Proposal for Determining the Pion-Nucleon Coupling Constant from the Angular Distribution for Nucleon-Nucleon Scattering*

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A plausibility argument is made that the real and imaginary parts of the nucleon-nucleon scattering amplitude as a function of the cosine of the barycentric system scattering angle, for fixed energy, are analytic in the complex plane, with singularities confined to the real axis. It is conjectured that in the real part there are poles at $\cos\theta=\pm(1+\mu^2/2k^2)$, where μ is the pion rest mass and k the barycentric momentum, branch points at $\cos\theta=\pm(1+2\mu^2/k^2)$, $\pm(1+9\mu^2/2k^2)$, etc. The residues of the poles are related directly to g^2, the pion nucleon coupling constant, and a procedure is outlined for determining g^2 by an extrapolation of experimental data on either backward or forward nucleon-nucleon scattering.

I. INTRODUCTION

DISPERSION relations have made feasible a systematic determination of the pion-nucleon coupling constant g^2 from experimental measurements of the scattering of π mesons by nucleons.[1] The method depends on the fact that g^2 is the residue of a pole in the scattering amplitude for fixed momentum transfer as a function of W^2, the square of the total energy in the barycentric system. Except for this residue, all other quantities occurring in the dispersion relations for pion-nucleon scattering are physically measurable. It has been pointed out[2] that a somewhat similar situation exists for nucleon-nucleon scattering at fixed momentum transfer where again g^2 occurs as the residue of a pole in the variable W^2. However, in this case, there are extensive nonphysical contributions to the dispersion relations which make practical applications difficult. It is the purpose of this paper to point out that if one considers instead the real part of the $N-N$ scattering amplitude at fixed energy as a function of Δ^2, the square of the momentum transfer, then there is probably a pole of residue g^2 located at $\Delta^2=-\mu^2$, where μ is the pion rest mass. Lehmann has recently demonstrated that there exists a region of analyticity in the complex Δ^2 plane which includes the physical region.[3] We are conjecturing that once the pole in question has been removed, this region of analyticity includes the point $\Delta^2=-\mu^2$, and that an extrapolation to determine the required residue is possible.

The physical idea underlying the present proposal is an old one, although it is usually stated in a different way, namely, that the nucleon-nucleon interaction at large distances is dominated by single pion exchange, which in turn is uniquely related to g^2 and μ.[4] A rough correspondence of this statement to the existence of a pole at $\Delta^2=-\mu^2$ may be seen if one believes that at high energies the outer fringe of the interaction determines the real part of the forward scattering amplitude. Remembering the relation between momentum transfer and the angle of scattering in the barycentric system,

$$\cos\theta=1-\Delta^2/2k^2, \tag{I.1}$$

where k is the magnitude of the momentum of either particle in this system, one sees that although the point $\Delta^2=-\mu^2$ always is unphysical, corresponding to $\cos\theta=1+\mu^2/2k^2>1$, this point comes nearer and nearer to physical forward scattering as k^2 increases. If, therefore, there is a pole at this point as a function of $\cos\theta$ and no other singularities in the immediate neighborhood, then the residue of this pole determines the asymptotic behavior of the real part of the forward scattering amplitude.

For practical reasons, some of which are discussed below in Sec. IV, the region of very high energy may not be most suitable for the determination of g^2, so we are basing our hopes on the possibility of bridging by analytic continuation an appreciable gap between the physical region and the position of the pole. The practical accuracy of such a continuation presumably requires that the gap in $\cos\theta$, which is $\mu^2/2k^2$, be small compared with the interval in $\cos\theta$ where the amplitude is experimentally known. Since we have $2k^2=MT_l$, if M is the nucleon mass and T_l the laboratory kinetic energy, we have the requirement

$$\mu^2/MT_l\ll 2,$$

or

$$T_l\gg\mu^2/2M=10\text{ Mev}. \tag{I.2}$$

This means the energy must be high enough so that nonzero orbital angular momenta are important. In the old way of describing the principle exploited here such a circumstance would be obvious. The "tail" of the interaction can be isolated only in states that are

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[1] U. Haber-Schaim, Phys. Rev. **104**, 1113 (1956); W. D. Davidon and M. L. Goldberger, Phys. Rev. **104**, 1119 (1956); W. Gilbert, Phys. Rev. **108**, 1078 (1957).

[2] Goldberger, Nambu, and Oehme, Ann. Phys. (N. Y.) **2**, 226 (1957).

[3] H. Lehmann, Institute for Advanced Study, Princeton, New Jersey (to be published).

[4] See, for example, the Supplement to Progr. Theoret. Phys. Japan **3**, (1956).

