Semiclassical Treatment of Direct Nuclear Reactions

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A semiclassical method is employed to discuss the angular distributions of direct nuclear reactions. The method considers the incident and outgoing particles to be described by rays which follow classical paths; these rays can be refracted, reflected, and absorbed by a "clouded-crystal-ball" nucleus. Any given incident ray is considered to have a certain probability of suffering a major "scattering" at each point within a nucleus, this "scattering" not only changing the direction of the ray but also its wavelength. A condition of angular momentum conservation is imposed on this direct-scattering event, such that the angular momentum change in this process is equal to that required by the direct reaction under consideration.

On this picture it is seen that the oscillations in the angular distributions which often occur with direct reactions arise as a result of interference between direct scatterings in different regions of the nucleus. Thus these oscillations are to be likened to the interference maxima and minima which arise when light is scattered, for example from a soap film. Effects due to compound nucleus absorption and of refraction and reflection of the incident and outgoing rays can be discussed fairly simply on this model.

An investigation of the basis of the semiclassical method indicates that it should be reliable over a wide range of interesting conditions.

I. INTRODUCTION

EXPERIMENTAL evidence is accumulating to the effect that a very large number of nuclear reactions which proceed at least to low-lying sharp energy levels of the final nucleus receive predominant contributions from a direct process.[1] The theoretical interpretation of the experimental differential cross sections for such reactions is dependent on the spins and parities of the nuclear energy levels involved, and also on the optical-model properties of the initial and final nuclei as seen by the incident and outgoing particles, respectively.[2,3] Thus a complete analysis of a given direct reaction differential cross section is capable of yielding a considerable amount of information not only in the realm of nuclear spectroscopy, but also as regards nuclear matter itself.

Unfortunately the theoretical evaluation of a direct-reaction cross section can be simply performed only under fairly extreme approximations.[2,3] A general evaluation for any one particular reaction involves a major computational program in which agreement with the experimental differential cross section may finally be achieved by appropriate choice of optical-model, and other, parameters.[4] In such a calculation, one tends often to achieve little real insight as to why the particular values of the parameters actually chosen should lead to the angular distributions finally obtained. Of course accurate calculations must be performed, but it is undoubtedly preferable to obtain a good qualitative and even semiquantitative understanding of any given angular distribution from simple physical considerations, leaving to a full-scale computational program only questions of detailed accuracy.

In this paper we discuss a simple semiclassical treatment of direct reactions, yielding a qualitative and in many cases quantitative understanding of the cross sections of these reactions. The method considers the incident and outgoing particles to be described by rays which follow classical paths (i.e., it may be thought of as employing WKB wave functions). A given incident ray is considered to be refracted and absorbed by the "clouded-crystal-ball" nucleus, and then at some well-defined point in the nucleus to undergo a major "scattering" event. This "scattering," which gives the direct reaction, may change the direction, wavelength, and particle type of the ray. The outgoing ray subsequent to such a scattering may then see the nucleus as a medium with different refraction and absorption properties from those seen by the ingoing ray.

Angular momentum conservation is imposed on the direct scattering event, the special assumption of the semiclassical method being that this selection rule applies locally, to each possible reaction point within the nucleus. As is well known for direct reactions the conservation laws of angular momentum and of parity, involving the spins and parities of the initial and final

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[1] Schrank, Gugelot, and Dayton, Phys. Rev. **96**, 1156 (1954); Levine, Bender, and McGruer, Phys. Rev. **97**, 1249 (1955); Benveniste, Finke, and Martinelli, Phys. Rev. **101**, 655 (1956); P. C. Gugelot and M. Rickey, Phys. Rev. **101**, 1613 (1956); H. J. Watters, Phys. Rev. **103**, 1763 (1956); C. E. Hunting and N. S. Wall, Phys. Rev. **108**, 901 (1957), and also Bull. Am. Phys. Soc. Ser. II, **2**, 181 (1957); R. A. Peck, Phys. Rev. **106**, 965 (1957); W. Blanpied and R. Sherr, Bull. Am. Phys. Soc. Ser. II, **2**, 303 (1957); R. W. Peelle, Phys. Rev. **105**, 1311 (1957); G. F. Pieper and N. P. Heydenberg, Bull. Am. Phys. Soc. Ser. II, **2**, 182 (1957), and Phys. Rev. **111**, 264 (1958); M. Rickey and R. Sherr (to be published).

