)^2\mu$ in (M_0+M_1).

It is now interesting to determine the amount by which the energy level of an atomic state is displaced by the contribution of Eq. (2.15). For simplicity, consider the $1S$ state of hydrogen and omit the magnetic moment term; one then wishes to evaluate

$$\Delta E_0'=-2\alpha\int d^3p_1d^3p\;\varphi_0^+(p_1)\Big\{-2\pi^2a_0\Big(-\frac{3}{4}\frac{\Delta_1}{\mu^2}+\frac{q^2}{\mu^2}\Big[-\frac{1}{12}+4\int_0^1 dy\,y(1-y)\ln\Big(\frac{\Delta y}{\mu^2}\Big)\Big]\Big)\Big\}\varphi_0(p),$$

where the Dirac wave functions have been replaced by their nonrelativistic Schrödinger limits ($\Delta E_0'$ is then unchanged to the order $\alpha(Z\alpha)^4\mu$). One finds

$$\Delta E_0'=\frac{4}{3\pi}\alpha(Z\alpha)^4\mu\Big\{\frac{30}{24}-\ln\Big(\frac{2\bar{\epsilon}}{\mu}\Big)\Big\}, \quad (2.16)$$

where the mean energy $\bar{\epsilon}$ introduced here is defined by

$$\int d^3p_1d^3p\;\varphi_0^+(p_1)\Big(4\int_0^1 dyy(1-y)\ln\Big(\frac{\Delta_y}{\mu^2}\Big)\Big)\varphi_0(p)=\frac{2}{3}\Big[\int d^3p\;\varphi_0(p)\Big]^2\ln\Big(\frac{2\bar{\epsilon}}{\mu}\Big). \quad (2.17)$$

[15] J. M. Jauch and F. Rohrlich, Helv. Phys. Acta **27**, 613 (1954); S. N. Gupta, Phys. Rev. **98**, 1507 (1955); **99**, 1015 (1955); D. R. Yennie and H. Suura, Phys. Rev. **105**, 1378 (1957).

Introducing dimensionless variables, Eq. (2.17) becomes

$$\frac{2}{3}\left(\frac{\pi}{4}\right)^2 \ln\left(\frac{2\bar{\epsilon}}{\mu(Z\alpha)^2}\right) = 4\int_0^1 dy\, y(1-y)\int_0^\infty \frac{dt_1 dt t_1^2 t^2}{(t_1^2+1)^2(t^2+1)^2} \times \ln[1+yt_1^2+(1-y)t^2] = \tfrac{2}{3}(\pi/4)^2\{\ln 16+\tfrac{1}{6}\}.$$

Equation (2.16) is to be compared to the complete value for this portion of the level shift

$$\Delta E_0=\frac{4}{3\pi}\alpha(Z\alpha)^4\mu\left\{\frac{11}{24}-\ln\left(\frac{2k_0}{\mu}\right)\right\}, \qquad (2.18)$$

where k_0 is the average excitation energy defined by Bethe[3] and calculated by Harriman[16] to be approximately 19.7 ry for the $1S$ state. It is noteworthy that the approximate expression contains the right coefficient for the logarithmic term. This shows, as will be confirmed in the next section, that the many-potential terms do not contribute a shift of order $\alpha(Z\alpha)^4\mu\ln(Z\alpha)$. Equation (2.16) yields a result about 10% larger than that of Eq. (2.18); the error is mainly an over-estimate of the additive constants to the logarithm. If the term $(\boldsymbol{p}_1-i\mu)\boldsymbol{a}$ of Eq. (2.15) is neglected in this calculation, the 30/24 of Eq. (2.16) becomes, instead, 3/24; this then yields a level shift roughly 10% smaller than that given by Eq. (2.18). As mentioned previously, any such $(\boldsymbol{p}_1-i\mu)\boldsymbol{a}$ term is not gauge invariant, and all contributions of this form have been shown to cancel. It is also worth noting that, while the additive constants are not obtained correctly, the large value of the excitation energy is in a sense contained in the present approximation. The large value can be "explained" by noting that in the above expression for $\bar{\epsilon}$ the integrand is not a rapidly decreasing function of t and t_1. Thus even though the integrand begins to drop off when t_1 and t exceed 1 (p_1 and p exceed $Z\alpha\mu$), a substantial contribution to the integral comes from larger values of t_1 and t (t_1, $t\sim 4$ or 5). In summary, the ZOP contributions yield the major part of the Lamb shift; the MP terms contribute only to the additive constants.

3. "MANY-POTENTIAL" CONTRIBUTIONS

We now calculate all the $\alpha(Z\alpha)^4\mu$ dependence of the nth term of the combination $M_n=I_n+J_n$. The following definitions will prove convenient:

$$\mathfrak{D}_n=(k^2-2k\cdot p_1+\Delta_1)^{-1}(k^2-2k\cdot p+\Delta)^{-1} \times(k^2-2k\cdot p_1'+\Delta_1')^{-1}\cdots(k^2-2k\cdot p_{n-1}'+\Delta_{n-1}')^{-1},$$

$$\Gamma_n=\boldsymbol{a}(\mathbf{p}_1-\mathbf{p}_1')(\boldsymbol{p}_1'-\boldsymbol{k}+i\mu)\boldsymbol{a}(\mathbf{p}_1'-\mathbf{p}_2')\cdots \times(\boldsymbol{p}_{n-1}'-\boldsymbol{k}+i\mu)\boldsymbol{a}(\mathbf{p}_{n-1}'-\mathbf{p}),$$

$$N_n=\gamma_\mu(\boldsymbol{p}_1-\boldsymbol{k}+i\mu)\Gamma_n(\boldsymbol{p}-\boldsymbol{k}+i\mu)\gamma_\mu +2k^{-2}\boldsymbol{k}(\boldsymbol{p}_1-\boldsymbol{k}+i\mu)\Gamma_n(\boldsymbol{p}-\boldsymbol{k}+i\mu)\boldsymbol{k}.$$

[16] J. M. Harriman, Phys. Rev. **101**, 594 (1956).

The nth matrix element ($n\geqslant 2$) is then given by

$$M_n=\int d^3p_1'\cdots d^3p_{n-1}'\int_F \frac{d^4k}{k^2}\,\mathfrak{D}_nN_n,$$

where the integrations over the $n-1$ intermediate momenta are to be understood whenever, in the interests of simplicity, they are omitted. The denominator combinations used here represent an immediate extension of those appearing in the calculation of I_1, and are presented, together with several necessary integrals, in the appendix.