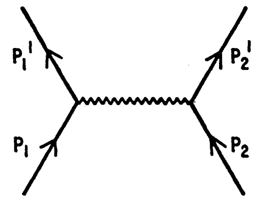

FIG. 1. The single-pion exchange diagram for nucleon-nucleon scattering.

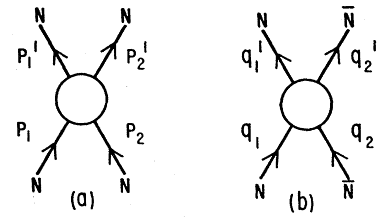

FIG. 2. Diagrams showing the relation between (a) nucleon-nucleon and (b) nucleon-antinucleon scattering.

prevented by an angular momentum barrier from penetrating to short distances.

II. THE SINGLE PION EXCHANGE TERM

A motivation for our conjecture concerning the pole at $\Delta^2=-\mu^2$ can be given in terms of Feynman diagrams. The diagram shown in Fig. 1, when renormalized, yields a term

$$g^2\frac{(\bar{u}_{p'1}\tau_\alpha\gamma_5 u_{p1})(\bar{u}_{p'2}\tau_\alpha\gamma_5 u_{p2})}{\Delta^2+\mu^2}, \qquad \text{(II.1)}$$

where

$$\Delta^2=(p_1-p_1')^2=(p_2-p_2')^2.$$

A little thought about other diagrams shows that none of them becomes infinite for $\Delta^2\rightarrow-\mu^2$; also the modification of the pion propagator and of the vertex functions in this limit are entirely absorbed by the renormalization of g^2 and μ. Thus, in perturbation theory, the scattering amplitude when evaluated in the neighborhood of $\Delta^2=-\mu^2$ is exactly represented by the renormalized Born approximation.

Confirmation is given by the circumstance that the term obtained from Eq. (II.1) by exchanging p_1' and p_2' has been singled out for a special role in the dispersion-relation discussion of the scattering amplitude as a function of

$$W^2=-(p_1+p_2)^2=-(p_1'+p_2')^2,$$

for fixed momentum transfer.[2] Here the denominator is

$$(p_1-p_2')^2+\mu^2=W^2-4M^2-\Delta^2+\mu^2, \qquad \text{(II.2)}$$

so that there is a pole at $W^2=4M^2-\mu^2+\Delta^2$, and in the neighborhood of this pole the renormalized Born approximation is exact. In the present discussion, where W^2 is fixed, we regard this exchange term as giving rise to a second pole in Δ^2. In terms of $\cos\theta$, the location of this pole is very natural: whereas the denominator in Eq. (II.1) vanishes at $\cos\theta=1+\mu^2/2k^2$, (II.2) vanishes at $\cos\theta=-1-\mu^2/2k^2$. With identical particles one of course expects symmetry (or antisymmetry) with respect to $\cos\theta$.

It seems very plausible, therefore, that a determination of the scattering amplitude near either of these two poles corresponds to a measurement of the value of g^2. We need information, however, about the location of other singularities in the Δ^2 complex plane before we can formulate an appropriate procedure for continuation from the physical region.

III. THE LOCATION OF SINGULARITIES IN THE Δ^2 PLANE

To make a guess about the singularities of the scattering amplitude as a function of Δ^2, let us consider, instead of nucleon-nucleon scattering as indicated in Fig. 2(a), the process of nucleon-antinucleon scattering, as indicated in Fig. 2(b). In perturbation theory we could obtain the $N\bar{N}$ amplitude from the NN by making the substitutions[5]

$$q_1=p_1,\quad q_2=-p_1',\quad q_1'=p_2',\quad q_2'=-p_2. \qquad \text{(III.1)}$$

Notice that the variable,

$$\Delta^2=(p_1-p_1')^2=(q_1+q_2)^2, \qquad \text{(III.2)}$$

becomes the negative energy squared in the $N\bar{N}$ case, while

$$W^2=-(p_1+p_2)^2=-(q_1-q_2')^2 \qquad \text{(III.3)}$$

becomes a negative (exchange) momentum transfer squared. In the NN case Δ^2 and W^2 are both positive, whereas they become both negative in the $N\bar{N}$ case. Nevertheless, we shall optimistically assume that the location of singularities of the scattering amplitude considered as a function of Δ^2 for fixed W^2 in the $N\bar{N}$ case can be carried over to the NN case.