[2] Austern, Butler, and McManus, Phys. Rev. **92**, 350 (1953).

[3] S. T. Butler, Phys. Rev. **106**, 272 (1957); C. A. Levinson and M. K. Banerjee, Ann. Phys. **1**, 471 (1957); M. K. Banerjee and C. A. Levinson, Ann. Phys. **1**, 499 (1957).

[4] C. A. Levinson and M. K. Banerjee, Ann. Phys. **3**, 67 (1958).

nuclei, place severe limitations on the orbital angular momentum change which can occur as a result of the reaction. If the angular momentum change be denoted by $l\hbar$, the usual circumstance is that only one specific integral value of l is allowed. Hence in the semiclassical treatment the condition is imposed that when any given incident ray "scatters" at a certain point into an outgoing ray, it must do so in such a way that the orbital angular momentum change be equal in magnitude to $l\hbar$. For a given angle of scattering this condition imposes restrictions on the region of the nucleus which can contribute to the direct reaction.

In Sec. II this semiclassical method will be defined in more detail, and its consequences evaluated under the same approximations as have previously been employed in obtaining closed-form results from a quantum-mechanical calculation.[2,3] It is found that the results of the semiclassical method closely resemble those of the corresponding wave-mechanical calculation for all values of the parameters involved. Moreover one gains an insight into the factors contributing to the well-known oscillatory nature of the angular distributions which are obtained. Thus the oscillations in the angular distributions are seen to arise from interference between rays which suffer their direct scattering in different regions of the nucleus, in close analogy to the interference maxima of intensity which occur when light is reflected from a soap film. Also, for those reactions in which the angular momentum transfer l in nonzero, it is readily to be understood why the differential cross sections are often small at small angles of scattering, with a first maximum displaced from the forward direction. This appears as a direct consequence of the angular momentum selection rule, when for small angles of scattering there is no point within the nucleus at which an ingoing ray can be "scattered" into an outgoing ray such as to yield the required orbital angular momentum change (see Appendix of first paper of reference 3).

As well as providing considerable insight into the factors contributing to the usual simplified closed-form formulas for direct reaction angular distributions, the semiclassical method also serves to show the shortcomings of some of the approximations employed in the derivation of these formulas. The present method does not depend for its simplicity on these approximations, and results by this method can be evaluated with absorption and refraction of the incident and outgoing rays considered in a general manner. In Sec. III we present preliminary results of such an investigation, in which compound-nucleus absorption is considered more realistically, and some effects of refraction are discussed qualitatively. The results of this section will show, for example, that absorption of the incident and outgoing rays tends to fill in the valleys between oscillations in the angular distributions without affecting the positions of the maxima to any great extent. In some cases when absorption is very strong, maxima subsequent to the first peak can be washed out almost entirely so that the whole *direct-reaction* angular distribution at large angles becomes slowly varying.

It is also seen that a significant effect of refraction is the possibility of an angular distribution being peaked forward, even if on the simple theory there is a "forbidden region" in the forward direction. Such a peak can arise, however, only if the compound-nucleus absorption of either the incident or outgoing particles is quite weak. Under these conditions we also see in Sec. III that it is possible for a forward peak to be accompanied by a backward peaking, reflection of the rays producing an "image" of the main forward peak in the backward direction.

A more detailed study of refraction as well as absorption effects, together with detailed comparisons with experiment, is deferred for a later publication.

In Sec. IV of the present paper an investigation of the reliability of the semiclassical method is carried out. It is derived from the full quantum-mechanical calculation, the accuracy of the required approximation then representing the accuracy of the semiclassical method. The derivation is actually carried through for a somewhat restricted case, *viz.*, under the simplifying assumptions regarding direct reactions discussed in Sec. II. Under these restrictions, however, it can be seen that the semiclassical results should be reliable over a wide range of interesting experimental conditions, and are at least qualitatively reliable in most cases.