The numerator N_n, of the integrand of M_n, may be rewritten so that the outer γ_μ and $\boldsymbol{k}$ factors stand directly on either side of Γ_n. From the array of terms so obtained it is convenient to list the following subset, each of which (neglecting the $\boldsymbol{k}$ dependence of Γ_n) is homogeneous in k

$$N_n^{(1)}=\left\{4\boldsymbol{p}_1\cdot\boldsymbol{p}+\frac{8}{k^2}(k\cdot p_1)(k\cdot p)\right\}\Gamma_n,$$

$$N_n^{(2)}=-2\{\boldsymbol{p}\boldsymbol{k}\Gamma_n+\Gamma_n\boldsymbol{k}\boldsymbol{p}_1+2k\cdot(p_1+p)\Gamma_n\},$$

$$N_n^{(3)}=\boldsymbol{k}\gamma_\mu\Gamma_n\gamma_\mu\boldsymbol{k}+2k^2\Gamma_n,$$

$$N_n^{(4)}=-2\left\{(\boldsymbol{p}_1-i\mu)\left[\boldsymbol{p}+\frac{2}{k^2}(k\cdot p)\boldsymbol{k}\right]\Gamma_n +\Gamma_n\left[\boldsymbol{p}_1+\frac{2}{k^2}(k\cdot p_1)\boldsymbol{k}\right](\boldsymbol{p}-i\mu)\right\}.$$

To the order $\alpha(Z\alpha)^4\mu$ these are the only terms that enter; this will become apparent below. Thus $N_n\rightarrow\sum_j N_n^{(j)}$; to each $N_n^{(j)}$ there corresponds an $M_n^{(j)}$, and to the order $\alpha(Z\alpha)^4\mu$, $M_n\rightarrow\sum_j M_n^{(j)}$.

It is also convenient to separate Γ_n into a part independent of $\boldsymbol{k}$, a part containing one $\boldsymbol{k}$ term, etc. Thus

$$\Gamma_n=\Gamma_n^{(0)}+\Gamma_n^{(1)}+\Gamma_n^{(2)}+\cdots,$$

$$\Gamma_n^{(0)}=\boldsymbol{a}(\mathbf{p}_1-\mathbf{p}_1')(\boldsymbol{p}_1'+i\mu)\boldsymbol{a}(\mathbf{p}_1-\mathbf{p}_2')(\boldsymbol{p}_2'+i\mu) \times\boldsymbol{a}(\mathbf{p}_2'-\mathbf{p}_3')\cdots\boldsymbol{a}(\mathbf{p}_{n-1}'-\mathbf{p}),$$

$$\Gamma_n^{(1)}=-k_\mu\{\boldsymbol{a}(\mathbf{p}_1-\mathbf{p}_1')\gamma_\mu\boldsymbol{a}(\mathbf{p}_1'-\mathbf{p}_2')(\boldsymbol{p}_2'+i\mu) \times\boldsymbol{a}(\mathbf{p}_2'-\mathbf{p}_3')\cdots\boldsymbol{a}(\mathbf{p}_{n-1}'-\mathbf{p})+\cdots\}.$$

The terms $\Gamma_n^{(2)}$ and those with a higher number of $\boldsymbol{k}$ factors will not actually be needed. One therefore considers the integrands $N_n^{(j)}$ depending on that portion of Γ_n independent of $\boldsymbol{k}$, $N_n^{(j)}(\Gamma_n^{(0)})$; the $N_n^{(j)}$ depending on that portion of Γ_n containing one $\boldsymbol{k}$, $N_n^{(j)}(\Gamma_n^{(1)})$; etc. The contributions obtained by performing the k integral will then be denoted by $M_n^{(j)}(\Gamma_n^{(0)})$, $M_n^{(j)}(\Gamma_n^{(1)})$, etc.

The calculation of $M_n^{(1)}(\Gamma_n^{(0)})$ will be presented in some detail, to illustrate the method and prove the cancellation of the spurious $\alpha(Z\alpha)^2\mu$ terms. Using the

definitions of $\bar{\Delta}_{n-1}$, $\bar{p}_{n-1}$ and formulas (1) and (4) of the appendix, one obtains

$$M_n^{(1)}(\Gamma_n^{(0)})=\frac{\pi^2 i}{n}\int dY_{n-1}\,\Gamma_n^{(0)}(\bar{p}_{n-1}{}^2)^{-1}\times\int_0^1 dx(\bar{\Delta}_{n-1}-x\bar{p}_{n-1}{}^2)^{-n}T_n(x),$$

with

$$T_n(x)=8[(p_1\cdot p)(\bar{p}_{n-1})^2-(p_1\cdot\bar{p}_{n-1})(p\cdot\bar{p}_{n-1})]+4x[4(p_1\cdot\bar{p}_{n-1})(p\cdot\bar{p}_{n-1})-(p_1\cdot p)\bar{p}_{n-1}{}^2]. \quad (3.1)$$

To determine the $Z\alpha$ order of these terms recall that every momentum which appears has the same fourth component, $i(\mu-\epsilon_a)$, and consequently this is also the fourth component of $\bar{p}_{n-1}$. A lowest $Z\alpha$ order may be assigned to the integral

$$\int_0^1 dx(\bar{\Delta}_{n-1}-x\bar{p}_{n-1}{}^2)^{-n}=[(n-1)\bar{p}_{n-1}{}^2]^{-1}\times\{(\bar{\Delta}_{n-1}-\bar{p}_{n-1}{}^2)^{-(n-1)}-(\bar{\Delta}_{n-1})^{-(n-1)}\} \quad (3.2)$$

by recognizing that the first term on the right-hand side of Eq. (3.2) will yield a result, after all the momentum integrations are performed, of order higher than $\alpha(Z\alpha)^4\mu$ [actually $\alpha(Z\alpha)^5\mu$ for $n=2$]. On the other hand, the second term on the right-hand side of Eq. (3.2) will yield a convergent matrix element of lowest nominal order $\alpha(Z\alpha)^2\mu$. If we consider only this portion of Eq. (3.2) when evaluating the integral of Eq. (3.1), and expand the remaining spatial momentum dependence of the latter only up to the point in which the resulting integrals are convergent, then the spatial components of all momenta will have the order $Z\alpha\mu$. Each Δ', as well as Δ_1, Δ, and $\bar{\Delta}_{n-1}$, is then of order $(Z\alpha\mu)^2$, and the second term of Eq. (3.2) has the order $(Z\alpha\mu)^{-2(n-1)}\mu^{-2}$.

