Consistency of Quantum Electrodynamics

KENNETH A. JOHNSON*†
Institute for Theoretical Physics, University of Copenhagen, Copenhagen, Denmark
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The proof of the statement "At least one of the renormalization constants in electrodynamics is infinite" is examined in the light of perturbation theory and the gauge invariance of electrodynamics. The essential result used to derive the statement is found not to reproduce perturbation theory at least in a simple way. On the basis of gauge considerations a conjecture is proposed which provides a modified essential result and which is found to reproduce perturbation theory. Even if the modified result could be rigorously established, it would not lead to the statement that any gauge-independent quantity is infinite. In fact, the combined results would establish only the statement that the use of gauges where the exact electron "wave function" relative to the "wave functions" for a free electron is a constant, is not consistent.

I

THE question of the consistency of quantum electrodynamics arose because of the divergent character of some of the parameters of the theory ("renormalization constants") when calculated with perturbation solutions of the field equations. The problem may then be stated: Is this inconsistency a property of the method of solution or is it a characteristic of the theory itself? Some years ago it was proposed[1] that this question might be settled by studying the contribution of a certain set of exact energy eigenstates, namely the "physical pair states," to the vacuum polarization of the electron field. It is the property of this set of states that it provides a lower bound on the part of the vacuum polarization which "renormalizes" the charge. If it could be shown that this lower bound yields a divergent contribution under the assumption that the complete charge renormalization and the other parameters of the theory are finite, then a fundamental inconsistency in quantum electrodynamics would be indicated.

The vacuum polarization of the field is a gauge-invariant notion and the bound on it given by the physical pair states should likewise be gauge independent. Thus, it is rather surprising that it was found that the contribution to the charge renormalization coming from very high-energy physical pairs is expressed in terms of the quantity which renormalizes the electron field and which is well known not to be a gauge-invariant parameter. It was implicitly asserted that the particular gauge in which the renormalization constant of the electron field should be calculated is that used in the quantization of the electromagnetic field by the method of the "indefinite metric."[2] Thus, the implicit assertion is made that no matter what gauge is used to formulate the theory, one should arrive at the "wave-function" renormalization factor of the "Gupta-Bleuler" gauge, in the gauge-invariant expression for the vacuum polarization. This is surprising in view of the fact that this physical quantity would then apparently be expressed in terms of states of the coupled fields which contain "longitudinal" and "scalar" quanta in the particular proportion characteristic of the Gupta-Bleuler gauge and that as a further consequence this gauge is singled out of the infinite class of gauges.

Because of this rather unexpected dependence of the vacuum polarization on a special gauge, it was decided to investigate this important assertion with perturbation theory. It is found that the "exact" expression for the contribution of the high-energy physical pairs to the charge renormalization when expanded in powers of α does not reproduce the perturbation expression of the same quantity (in the fourth order).

To gain further understanding of this matter, it was decided to investigate the question of whether agreement with perturbation theory is produced if one substitutes in the expression in place of the complete wave function amplitude, the part of it which includes only the "physical states" in its definition. By "physical states," we mean those states which satisfy the supplementary condition in the gauge in which the amplitude is calculated. One would anticipate that this quantity is in a simple way gauge invariant and identical to the amplitude calculated by considering the electron field coupled to only transverse light quanta and to the Coulomb field when there is no supplementary condition on the states. It is found that if this is done, the modified form when expanded in powers of α reproduces perturbation theory (in the fourth order) in the high-energy domain. Unfortunately, even if the modified form could be rigorously established, no conclusion about the consistency of electrodynamics is possible. For in contrast to the energy-independent character of the complete wave-function renormalization constant in the Gupta-Bleuler gauge, the part of the amplitude which is governed only by the states which satisfy the

* Natural Science Foundation Postdoctoral Fellow.
† Permanent address: Massachusetts Institute of Technology, Cambridge, Massachusetts.
[1] G. Källén, in *Handbuch der Physik* (Springer-Verlag, Berlin, 1958), Vol. 5, p. 169. Reference to the original papers is given here.
[2] S. N. Gupta, Proc. Phys. Soc. (London) **A63**, 681 (1950); K. Bleuler, Helv. Phys. Acta **23**, 567 (1950).

supplementary condition is energy dependent. Consequently, the conclusion that the bound on the charge renormalization given by the physical electron pairs is infinite would no longer follow without an investigation of the energy dependence of the part of the electron wave-function amplitude governed by the states which satisfy the supplementary condition.