The derivation of the semiclassical method has not as yet been carried out with absorption and refraction effects fully included. Nevertheless the results of Sec. IV give us confidence that it can be useful for more general calculations.

II. SEMICLASSICAL METHOD

In a given direct reaction, let the wave vectors for the incident and outgoing particles be $\mathbf{k}_i$ and $\mathbf{k}_f$, respec-

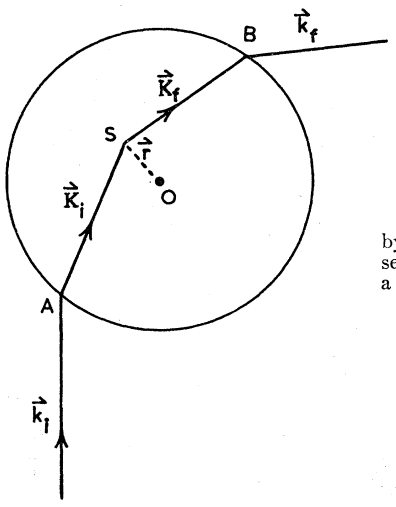

FIG. 1. Path traveled by a typical ray in the semiclassical picture of a direct reaction.

tively, the angle between these vectors being the angle of scattering, and let the angular momentum transfer be $l\hbar$. The semiclassical picture of such a reaction is depicted in Fig. 1. A particular incident ray enters the nucleus at A, where it is subjected to *refraction*, and thereafter its amplitude will suffer damping due to *absorption*. At some point S a direct scattering occurs in which the wavelength of the ray is changed. The outgoing ray penetrates to the nuclear surface at B, is refracted, and emerges with wave vector $\mathbf{k}_f$.

Let the wave vectors of the incident and outgoing rays within the nucleus be $\mathbf{K}_i$ and $\mathbf{K}_f$, respectively, and let the position vector of the point S be $\mathbf{r}$. Then the orbital angular momentum of the incident ray may be written $\hbar\mathbf{K}_i\times\mathbf{r}$ (since refraction at the nuclear surface cannot change the angular momentum), and of the outgoing ray $\hbar\mathbf{K}_f\times\mathbf{r}$. Thus the orbital angular momentum change $\Delta\mathbf{L}$ as a result of the direct scattering is given in magnitude by

$$\Delta L=\hbar|\mathbf{Q}^*\times\mathbf{r}|, \tag{1}$$

where

$$\mathbf{Q}^*=\mathbf{K}_i-\mathbf{K}_f.$$

By the condition therefore that the orbital angular momentum change be $l\hbar$ with l a particular integral value, we have from (1) that

$$|\mathbf{Q}^*\times\mathbf{r}|=l. \tag{2}$$

It is the condition (2) which is to be imposed on each direct scattering event, and which for a given angle of scattering severely limits the regions of the nucleus which can contribute to the reaction.

Illustration in Simple Case

In order to obtain simplified closed-form expressions from a quantum-mechanical calculation of direct reaction cross sections, the following two approximations have previously been made (see, e.g., references 2, 3): (a) compound-nucleus absorption of the incident and outgoing particles is assumed to be very strong in the interior of the nucleus, thereby limiting the reaction to the surface; (b) the wave functions for the incident and outgoing particles are taken simply to be undistorted plane waves, and all parts of the nuclear surface are considered to be able to contribute with equal probabilities to the reaction. If the same assumptions are made here for comparison, our picture of the reaction simplifies to that of Fig. 2, and the results of the semiclassical method may be written down almost at once.