For clarity, consider the lowest $Z\alpha$ order of a term such as

$$\int d^3p_1 d^3p\,\varphi_a{}^+(\mathbf{p}_1)\int d^3p_1{}'\cdots d^3p_{n-1}{}'\Gamma_n{}^{(0)}\times\int_0^1 dx(\bar{\Delta}_{n-1}-x\bar{p}_{n-1}{}^2)^{-n}(p_1\cdot p)\varphi_a(\mathbf{p}), \quad (3.3)$$

where all the momentum integrals are explicitly shown; the wave functions used are the Schrödinger limits of the Dirac ϕ_a. Also to lowest order, $\Gamma_n{}^{(0)}\to(2i\mu)^{n-1}\boldsymbol{a}(\mathbf{p}_1-\mathbf{p}_1{}')\cdots\boldsymbol{a}(\mathbf{p}_{n-1}{}'-\mathbf{p})$, since each $\boldsymbol{p}'\to i\mu$. (The Dirac notation is superfluous, but will be retained.) Using the substitution of the previous section ($\mathbf{p}=Z\alpha\mu\mathbf{t}$), we find $\boldsymbol{a}(q)\sim(Z\alpha\mu^2)^{-1}$ and then $\Gamma_n{}^{(0)}\sim\mu^{-1}(Z\alpha\mu)^{-n}$; further, $p_1\cdot p$ is of order μ^2. Finally, each momentum integral yields a factor $(Z\alpha\mu)^3$, and the wave functions a factor $(Z\alpha\mu)^{-\frac{3}{2}}$. The lowest order contribution of Eq. (3.3), and therefore of Eq. (3.1), is then

$$\alpha\mu^{-1}(Z\alpha\mu)^{[-3+3(n+1)-2(n-1)-n]}=\alpha(Z\alpha)^2\mu.$$

A result of order $\alpha(Z\alpha)^4\mu$ may be obtained from Eq. (3.3) by retaining the spatial components of any two momenta in the numerator, or alternatively by inserting a factor of x in the integrand.

Returning to Eq. (3.1), it is now evident that only those terms in the integrand which are not multiplied by x can yield an $\alpha(Z\alpha)^2\mu$ contribution. The statement that the effect of the J_n is to cancel all $\alpha(Z\alpha)^2\mu$ dependence from M_n is then verified by noting that, to lowest order (replacing all momenta by $i\mu$), the first bracket of Eq. (3.1) vanishes. From the above remarks it follows that none of the other $M_n{}^{(j)}(\Gamma_n{}^{(0)})$ or $M_n{}^{(j)}(\Gamma_n{}^{(1)})$ can yield an $\alpha(Z\alpha)^2\mu$ result; thus all $\alpha(Z\alpha)^2\mu$ dependence has vanished from the complete matrix element. The $\alpha(Z\alpha)^4\mu$ contribution of Eq. (3.1) is obtained by retaining the lowest order spatial momentum portion of the first bracket and the lowest order part of the second; denoting $\boldsymbol{a}(\mathbf{p}_1-\mathbf{p}_1{}')\boldsymbol{a}(\mathbf{p}_1{}'-\mathbf{p}_2{}')\cdots\boldsymbol{a}(\mathbf{p}_{n-1}{}'-\mathbf{p})$ by the symbol $[\boldsymbol{a}]^n$, one then finds

$$M_n^{(1)}(\Gamma_n^{(0)})\to\frac{8\pi^2 i}{n(n-1)}\frac{(2i\mu)^{n-1}}{\mu^2}[\boldsymbol{a}]^n\int dY_{n-1}(\bar{\Delta}_{n-1})^{-(n-1)}\times\{\mathbf{p}_1\cdot\mathbf{p}-\mathbf{p}_{n-1}\cdot(\mathbf{p}+\mathbf{p}_1)+\bar{\mathbf{p}}_{n-1}{}^2\}-\frac{12\pi^2 i}{n}\mu^2(2i\mu)^{n-1}[\boldsymbol{a}]^n\times\int dY_{n-1}\int_0^1 dx(\bar{\Delta}_{n-1}-x\bar{p}_{n-1}{}^2)^{-n}x$$

Using the formulas listed in the appendix, the limits of the remaining $M_n{}^{(j)}$ terms are similarly found to be

$$M_n^{(2)}(\Gamma_n^{(0)})\to\frac{12\pi^2 i}{n}\mu^2(2i\mu)^{n-1}[\boldsymbol{a}]^n\times\int dY_{n-1}\int_0^1 dx(\bar{\Delta}_{n-1}-x\bar{p}_{n-1}{}^2)^{-n}x,$$

$$M_n^{(3)}(\Gamma_n^{(0)})\to M_2^{(3)}(\Gamma_2^{(0)})\to 6\pi^2 i\left(\frac{i\mu}{\mu^2}\right)\boldsymbol{aa},$$

$$M_n^{(4)}(\Gamma_n^{(0)})\to 0,$$

$$M_n^{(1)}(\Gamma_n^{(1)})\to M_2^{(1)}(\Gamma_2^{(1)})\to 6\pi^2 i\left(\frac{i\mu}{\mu^2}\right)\boldsymbol{aa},$$

$$M_n^{(2)}(\Gamma_n^{(1)})\to M_2^{(2)}(\Gamma_2^{(1)})\to -6\pi^2 i\left(\frac{i\mu}{\mu^2}\right)\boldsymbol{aa},$$

$$M_n^{(3)}(\Gamma_n^{(1)})\to M_2^{(3)}(\Gamma_2^{(1)})\to -3\pi^2 i\left(\frac{i\mu}{\mu^2}\right)\boldsymbol{aa},$$

$$M_n^{(4)}(\Gamma_n^{(1)})\to 0.$$

(In the cases indicated, a contribution of order $\alpha(Z\alpha)^4\mu$ occurs only for $n=2$.) The contribution of $M_n{}^{(4)}(\Gamma_n{}^{(1)})$, as is easily seen, contains no terms which can yield an

$\alpha(Z\alpha)^4\mu$ result. For the latter reason all the remaining terms of N_n, including those containing $\Gamma_n^{(3)}$, $\Gamma_n^{(4)}$, etc., were initially omitted. To obtain an expression for the entire matrix element, the sum of all the $\alpha(Z\alpha)^4\mu$ contributions from the MP terms must be added to that of Eq. (2.13). As anticipated, one then sees that the non-gauge-invariant terms exactly cancel, leaving

$$\sum_{n=0}^{\infty} M_n \to 2\pi^2 i\left(\frac{q^2}{\mu^2}\right)a\left\{-\frac{1}{12}+4\int_0^1 dy\, y(1-y)\ln\left(\frac{\Delta_y}{\mu^2}\right)\right\}$$
$$+\sum_{n=2}^{\infty}\frac{8\pi^2 i}{n(n-1)}\frac{(2i\mu)^{n-1}}{\mu^2}[a]^n\int dY_{n-1}(\bar{\Delta}_{n-1})^{-(n-1)}$$
$$\times\{\mathbf{p}_1\cdot\mathbf{p}+\bar{\mathbf{p}}_{n-1}{}^2-\bar{\mathbf{p}}_{n-1}\cdot(\mathbf{p}+\mathbf{p}_1)\},\quad(3.4)$$

which is the $\alpha(Z\alpha)^4\mu$ portion of the entire matrix element, excluding the magnetic moment contribution.