Following the by now familiar approach of contraction of the S-matrix element,[6] applied here to $N\bar{N}$ scattering, one sees that *if* a "normal" dispersion relation exists for fixed $W^2=-(q_1-q_2')^2$, then the singularities in Δ^2 are associated with the possible vanishing of the two expressions

$$\begin{aligned}(q_1+q_2)^2+m^2&=\Delta^2+m^2,\\(q_2-q_2')^2+m^2&=W^2-4M^2-\Delta^2+m^2,\end{aligned} \qquad \text{(III.4)}$$

where m ranges over the mass values of states that can be reached from the $N\bar{N}$ system. The lowest-mass state of this kind is the single π meson, and it gives rise precisely to the two poles discussed above. The next lowest masses belong to the two-pion state. We expect then two branch points corresponding to $m^2=4\mu^2$, with cuts to $\mp\infty$. More complicated states give rise to further pairs of branch points on the real axis. Changing

[5] See, for example, J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Press, Cambridge, 1955), p. 161.

[6] Lehmann, Zimmermann, and Symanzik, Nuovo cimento **1**, 217 (1955).

variables from Δ^2 and W^2 to $\cos\theta$ and k^2 according to

$$k^2=\tfrac{1}{4}W^2-M^2, \quad \text{(III.5)}$$

$$\cos\theta=1-\Delta^2/2k^2, \quad \text{(III.6)}$$

we find the corresponding singularities in $\cos\theta$ for fixed k^2:

$$\begin{aligned} &\text{poles at} && \cos\theta=\pm(1+\mu^2/2k^2),\\ &\text{branch points at} && \cos\theta=\pm(1+2\mu^2/k^2),\\ &&& \pm(1+9\mu^2/2k^2), \text{ etc.}\end{aligned} \quad \text{(III.7)}$$

This whole procedure of course amounts to nothing more than an optimistic conjecture. Our guess (III.7) may be partially checked, however, against recent rigorous results obtained by Lehmann.[3] Lehmann proves that the NN scattering amplitude as a function of $\cos\theta$ for fixed W^2 is free from singularity, at least within an ellipse whose foci are at ± 1 and whose semimajor axis is of length

$$X_0(k^2)=\left[1+\frac{(2M\mu+\mu^2)^2}{4k^2(k^2+M^2)}\right]^{\frac{1}{2}}. \quad \text{(III.8)}$$

One may confirm that this region never includes the singularities whose existence we have conjectured, although at $k^2=\tfrac{1}{2}M\mu$ our poles lie exactly on the boundary. At all other energies they are outside Lehmann's ellipse.

Lehmann proves further that the imaginary part of the scattering amplitude as a function of $\cos\theta$ is analytic within a larger ellipse, whose semimajor axis is $2X_0^2-1$ or

$$1+(2M\mu+\mu^2)^2/[2k^2(k^2+M^2)]. \quad \text{(III.9)}$$

For small k^2 this ellipse includes not only our poles but also our first conjectured branch points, which at first sight is somewhat surprising. (Our poles have real residues and therefore are singularities only in the real part of the amplitude.) If one, however, believes that a two-dimensional spectral representation exists, which exhibits simultaneously the behavior of W^2 and Δ^2 in the complex plane, then it becomes plausible that the imaginary part of the amplitude has a wider domain of analyticity than does the real part.

For example, suppose the scattering amplitude (excluding the poles) could be expressed by representations of the form

$$T(W^2,\Delta^2)=\int\int dm_1^2 dm_2^2 \frac{\rho(m_1^2,m_2^2)}{(m_1^2-W^2)(m_2^2+\Delta^2)}, \quad \text{(III.10)}$$

where $\rho(m_1^2,m_2^2)$ is real, as suggested by Mandelstam[7] in connection with pion-nucleon scattering. Then, for fixed W^2 in the physical region,

$$T(W^2,\Delta^2)=\int dm_2^2 \frac{G(W^2,m_2^2)}{m_2^2+\Delta^2}, \quad \text{(III.11)}$$

where

$$\mathrm{Re}G(W^2,m_2^2)=\mathrm{P}\int_{(2M)^2}^{\infty} dm_1^2 \frac{\rho(m_1^2,m_2^2)}{m_1^2-W^2}, \quad \text{(III.12)}$$

and

$$\mathrm{Im}G(W^2,m_2^2)=\pi\rho(W^2,m_2^2). \quad \text{(III.13)}$$

It follows from (III.12) and (III.13) that $\mathrm{Re}G(W^2,m_2^2)$ is in general nonzero for a wider range of m_2^2 than is $\mathrm{Im}G(W^2,m_2^2)$. Correspondingly, from (III.11), the branch points of $\mathrm{Re}T(W^2,\Delta^2)$ extend over a wider range of the Δ^2 real axis then they do for $\mathrm{Im}T(W^2,\Delta^2)$.