Finally, it is discussed how the expression and its modification could be reconciled if it could be agreed to ignore the evidence of perturbation theory in divergent expressions. The two statements taken together could imply that it is the wave-function amplitude that is the source of the divergence in all gauges where it is a constant. It is pointed out that this does not carry the implication that electrodynamics is inconsistent, but only would mean that gauges where the amplitude is a constant are not self-consistent. It is still an open question as to whether or not any gauge-independent quantity is infinite.

II

The charge renormalization is given by the relation[1]

$$\alpha=\alpha_0(1-L), \quad \alpha=e^2/4\pi, \tag{1}$$

where

$$(1-L)^{-1}=1+\int_0^\infty \frac{\Pi(-a)}{a}da, \tag{2}$$

and where the positive, gauge-invariant function, $\Pi(-a)$, is defined by

$$\Pi(-a)=\frac{V}{3a}\sum\langle 0|j_\mu|z\rangle\langle z|j_\mu|0\rangle. \tag{3}$$

The sum is taken over all exact eigenstates where the total energy momentum vector is $-p^2=a=K^2$. It was shown that if only states which correspond asymptotically to a physical pair are included in (3), the resulting quantity, $\Pi_p(-a)$, is a lower bound to $\Pi(-a)$,

$$\Pi(-a)\geqslant\Pi_p(-a). \tag{4}$$

It was then shown[1] with a somewhat intricate proof that

$$\lim_{a\to\infty}\Pi_p(-a)=\frac{\alpha_0}{\pi}\frac{1}{3}(N^2)^2(1-L)^{-1}, \tag{5}$$

where N^2 is the constant which renormalizes the electron field, *as defined in the Grupta-Bleuler gauge*. Accordingly, it is expressed in terms of states which contain longitudinal and scalar quanta. Since N^2 is a constant, and Π_p is a lower bound to Π, the integral (2) must diverge in contradiction to its assumed convergence. The latter assumption was required to establish (5). Hence the assertion of the inconsistency follows. It is the nature of the limit (5) we wish to investigate.

If N^2 and L are calculated in the lowest order[3] of perturbation theory, we find

$$N^2=1-\frac{\alpha_0}{\pi}\left(\frac{1}{2}\ln\frac{K_c}{m}-\ln\frac{m}{\mu}\right), \quad L=\frac{2\alpha_0}{3\pi}\ln\frac{K_c'}{m}, \tag{6}$$

where μ is an infrared cutoff introduced by the prescription $k^2\to k^2+\mu^2$ in the photon Green's function, K_c' in an ultraviolet cutoff in the particle Green's function, and K_c a similar cutoff in the photon Green's function. Consequently, cutoff perturbation theory to order α_0^2 yields for the exact theorem (5)

$$\lim_{a\to\infty}\Pi_p(-a)=\frac{\alpha_0}{3\pi}\left[1-\frac{\alpha_0}{\pi}\left(\ln\frac{K_c}{m}-\frac{2}{3}\ln\frac{K_c'}{m}-2\ln\frac{m}{\mu}\right)\right]. \tag{7}$$