It is worthwhile to note that in this procedure the separation of the matrix element into high- and low-energy parts has not been required, nor has the dipole approximation been utilized. In fact, except for the case $n=2$, our procedure amounts to using the non-relativistic approximation to the electron propagator between potential scatterings ($\not{p}_j'-\not{k}+i\mu\to 2i\mu$ in the numerator). Retardation effects are, however, properly taken into account in this nonrelativistic propagator; i.e., the spatial components of k_μ are not neglected. To the given order in $Z\alpha$, all the logarithmic terms occur in the one-potential contribution; this can be seen most easily by considering the x integration of Eq. (3.1), which for $n>1$ yields Eq. (3.2) and for $n=1$ leads to $\ln(\bar{\Delta}/\mu^2)$.

4. DEDUCTION OF THE LOWEST ORDER LAMB SHIFT FORMULA

Equation (3.4) will now be shown to be identical in content to the familiar $\alpha(Z\alpha)^4\mu$ level shift formula[4]; the method used to display this equivalence will permit the development of an alternative expression for the Bethe logarithm. The second line of Eq. (3.4) may be written as $\sum_{n=2}^{\infty}\mathfrak{M}_n$, where

$$\mathfrak{M}_n=\frac{4\pi^2}{\mu^3}\frac{(2i\mu)^n}{n(n-1)}\int d^3p_1'\cdots d^3p_{n-1}'a(\mathbf{p}_1-\mathbf{p}_1')\cdots$$
$$\times a(\mathbf{p}_{n-1}'-\mathbf{p})\int dY_{n-1}(\bar{\Delta}_{n-1})^{-(n-1)}$$
$$\times\{\mathbf{p}_1\cdot\mathbf{p}-\bar{\mathbf{p}}_{n-1}\cdot(\mathbf{p}+\mathbf{p}_1)+\bar{\mathbf{p}}_{n-1}{}^2\}.\quad(4.1)$$

The problem here is to find a simplified representation for Eq. (4.1). To this end, consider the quantity $\Pi_n(z)$ defined by

$$\Pi_n(z)=[D(p_1)D(p)D(p_1')\cdots D(p_{n-1}')]^{-1},\quad(4.2)$$

where $D(p_j')=(z+\Delta_j'+\mathbf{v}\cdot\mathbf{p}_j')$. The vector $\mathbf{v}$ used here is independent of z and of all the momenta, but is otherwise arbitrary. By exactly the same denominator combinations as previously used for the M_n, it follows that

$$\Pi_n(z)=\int dY_{n-1}[z+\bar{\Delta}_{n-1}+\mathbf{v}\cdot\bar{\mathbf{p}}_{n-1}]^{-(n+1)}.\quad(4.3)$$

Using the identity (valid for $b>0$, $n\geqslant 2$)

$$\frac{1}{n(n-1)b^{n-1}}=\int_0^\infty\frac{zdz}{(z+b)^{n+1}}=\frac{(n+1)}{2!}\int_0^\infty\frac{z^2dz}{(z+b)^{n+2}}$$
$$=\frac{(n+1)(n+2)}{3!}\int_0^\infty\frac{z^3dz}{(z+b)^{n+3}},$$

and associating the quantity (b) with $(\bar{\Delta}_{n-1})$, Eq. (4.3) can be used to obtain the relation

$$\frac{1}{n(n-1)}\int\frac{dY_{n-1}}{(\bar{\Delta}_{n-1})^{n-1}}\{\mathbf{p}_1\cdot\mathbf{p}-\bar{\mathbf{p}}_{n-1}\cdot(\mathbf{p}+\mathbf{p}_1)+\bar{\mathbf{p}}_{n-1}{}^2\}$$
$$=\int_0^\infty dz\left\{z(\mathbf{p}_1\cdot\mathbf{p})+\frac{z^2}{2}(\mathbf{p}_1+\mathbf{p})\cdot\frac{\partial}{\partial\mathbf{v}}+\frac{z^3}{3!}\frac{\partial^2}{\partial\mathbf{v}^2}\right\}\Pi_n(z)\Big|_{\mathbf{v}\to 0}.$$

The vector $\mathbf{v}$ is set equal to zero after the differentiation is performed; the derivatives are to be taken before the z integration is carried out. Upon using Eq. (4.2), Eq. (4.1) can then be written as

$$\sum_{n=2}^{\infty}\mathfrak{M}_n=\frac{4\pi^2}{\mu^3}\int_0^\infty dz\left\{z(\mathbf{p}_1\cdot\mathbf{p})+\frac{z^2}{2!}(\mathbf{p}_1+\mathbf{p})\cdot\frac{\partial}{\partial\mathbf{v}}+\frac{z^3}{3!}\frac{\partial^2}{\partial\mathbf{v}^2}\right\}\sum_{n=2}^{\infty}M_n'(\mathbf{v})\Big|_{v\to 0},\quad(4.4)$$

$$M_n'(\mathbf{v})=[D(p_1)D(p)]^{-1}(2i\mu)^n\int d^3p_1'\cdots$$
$$\times d^3p_{n-1}'a(\mathbf{p}_1-\mathbf{p}_1')D^{-1}(p_1')\cdots$$
$$\times D^{-1}(p_{n-1}')a(\mathbf{p}_{n-1}'-\mathbf{p}).$$

Thus all the parametric integrations $\int dY_{n-1}$ have been replaced by a set of differentiation operations and one z integration; the latter will play a role analogous to that of the nonrelativistic photon momentum integral of previous treatments.[3,4,6]