Mandelstam[7] has shown that representations of the type (III.10) are compatible with fourth-order perturbation theory. We do not here rest our case on the validity of Mandelstam's particular conjecture, but believe that the general feature described is likely to be present in a correct two-dimensional representation.

IV. POSSIBLE CONTINUATION PROCEDURES

One can think of many possible procedures for performing the required continuation from the physical region, $-1<\cos\theta<1$, to the poles at $\cos\theta=\pm(1+\mu^2/2k^2)$. A detailed investigation of the kind of information available from experiment, as well as of the theoretical complications due to spin and isotopic spin, must be made before one can say which procedures are practical. Such an investigation is under way and the results will be reported at a later time. Here we merely mention a few general considerations.

In principle it is possible to work directly with measured differential elastic-scattering cross sections. That is to say, the location of singularities of the absolute square of the scattering amplitude is the same as for the amplitude itself. The poles are of second order but their strength is still simply related to g^2. One might think that this method of approach allows the use of experiments at very high energy where the poles come close to the physical region. However, when inelastic processes are frequent the imaginary part of the elastic-scattering amplitude has a strong maximum at small angles (the familiar diffraction peak), which tends to obscure the behavior of the real part. At backward angles for $n-p$ scattering the maximum in the imaginary part should be less pronounced.

Roughly speaking, in order to find g^2 directly from a forward angular distribution, one could multiply the experimental distribution by

$$[\cos\theta-(1+\mu^2/2k^2)]^2, \quad \text{(IV.1)}$$

plot against $\cos\theta$, and hope that the resulting function is smooth enough to be extrapolated to the required point. The procedure for backward scattering would be

[7] S. Mandelstam, Phys. Rev. **112**, 1344 (1958). Mandelstam's representation is a sum of three terms of the type (III.10). We are indebted to Dr. Mandelstam for the argument following (III.10).

analogous. The difficulty with this approach is that the complete NN amplitude, because of spin and isotopic spin, is made up of ten scalar amplitudes which may interfere with one another so as to obscure the behavior of the individual functions. For example, $p-p$ scattering in the region of a few hundred Mev is roughly isotropic —showing little of the forward and backward peaking which is expected from the influence of our poles and which is present in $n-p$ angular distributions. Evidently cancellations are at work in the $p-p$ amplitude.

Should the pole in the backward $n-p$ scattering be of sufficient prominence to warrant detailed analysis, a possible specific procedure is the following: Defining

$$x=\left(1+\frac{\mu^2}{2k^2}\right)+\cos\theta, \tag{IV.2}$$

one may, after multiplying the experimental points for $d\sigma/d\Omega$ by x^2, attempt a least squares fit with a form

$$A+Bx+Cx^2+\cdots. \tag{IV.3}$$

The coefficient A is the desired quantity, being proportional to g^4.[8] A minimum range of experimental points which may be used is determined by the distance from the pole to the nearest branch point as given by (III.7). Since a power series converges within a circle whose boundary contains the nearest singularity, experimental points in the range

$$-1>\cos\theta>-1+\mu^2/k^2 \tag{IV.4}$$

are certainly suitable. In other words, the physical range available is at least twice the distance of extrapolation. No doubt more sophisticated extrapolation procedures are possible, which further extend the useful physical range.

If at some energy a complete set of phase shifts and mixing parameters were available, one could construct the ten separate scalar amplitudes for each of which the residue of the poles is related to g^2. Multiplying any one by

$$\cos\theta\mp(1+\mu^2/2k^2),$$

one would hope to find a smooth function in the small-angle (or large-angle) region that could be extrapolated.

Evidently the program outlined here may be regarded as a check on local field theory as applied to strong-coupling phenomena. We are predicting that the constant g^2, determined by the proposed extrapolation procedure, will not only be the same for all ten scalar amplitudes but will also be independent of energy and have the same value as that determined by pion-nucleon and photopion dispersion relations.

ACKNOWLEDGMENTS

The author is indebted to Dr. S. Mandelstam and Dr. John G. Taylor for illuminating discussions concerning the spectral representations of scattering amplitudes.

[8] The precise relation for backward $n-p$ scattering is $A=f_c^4M^4/[k^4(M^2+k^2)]$, where $f_c^2=(\mu g_c/2M)^2\approx0.08$ is the charged pion coupling constant. The two neutral pion coupling constants occur in forward $n-p$ and in $p-p$ scattering; therefore in principle these three constants can be independently measured. Because of electromagnetic effects that violate charge independence, differences of a few percent between the three are to be expected.