However, the same function has been calculated directly in perturbation theory to the same order.[4] If we take this result and let $a=K^2\to\infty$, we find

$$\lim_{a\to\infty}\Pi_p(-a)\cong\frac{\alpha_0}{3\pi}\left[1-\frac{\alpha_0}{\pi}\left\{2\ln\frac{K}{m}\left(\ln\frac{K}{m}+2\ln\frac{m}{\mu}\right)+\frac{2}{3}\ln\frac{K_c'}{m}-\frac{13}{3}\ln\frac{K}{m}\right\}\right], \tag{8}$$

where we have kept all terms of order $\ln(K/m)$ or larger, and we have omitted constant terms, terms of order $(m/K)\ln(K/m)$, m/K_c, m/K_c', and those of smaller orders. K, K_c, K_c' are considered to be of the same order of magnitude. The lack of a simple correspondence between (7) and (8) is evident. Further, (8) has a simple physical character not expressed in (7). Namely, as $\mu\to 0$, the limit in which the "physical pair states" with no photons should formally give a smaller contribution (all states have photons in them in this limit), the right side of (8) indeed decreases while in (7) we find an increasing function of μ^{-1}. Thus, the theorem (5) does not reproduce perturbation theory, at least in an elementary way.

Finally, we ask what might be an acceptable form for (5) on the basis of gauge-invariance considerations. It is conjectured that one might replace N^2 by the similar, but gauge-invariant, quantity which results when only states which satisfy the supplementary condition in any gauge are kept in its definition. This may be done by calculating the amplitude in a gauge where only physical states are considered and hence no supplementary condition is required. We may use the mass operator[5] of the electron field to generate an expression for the amplitude in the customary way. Consequently, we have calculated with standard

[3] See, for example, J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Cambridge, 1955).
[4] G. Källén and A. Sabry, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 17 (1955).
[5] J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452 (1951).

methods the mass operator to the lowest order in perturbation theory, using for the photon Green's function the one which describes the coupling of the electron to only the transverse field and to the Coulomb field,

$$\begin{aligned} \mathcal{G}_{\mu\nu} &= \left(\delta_{ij}-\frac{k_ik_j}{|\mathbf{k}|^2}\right)\frac{1}{k^2}, \quad \mu,\,\nu=i,\,j=1,\,2,\,3; \\ &= -1/|\mathbf{k}|^2, \qquad \mu,\,\nu=0,\,0. \end{aligned} \tag{9}$$

In this case we find that N_{phys}^2 depends upon the momentum of the electron in the frame in which the quantization of the electromagnetic field was performed, thus,

$$N_{\text{phys}}^2 = 1-\frac{\alpha_0}{2\pi}\ln\frac{K_c}{m}-\frac{\alpha_0}{2\pi}\int_0^1 dx \times\frac{p^2x^2}{m^2+p^2(1-x^2)}\ln\frac{m^2+p^2(1-x^2)}{\mu^2}, \tag{10}$$

where $p=|\mathbf{p}|$, and μ is an infrared cutoff introduced in the same way as before. The conjecture is that $N_{\text{phys}}^2(K/2)$ should be present in (5) instead of N^2, if we are to reproduce perturbation theory. Thus, the field is quantized in the frame of the center of mass of the virtual pair which is the proper frame of the electron whose charge is being shielded. If we make this replacement and approach the limit, $K\rightarrow\infty$, we obtain

$$\begin{aligned} \lim_{a\to\infty}\Pi_p(-a) &= \lim_{K\to\infty}\frac{\alpha_0}{3\pi}\left[N_{\text{phys}}^2\left(\frac{K}{2}\right)\right]^2(1-L)^{-1} \\ &\cong \frac{\alpha_0}{3\pi}\left[1-\frac{\alpha_0}{\pi}\left\{2\ln\frac{K}{m}\left(\ln\frac{K}{m}+2\ln\frac{m}{\mu}\right)\right.\right. \\ &\qquad \left.\left.-4\ln\frac{K}{m}-\frac{2}{3}\ln\frac{K_c'}{m}+\ln\frac{K_c}{m}\right\}\right], \end{aligned} \tag{11}$$

where again all terms of order $\ln(K/m)$ or larger are kept, and terms of the same orders as those omitted in Eq. (8) are dropped. If we assume that K, K_c, and K_c' are of the same order of magnitude, which is in accordance with the interpretation of the cutoff as the limit on the energy of the intermediate states, we see that the $\ln(K/m)$ term in (8) is

$$\left(-\frac{13}{3}+\frac{2}{3}\right)\ln\frac{K}{m}=-\frac{11}{3}\ln\frac{K}{m},$$

and in (11) the same term is

$$(-4-\tfrac{2}{3}+1)\ln\frac{K}{m}=-\frac{11}{3}\ln\frac{K}{m}.$$

Consequently, there is exact agreement between (8) and (11) in all the divergent terms. This agreement would be rather remarkable if accidental.‡