Now consider the operator

$$\delta(\mathbf{p}_1-\mathbf{p})p_j[z+\Delta+\mathbf{v}\cdot\mathbf{p}-2i\mu V]^{-1}p_j$$
$$\equiv\delta(\mathbf{p}_1-\mathbf{p})p_j[D+\mathbf{v}\cdot\mathbf{p}]^{-1}p_j,\quad(4.5)$$

where V is the formal operator as defined in Sec. 2, and the sum on the subscript j is to be understood (in the sense of $\sum_j p_j{}^2=\mathbf{p}^2$). If Eq. (4.5) is expanded in powers of V and each term of the resulting sequence is inserted between the wave functions $\bar{\phi}_a(\mathbf{p}_1)$, $\phi_a(\mathbf{p})$, the

result is equivalent to

$$\delta(\mathbf{p}_1-\mathbf{p})\mathbf{p}^2D^{-1}(p)+2i\mu(\mathbf{p}_1\cdot\mathbf{p})D^{-1}(p_1)\boldsymbol{a}(\mathbf{p}_1-\mathbf{p})D^{-1}(p)+\mathbf{p}_1\cdot\mathbf{p}\sum_{n=2}^{\infty}M_n'(\mathbf{v})$$

between the same wave functions. The last term of this expression, which occurs in Eq. (4.4), may therefore be expressed in terms of the operator defined by Eq. (4.5). In similar fashion, analogous replacements can be made for the remaining integrands of Eq. (4.4). The result can then be written in the form

$$\sum_{n=2}^{\infty}\mathfrak{M}_n=A+B+C,$$

where, upon performing the necessary $\mathbf{v}$ differentiations, one finds

$$A=\frac{4\pi^2}{\mu^3}\delta(\mathbf{p}_1-\mathbf{p})\int_0^\infty dz\left\{zp_jD^{-1}p_j-\frac{z^2}{2}(p_jD^{-1}p_jD^{-1}+D^{-1}p_jD^{-1}p_j)+\frac{z^3}{3}D^{-1}p_jD^{-1}p_jD^{-1}\right\},$$

$$B=-\frac{4\pi^2}{\mu^3}\mathbf{p}^2\delta(\mathbf{p}_1-\mathbf{p})\int_0^\infty dz\left\{1-\frac{\Delta^3}{(z+\Delta)^3}\right\},$$

$$C=-\frac{8\pi^2i}{\mu^2}\boldsymbol{a}\int_0^1 dy\int_0^\infty dz\left\{\frac{z(\mathbf{p}_1\cdot\mathbf{p})}{(z+\Delta_y)^2}-\frac{z^2p_y\cdot(\mathbf{p}+\mathbf{p}_1)}{(z+\Delta_y)^3}+\frac{z^3p_y^2}{(z+\Delta_y)^4}\right\}.$$

The sum of these three groups is perfectly finite; separately, however, each is divergent on performing the z integration, and accordingly an upper cutoff will be introduced. The expectation value of group A can be simplified by noting that, to lowest order,

$$\frac{1}{D}\rightarrow\frac{1}{2\mu}\left(\frac{1}{H-E_a+k}\right)=\frac{1}{2\mu}\sum_m\frac{|m\rangle\langle m|}{E_m-E_a+k},$$

where $H=V_0+\mathbf{p}^2/2\mu$, $k=z/2\mu$, $E_a=-\epsilon_a$, and $\sum_m$ stands for the sum over all Schrödinger states of H. Forming $\langle a|A|a\rangle$, and using orthogonality, one obtains

$$\langle a|A|a\rangle=\frac{8\pi^2}{3\mu^2}\int_0^{k_m}kdk\sum_m\frac{|\langle a|p_j|m\rangle|^2}{[E_m-E_a+k]},$$

where the number k_m is the required cutoff. The expectation value of group B is

$$\langle a|B|a\rangle=-\frac{8\pi^2}{3\mu^2}\langle a|\mathbf{p}^2|a\rangle\int_0^{k_m}dk+\frac{2\pi^2}{3\mu^3}\langle a|\mathbf{p}^2\Delta|a\rangle,$$

while that of group C yields

$$C=\frac{8\pi^2i}{\mu^2}\boldsymbol{a}\left\{\mathbf{q}^2\int_0^1 dy\,y(1-y)\ln\left(\frac{2\mu k_m}{\Delta_y}\right)-\frac{1}{12}(\mathbf{p}_1^2+\mathbf{p}^2)-\frac{1}{18}\mathbf{q}^2\right\}.$$

The integration of C over the wave functions $\bar{\phi}_a(\mathbf{p}_1)$, $\phi_a(\mathbf{p})$ is to be understood. Combining these expressions, one obtains

$$\sum_{n=2}^{\infty}\mathfrak{M}_n=-\frac{8\pi^2}{3\mu^2}\ln\left(\frac{k_m}{k_0}\right)\langle a,0|p_j[H,p_j]|a,0\rangle+\frac{2\pi^2}{3\mu^3}\langle a|\mathbf{p}^2\Delta|a\rangle-\frac{8\pi^2i}{6\mu^2}\mathbf{p}_1^2\boldsymbol{a}+\frac{8\pi^2i}{\mu^2}\mathbf{q}^2\boldsymbol{a}\left\{\int_0^1 dy\,y(1-y)\ln\left(\frac{2\mu k_m}{\Delta_y}\right)-\frac{1}{18}\right\},\quad(4.6)$$

where $k_0(n,l)$ is the average excitation energy as defined by Bethe[3] ($|a,0\rangle$ is that atomic state a with $l=0$), and $(\mathbf{p}_1^2+\mathbf{p}^2)\boldsymbol{a}$ has been replaced by the equivalent $2\mathbf{p}_1^2\boldsymbol{a}$.

Upon using the relation $\Delta=(p+i\mu)(p-i\mu)$, to the order $(Z\alpha)^4$, the second and third terms of Eq. (4.6) exactly cancel. Further, it is easy to see that $\langle a,0|p_j[H,p_j]|a,0\rangle$ may be replaced by $\frac{1}{2}i\mathbf{q}^2\boldsymbol{a}$ (using the again superfluous Dirac notation), the latter having a nonvanishing integral over $\phi_a{}^+(\mathbf{p}_1)$, $\phi_a(\mathbf{p})$ for states with $l=0$ only. All the k_m dependence then vanishes from Eq. (4.6); combining the latter with the first line of Eq. (3.4), one finds that the terms involving $\ln(\Delta_y/\mu^2)$ cancel, and the entire lowest order matrix element becomes

$$\sum_{n=0}^{\infty}M_n=-\frac{4\pi^2i}{3\mu^2}q^2\boldsymbol{a}\left\{\ln\frac{\mu}{2k_0}+\frac{11}{24}\right\}.\quad(4.7)$$

Inserting Eq. (4.7) between the S-state wave functions $\phi_{a,0}{}^+$, $\phi_{a,0}$, one then multiplies by -2α to obtain the $l=0$ energy-level shift

$$\Delta E_{a,\,l=0}=\frac{4}{3\pi n_a{}^3}\alpha(Z\alpha)^4\mu\left\{\ln\frac{\mu}{2k_0}+\frac{11}{24}\right\},$$

which is the correct lowest order Lamb shift formula, excluding the magnetic moment and vacuum polarization contributions. As is evident from the definition of k_0 and the above reduction, the factor $11/24$ is absent in the corresponding portion of the P-state level shift.

5. ALTERNATIVE EXPRESSION FOR THE BETHE LOGARITHM

We now return to the original idea underlying this presentation, that of the analogy between the scattering

$\ln(\lambda/\mu)$ and the bound state $\ln(\Delta_y/\mu^2)$. To simplify the notation to follow, rewrite the matrix element $\mathfrak{M}_n$ as

$$\mathfrak{M}_n=\mathfrak{M}_n\{\mathbf{p}_1\cdot\mathbf{p}\}+\mathfrak{M}_n\{\bar{\mathbf{p}}_{n-1}\cdot(\mathbf{p}_1+\mathbf{p})\}+\mathfrak{M}_n\{\bar{\mathbf{p}}_{n-1}{}^2\}, \quad (5.1)$$

where each of the terms on the right-hand side of Eq. (5.1) refers to the corresponding portion of M_n in Eqs. (4.1) and (4.4). If the reduction of the previous section had been applied only to the sum $\sum_{n=2}^{\infty}\mathfrak{M}_n\{\mathbf{p}_1\cdot\mathbf{p}\}$, the result would have been

$$\sum_{n=2}^{\infty}\mathfrak{M}_n\{p_1\cdot p\}=\frac{8\pi^2}{\mu^2}\int_0^{k_m}kdk\sum_m\frac{|\langle a|p_j|m\rangle|^2}{[E_m-E_a+k]}$$
$$-\frac{4\pi^2}{\mu^3}\mathbf{p}^2\int_0^{2\mu k_m}\frac{zdz}{(z+\Delta)}$$
$$-\frac{8\pi^2 i}{\mu^2}(\mathbf{p}_1\cdot\mathbf{p})\boldsymbol{a}\int_0^1 dy\left\{\ln\left(\frac{2\mu k_m}{\Delta y}\right)-1\right\}, \quad (5.2)$$

which is easy to verify by merely taking the first terms of each of the expectation values of A, B, and C. Comparing Eq. (5.2) with the sum of groups A, B, and C, one can eliminate the sum-over-states terms from Eqs. (4.6) and (5.2), and obtain

$$\sum_{n=2}^{\infty}\mathfrak{M}_n=\tfrac{1}{3}\sum_{n=2}^{\infty}\mathfrak{M}_n\{\mathbf{p}_1\cdot\mathbf{p}\}+\frac{4\pi^2}{3\mu^3}\mathbf{p}^2\int_0^{2\mu k_m}dz\left\{\frac{z}{(\Delta+z)}-1\right.$$
$$\left.+\frac{\Delta^3}{(\Delta+z)^3}\right\}+\frac{8\pi^2 i}{\mu^2}\boldsymbol{a}\int_0^1 dy\left\{[\mathbf{q}^2y(1-y)\right.$$
$$\left.+\tfrac{1}{3}(\mathbf{p}_1\cdot\mathbf{p})]\ln\left(\frac{2\mu k_m}{\Delta_y}\right)-\tfrac{1}{3}(\mathbf{p}_1\cdot\mathbf{p})\right\}$$
$$-\frac{8\pi^2 i}{\mu^2}\boldsymbol{a}\left\{\frac{1}{6}\mathbf{p}_1{}^2+\frac{1}{18}\mathbf{q}^2\right\}. \quad (5.3)$$

Using the relation $\mathbf{q}^2=\mathbf{p}_1{}^2+\mathbf{p}^2-2\mathbf{p}_1\cdot\mathbf{p}$, and the previous $\alpha(Z\alpha)^4\mu$ equivalence of $\langle a|\mathbf{p}^2\Delta|a\rangle$ with $2i\mu\mathbf{p}_1{}^2\boldsymbol{a}$, Eq. (5.3) may be rewritten as

$$\sum_{n=2}^{\infty}\mathfrak{M}_n\rightarrow\tfrac{1}{3}\sum_{n=2}^{\infty}\mathfrak{M}_n\{\mathbf{p}_1\cdot\mathbf{p}\}+\frac{4\pi^2}{3\mu^3}\mathbf{p}^2\Delta\ln\left(\frac{\Delta}{\mu^2}\right)-\frac{8\pi^2 i}{18\mu^2}\mathbf{q}^2\boldsymbol{a}$$
$$-\frac{8\pi^2 i}{\mu^2}\mathbf{q}^2\boldsymbol{a}\int_0^1 dy\,y(1-y)\ln\left(\frac{\Delta_y}{\mu^2}\right)$$
$$-\frac{8\pi^2 i}{3\mu^2}(\mathbf{p}_1\cdot\mathbf{p})\boldsymbol{a}\left\{1+\int_0^1 dy\ln\left(\frac{\Delta_y}{\mu^2}\right)\right\}, \quad (5.4)$$

where the $\ln(k_m)$ terms have cancelled. Adding to Eq. (5.4) the contribution of the first line of Eq. (3.4), the term proportional to $\int_0^1 dy\,y(1-y)\ln(\Delta_y/\mu^2)$ cancels and is effectively replaced by the simpler quantity proportional to $\int_0^1 dy\ln(\Delta_y/\mu^2)$, yielding for the entire matrix element

$$\sum_{n=0}^{\infty}M_n\rightarrow-\frac{4\pi^2 i}{3\mu^2}\mathbf{q}^2\boldsymbol{a}\left[\frac{11}{24}-\left\{1+\int_0^1 dy\ln\left(\frac{\Delta_y}{\mu^2}\right)\right\}\right]$$
$$-\frac{8\pi^2 i}{3\mu^2}\mathbf{p}_1{}^2\boldsymbol{a}\left\{1+\int_0^1 dy\ln\left(\frac{\Delta_y}{\Delta_1}\right)\right\}$$
$$+\tfrac{1}{3}\sum_{n=2}^{\infty}\mathfrak{M}_n\{\mathbf{p}_1\cdot\mathbf{p}\}. \quad (5.5)$$

Only the last term of Eq. (5.5) presents an obstacle to the immediate expectation value calculation, by straightforward integration, of this expression to obtain this portion of the $\alpha(Z\alpha)^4\mu$ level shift. It is possible to write Eq. (5.5) so that the entire matrix element is proportional to q^2, as in the usual Lamb shift formula.