It is to be noted that since N_{phys}^2 is a function of $K^2=a$, whose high-energy character is unknown, no conclusion about the convergence of (2) would be possible without a detailed investigation. However, it might be remarked that since only physical states are used to define N_{phys}^2 and these have a normal metric, one can prove that just as $1\geqslant L\geqslant 0$ so also $1\geqslant N_{\text{phys}}^2\geqslant 0$. Hence, it is at least possible that the convergence of the lower bound to (2) given by the pair states is better than it is in perturbation theory.

III

Finally, we come to the question of whether or not the remarks here lead one to the conclusion that the proofs given in reference 1 are incorrect. They do not necessarily, if one agrees that the evidence of perturbation theory in divergent expressions is not infallible. Alternatively, we could say that the very fact that one could obtain a relation such as (5) indicates a fundamental inconsistency in the theory.

If we assume that both results must agree, then it would be necessary that $\lim_{p\to\infty}N_{\text{phys}}^2(p)=N^2$. Further, if we would perform the identical calculations of reference 1 using the covariant photon Green's function

$$\mathcal{G}_{\mu\nu}=\left(\delta_{\mu\nu}+\epsilon\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2},$$

the additional ϵ-dependent term would introduce only angular factors which should not effect the energy denominators and consequently one should arrive at the limit (5) with the constant $N^2(\epsilon)$ replacing N^2. Again, because of the gauge invariance of the left side of (5) this would mean that $N^2(0)=N^2=N^2(\epsilon)$, for all values of ϵ. Such a statement is not in any way indicated by perturbation theory where $N^2(\epsilon)$ varies with ϵ. The only possible conclusion would seemingly be that $N^2=0=N^2(\epsilon)=\lim_{p\to\infty}N_{\text{phys}}^2(p)$, that is, that there is a divergence.

It is to be pointed out that this does not mean that electrodynamics itself is inconsistent, if we mean by the word electrodynamics a formulation of the theory in a truly gauge-invariant fashion. For the parameter N^2 is gauge dependent, and the only indication is that $N^2(\epsilon)$ is zero in the class of gauges where the amplitude of the electron wave function is a constant. We could renormalize the electron field with $N_{\text{phys}}^2(0)$, and

‡ *Note added in proof.*—The cutoff has been used to compute only the infinite terms. Consequently the limit $K\rightarrow\infty$ means, of course, $K\sim K_c$. Since neither in perturbation theory or in the expansion of the conjecture (11) does a *cutoff dependent* $(\ln K)^2$ term occur, there is no ambiguity in the coefficient of the $(\ln K)$ terms but of course there would be ambiguity in the constant terms, since $\ln(\beta K_c)=(\ln K_c)+(\ln\beta)$, and $\ln K_c$ and $\ln(\beta K_c)$ are counted the same.

nothing (except perturbation theory) indicates that this quantity is divergent. Further, nothing indicates that the amplitude in all gauges is divergent. It seems that the only conclusion which would be safe to draw is that the use of gauges where the amplitude of the wave function is a constant does not provide for a consistent formulation of electrodynamics. It is still an open, and interesting, question as to whether or not any physical (gauge-invariant) parameter is infinite.