Defining the function

$$M_n'=(2i\mu)^n\int_0^{\infty}\frac{zdz}{(z+\Delta_1)(z+\Delta)}\int d^3p_1'\cdots$$
$$\times d^3p_{n-1}'\,\frac{\boldsymbol{a}(\mathbf{p}_1-\mathbf{p}_1')\cdots\boldsymbol{a}(\mathbf{p}_{n-1}'-\mathbf{p})}{(z+\Delta_1')\cdots(z+\Delta_{n-1}')},$$

which differs from the previous $M_n'(0)$ of Eq. (4.4) by the inclusion of the factor $\int_0^{\infty}zdz$, one may then compare Eq. (5.5) with (4.7) and obtain

$$\ln\left(\frac{k_0}{\mathrm{ry}}\right)\doteq 1+\int_0^1 dy\ln\left(\frac{\Delta_y}{\alpha^2\mu^2}\right)$$
$$-2\frac{\mathbf{p}_1{}^2}{\mathbf{q}^2}\left\{1+\int_0^1 dy\ln\left(\frac{\Delta_y}{\Delta_1}\right)\right\}$$
$$+\frac{2\pi^2(\mathbf{p}_1\cdot\mathbf{p})}{(Z\alpha\mu)}\sum_{n=2}^{\infty}M_n'. \quad (5.6)$$

The symbol $\doteq$ stands for the equality of expectation values over the Schrödinger wave functions $\phi_a{}^+(\mathbf{p}_1)$, $\phi_a(\mathbf{p})$. Note that, due to the definition of k_0, the S-state ϕ_a are always to be used on the left-hand side of Eq. (5.6), while the right-hand side integrations are to be performed with the wave functions of that state whose level shift is desired. Thus Eq. (5.6) represents an exact extension of the order of magnitude contributions given by the ZOP terms of Sec. 2.

With the exception of the last term on the right-hand side of Eq. (5.6), the latter has been evaluated for the $1S$, $2S$, and $2P$ states of hydrogen. It is found that the neglect of the terms $\sum_{n=2}^{\infty}M_n'$ (which, incidentally, are positive[17]) yields an error of less than 2% in the

[17] That these terms are positive may be seen qualitatively by noting that integrands comprising the positive semidefinite M_n have their maximum value when their momenta are parallel; hence the largest contribution results when $\mathbf{p}_1\cdot\mathbf{p}$ is positive.

TABLE I. The results obtained by performing the indicated integrations of Eq. (5.7), as compared to the complete numerical values given by Harriman.[a]

State	$1+\int_0^1 dy \ln\left(\frac{\Delta_y}{\alpha^2\mu^2}\right)$	$-2\frac{p_1^2}{q^2}\left\{1+\int_0^1 dy \ln\left(\frac{\Delta_y}{\Delta_1}\right)\right\}$	$\ln\left(\frac{\bar{\epsilon}'}{\mathrm{ry}}\right)$	$\ln\left(\frac{k_0}{\mathrm{ry}}\right)$
$1S$	$\ln 4+5/2$	-1.02	2.87	2.98
$2S$	$4-1/12$	-1.37	2.55	2.81
$2P$	0	-0.38	-0.38	-0.03
$2S$–$2P$	$4-1/12$	-0.99	2.93	2.84

[a] See reference 16.

$1S$ level shift and $2S$–$2P$ level-shift splitting. Writing

$$\ln\left(\frac{\bar{\epsilon}'}{\mathrm{ry}}\right)\doteq 1+\int_0^1 dy \ln\left(\frac{\Delta_y}{\alpha^2\mu^2}\right) -2\frac{\mathbf{p}_1^2}{\mathbf{q}^2}\left\{1+\int_0^1 dy \ln\left(\frac{\Delta_y}{\Delta_1}\right)\right\}, \quad (5.7)$$

one can construct, with the aid of reference 16, the set of approximate numbers displayed in Table I. Only for the $2P$ state is there a large relative error in the value of $\bar{\epsilon}'$; since the correct contribution is quite small, this is not particularly significant. The contribution of the neglected $\sum_{n=2}^{\infty} M_n'$ terms then serves to lower the approximate $1S$ level shift and $2S$–$2P$ level splitting found here by roughly 140 Mc/sec and 17 Mc/sec, respectively.

Equation (5.6) may, of course, be derived readily from Bethe's original definition of k_0 in terms of a sum over states.[3] The present derivation indicates that the contribution of order $\alpha(Z\alpha)^4\mu\ln(Z\alpha)$ comes entirely from the ZOP contributions. The entries in Table I are additive constants (to be compared with $2\ln(Z\alpha)$, which is -9.85 for hydrogen); and by comparison with the results calculated numerically, we see that the bulk of their contribution comes also from the ZOP terms.

6. SUMMARY AND ACKNOWLEDGMENTS

The ease with which the lowest-order Lamb shift formula has been obtained using the above methods, as well as the similar calculational simplicity involved in deriving the next higher order $\alpha(Z\alpha)^5\mu$ terms, indicates that the scattering-type expansion of $S_F{}^e$ may prove quite useful in obtaining further higher order corrections. The method may be applied to similar problems, such as that of radiative corrections to hfs,[18] where corresponding calculational simplicity may be expected.

The ideas underlying this presentation grew out of several informal discussions with Professor M. Lévy; his suggestions are gratefully acknowledged. The material contained in this paper formed the basis of a dissertation submitted by one of us (HMF) to Stanford University; during the period of the writing of this paper and the calculation of the $\alpha(Z\alpha)^5\mu$ terms, this author had the pleasure of enjoying the warm hospitality of Professor Lévy's group at the Université de Paris, France, where he was the recipient of a National Science Foundation post-doctoral fellowship.

APPENDIX

We require the definitions

$$p_y=yp_1+(1-y)p,$$
$$\bar{p}_1=y_1p_y+(1-y_1)p_1',$$
$$\bar{p}_{n-1}=y_{n-1}\bar{p}_{n-2}+(1-y_{n-1})p_{n-1}',$$
$$\bar{\Delta}_1=y_1\Delta_y+(1-y_1)\Delta_1',$$
$$\bar{\Delta}_{n-1}=y_{n-1}\bar{\Delta}_{n-2}+(1-y_{n-1})\Delta_{n-1}',$$
$$\int dY_{n-1}=n!\int_0^1\cdots\int_0^1 dy\ y_1dy_1y_2^2dy_2\cdots(y_{n-1})^{n-1}dy_{n-1}.$$

Then

$$\mathfrak{D}_n=\int dY_{n-1}[k^2-2k\cdot\bar{p}_{n-1}+\bar{\Delta}_{n-1}]^{-(n+1)},$$

and with $D_n(x)=(\bar{\Delta}_{n-1}-x\bar{p}_{n-1}{}^2)$, one has

$$(k^2)^{-1}\mathfrak{D}_n=(n+1)\int dY_{n-1}\int_0^1 dx \times x^n[(k-x\bar{p}_{n-1})^2+xD_n(x)]^{-(n+2)},$$

$$(k^2)^{-2}\mathfrak{D}_n=(n+1)(n+2)\int dY_{n-1}\int_0^1 dx \times x^n(1-x)[(k-xp_{n-1})^2+xD_n(x)]^{-(n+3)}.$$

One then obtains the following formulas:

$$\int_F \frac{d^4k}{k^2}\mathfrak{D}_n=\frac{\pi^2i}{n}\int dY_{n-1}\int_0^1 dx[\mathfrak{D}_n(x)]^{-n}, \quad (A\text{-}1)$$

$$\int_F \frac{d^4k}{k^2}\mathfrak{D}_n(a\cdot k)=\frac{\pi^2i}{n}\int dY_{n-1}\int_0^1 xdx[D_n(x)]^{-n}(a\cdot\bar{p}_{n-1}), \quad (A\text{-}2)$$

$$\int \frac{d^4k}{k^2}\mathfrak{D}_n(a\cdot k)(b\cdot k)=\frac{\pi^2i}{n(n-1)}\frac{a\cdot b}{4}\int dY_{n-1}[D_n(1)]^{-(n-1)} +\frac{\pi^2i}{n}\int dY_{n-1}\int_0^1 x^2dx[D_n(x)]^{-n}\left\{(a\cdot\bar{p}_{n-1})(b\cdot\bar{p}_{n-1})-\bar{p}_{n-1}{}^2\left(\frac{a\cdot b}{4}\right)\right\}, \quad (A\text{-}3)$$

[18] N. M. Kroll and F. Pollock, Phys. Rev. 86, 876 (1952).

$$\int_F \frac{d^4k}{k^2k^2}(a\cdot k)(b\cdot k)\mathfrak{D}_n=\frac{\pi^2 i}{n}\int dY_{n-1}\int_0^1 dx[D_n(x)]^{-n}\left\{\tfrac{1}{2}(a\cdot b)(1-x)-(1-2x)\frac{(a\cdot\bar{p}_{n-1})(b\cdot\bar{p}_{n-1})}{\bar{p}_{n-1}{}^2}\right\}, \tag{A-4}$$

$$\int_F \frac{d^4k}{k^2k^2}(a\cdot k)(b\cdot k)(c\cdot k)\mathfrak{D}_n=\frac{\pi^2 i}{n}\int dY_{n-1}\int_0^1 dx[D_n(x)]^{-n}\left\{\tfrac{1}{2}x(1-x)[(a\cdot b)(c\cdot\bar{p}_{n-1})\right.$$
$$\left.+a_\mu(b\cdot\bar{p}_{n-1})c_\mu+(a\cdot\bar{p}_{n-1})(b\cdot c)]+x(3x-2)\frac{(a\cdot\bar{p}_{n-1})(b\cdot\bar{p}_{n-1})(c\cdot\bar{p}_{n-1})}{\bar{p}_{n-1}{}^2}\right\}. \tag{A-5}$$

The term $a_\mu(b\cdot\bar{p}_{n-1})c_\mu$ is written in this form, rather than $(a\cdot c)(b\cdot\bar{p}_{n-1})$, to insure obtaining the correct expression when any two or more of the a, b, c are, or contain, γ matrices.

$$\int_F \frac{d^4k}{k^2}(a\cdot k)(b\cdot k)(c\cdot k)\mathfrak{D}_n=\frac{\pi^2 i}{n}\int dY_{n-1}\int_0^1 x^3dx[D_n(x)]^{-n}(a\cdot\bar{p}_{n-1})(b\cdot\bar{p}_{n-1})(c\cdot\bar{p}_{n-1})$$
$$+\frac{\pi^2 i}{2n(n-1)}\int dY_{n-1}\int_0^1 x^2dx[D_n(x)]^{-(n-1)}[(a\cdot b)(c\cdot\bar{p}_{n-1})+a_\mu(b\cdot\bar{p}_{n-1})c_\mu+(a\cdot\bar{p}_{n-1})(b\cdot c)]. \tag{A-6}$$

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Elimination of Ghosts in Propagators*

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Within the general framework of perturbation theory a method for calculating modified propagators in terms of proper Feynman diagrams is derived. This method differs from previous approaches in that one insists that the propagator have the correct analytical behavior as a function of p^2. As a result one gets an expression for the propagator which is similar to a conventional term-by-term perturbation theory expansion except that it is only necessary to consider proper diagrams and that the iteration of the proper diagrams is represented by a damping factor. As an example, the meson propagator for a pseudoscalar meson coupled to nucleons with a pseudoscalar coupling is approximated by considering only the lowest order proper diagram, a nucleon-antinucleon bubble. The resulting expression for the propagator has the following interesting properties: (1) by construction it has the proper analytical behavior as a function of p^2, (2) the result has a singularity at $g^2=0$ when considered as a function of g^2, and (3) the wave function renormalization is finite. These three properties are intimately connected and when this connection is realized it is easy to understand why the usual methods of expressing propagators in terms of proper Feynman diagrams leads to ghosts. It is the purpose of this paper to understand this connection and to indicate how it is possible to take into account consistently the iteration of proper Feynman diagrams without ever having ghosts appear. It is also found that an asymptotic expansion valid in the region $g^2=0$ is possible and that this asymptotic expansion is identical with the perturbation theory series.

I. INTRODUCTION

FOR definiteness in what follows, and for simplicity in presenting the arguments, we shall consider a pseudoscalar boson field with mass μ represented by the renormalized Heisenberg operator $\varphi(x)$. The dependence on isotopic spin will be suppressed in what follows since for the propagator this dependence is a trivial $\delta_{\alpha\beta}$. The extension of the methods developed in this paper to fields with additional degrees of freedom such as the electromagnetic field, or to fermions, is straightforward and will not be discussed.

We wish to find an expression for the modified propagator $\Delta_F'(p^2)$ which is defined in terms of the vacuum expectation value of a time-ordered product:

$$\Delta_F'(p^2)=i\int d_4x\, e^{-ip(x-y)}{}_0\langle T(\varphi(x)\varphi(y))\rangle_0. \tag{1}$$

The lowest order approximation to this function is given by $\Delta_F(p^2)$ where $\Delta_F(p^2)=(p^2+\mu^2-i\epsilon)^{-1}$. Historically the first attempts to calculate the corrections terms were based on a term-by-term perturbation expansion in the coupling constant. In electrodynamics the corrections thus obtained were found to be small and reasonable in that, for example, they led to small terms in the Lamb shift which due to the precision of the experiments could be seen to be necessary.

For strong-coupling meson theories it is readily seen that such a simple procedure does not prove to be

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