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Dispersion Relations for Pion-Proton Scattering

D. A. GEFFEN*
Carnegie Institute of Technology, Pittsburgh, Pennsylvania
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The dispersion relations are used to predict the values at zero kinetic energy, of the derivatives, $\partial D_{\pm}(0)/\partial k^2$, of the real parts, $D_+(0)$ and $D_-(0)$, of the forward elastic scattering amplitudes for π^+ and π^- mesons scattered by protons. The experimental value of $\partial D_+(0)/\partial k^2$ is fairly well known, and, when compared with the predicted value, yields a determination of the coupling constant, $f^2=0.104\pm0.014$. The predicted value for $\partial D_-(0)/\partial k^2$ disagrees badly with experiment, especially with an f^2 as large as 0.10.

The dispersion relations are modified by introducing an extra energy denominator in such a way as to contain, as the additional constants, the derivatives $\partial D_{\pm}(0)/\partial k^2$. This enables us to check the values of $\partial D_{\pm}(0)/\partial k^2$ obtained from the usual dispersion relations as well as the assumption that $\omega^{-2}T_{\pm}(\omega)$ vanishes at infinity. It is found that as long as the agreement with experiment obtained for the π^+ relation is retained, no appreciable change in the values of $\partial D_{\pm}(0)/\partial k^2$ is possible and that the high-energy behavior of $T_{\pm}(\omega)$, usually assumed, is correct. The predicted value for $\partial D_-(0)/\partial k^2$ strongly suggests a nonzero effective range for α_1 and a relatively large α_{11}.

1. INTRODUCTION

THE discrepancy with experiment of the π^- dispersion relation, which was first pointed out by Puppi and Stanghellini[1] and subsequently discussed by several authors,[2-4] is examined in this paper by use of a slightly different approach. The dispersion relations are used to predict the values, at zero kinetic energy, of the derivatives, $\partial D_{\pm}(0)/\partial k^2$, of the real parts, $D_+(0)$ and $D_-(0)$, of the forward elastic scattering amplitudes for π^+ and π^- mesons scattered by protons. The experimental values of these derivatives depend very strongly on the P-wave scattering lengths and the S-wave effective ranges. These quantities are fairly well known for $T=\frac{3}{2}$, and yield a value for $\partial D_{\pm}(0)/\partial k^2$ which, when compared with the prediction of the dispersion relations, leads to a determination of the coupling constant, $f^2=0.104\pm0.014$. On the other hand, the dispersion relations predict a value for $\partial D_-(0)/\partial k^2$ which disagrees badly with present π^- experiments, especially with an f^2 as large as 0.10.

To eliminate the unknown high-energy contributions to the integrals appearing in the dispersion relations as well as to check the assumption that $\omega^{-2}T_{\pm}(\omega)$ vanishes as ω becomes infinite, the dispersion relations are modified by introducing an extra energy denominator in the integrals. This involves the added subtraction of the real part of the scattering amplitude at an arbitrary energy ω_0. By letting ω_0 approach 1, the relations can be simplified and contain, as added constants, the quantities $\partial D_{\pm}(0)/\partial k^2$. These new relations have the added advantage, then, of enabling us to check the values of $\partial D_{\pm}(0)/\partial k^2$ obtained from the usual dispersion relations. It is found that as long as the agreement with experiment obtained for the π^+ relation is retained, no appreciable change in the latter values of $\partial D_{\pm}(0)/\partial k^2$ is possible. This result indicates the correctness of the assumed high-energy behavior of $T_{\pm}(\omega)$ and reinforces the conviction that the values predicted for $\partial D_{\pm}(0)/\partial k^2$ are correct.

The value for $\partial D_-(0)/\partial k^2$ predicted by the dispersion relations is compared with experiment and the discrepancy between these two values is interpreted as being due to the very small $T=\frac{1}{2}$ scattering cross sections that have so far been observed. It will be shown that a resolution of the discrepancy between the theoretical and experimental values of $\partial D_-(0)/\partial k^2$ could very well involve changes in present experimental data which would also remove the discrepancy between the predicted and observed values of the real part of the $\pi^- - p$ forward elastic scattering amplitude. It would seem reasonable, therefore, to take the failure or success

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[4] J. Hamilton (to be published).