To start with, since refraction at the surface is being ignored, the wave vectors $\mathbf{k}_i$ and $\mathbf{K}_i$ are identical, as also are the vectors $\mathbf{k}_f$ and $\mathbf{K}_f$. Thus condition (2) becomes simply

$$|\mathbf{Q}\times\mathbf{r}|=l, \tag{3}$$

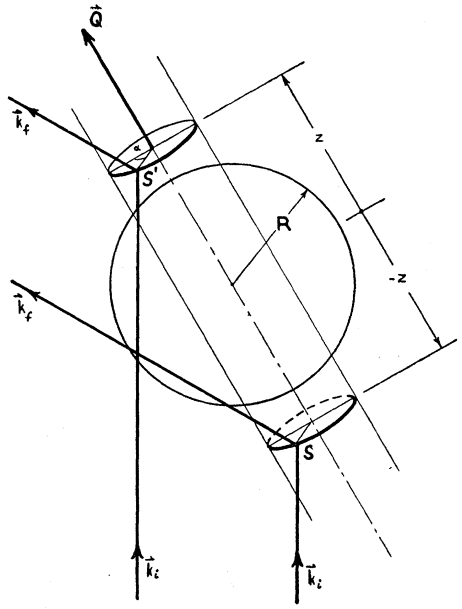

FIG. 2. Semiclassical picture of a direct surface reaction under the simplifying approximations of Sec. II. The two scattering events depicted occur at opposite ends of the active cylinder; it is the interference between the outgoing rays from such scatterings that gives rise to the "interference maxima and minima" of the angular distribution.

where $\mathbf{Q}$ is the recoil momentum of the target,

$$\mathbf{Q}=\mathbf{k}_i-\mathbf{k}_f.$$

For a given angle of scattering, the condition (3) restricts the possible points of scatter S to the surface of a cylinder of radius l/Q whose axis passes through the center of the nucleus in the direction of the vector $\mathbf{Q}$. We shall for conveniencec all this the "active surface."

We wish to add coherently all the outgoing rays, and to do this we must take into account the fact that the total path length differs for rays scattered at different points on the active surface. It may readily be ascertained that this leads to a phase-factor $e^{i(\mathbf{Q}\cdot\mathbf{r})}$ to be associated with a given outgoing ray, where $\mathbf{r}$ is the position vector of the point S. Thus we must form the sum of all factors $e^{i(\mathbf{Q}\cdot\mathbf{r})}$ over the active cylinder, weighting each point S by a probability amplitude $p(r)$ that a direct scatter could have occurred at that point. The total amplitude T for the direct reaction therefore becomes

$$T=\int_{\text{active cylinder } (r\leqslant R)} d\mathbf{r}\, p(r)e^{i(\mathbf{Q}\cdot\mathbf{r})}, \tag{4}$$

the nuclear radius being denoted by R.

Clearly the most convenient method for evaluating this integral is to employ cylindrical coordinates, say (ρ,α,z) with the z axis chosen as the axis of the active cylinder. The phase factor takes the simple form e^{iQz},

where Q is the magnitude of $\mathbf{Q}$; the contributions from $+z$ and $-z$ combine, giving the interesting oscillatory factor

$$e^{iQz}+e^{-iQz}\rightarrow\cos Qz.$$

The ρ and α integrations are entirely trivial; in particular ρ is the radius of the active cylinder, $\rho=l/Q$. The resulting one-dimensional integral on z is then conveniently transformed back in terms of the polar coordinate radius r, where $r^2=z^2+(l/Q)^2$. This yields

$$T=4\pi l\int_{r\leqslant\max(R,l/Q)} rdr\,p(r)\frac{\cos(Q^2r^2-l^2)^{\frac{1}{2}}}{(Q^2r^2-l^2)^{\frac{1}{2}}}. \quad (5)$$

The factor $p(r)$ of the integrand of (5) we have defined to be the probability amplitude for a direct scatter to have occurred at the point S, whose distance from the center of the nucleus is r. It clearly should be proportional to the availability at that point of a nucleon (or nucleon group) in an appropriate state to produce the scattering; it equally clearly should be proportional to the amplitude at that point for the nucleon (or nucleon group) which remains captured in the final nuclear state.

We shall obtain more insight into this factor $p(r)$ shortly, but for the moment it will suffice us to know that it must fall off with increasing $r(>R)$ as does the nucleon distribution in the initial and final nuclei. We therefore expect the main contributions to the integral of (5) to come from values of r close to the lower limit of the integral.

As long as l/Q is less than R, so that the active cylinder does in fact intersect the main core of the nucleus, the amplitude would therefore be expected to be represented accurately by the following approximation:

$$T\approx 4\pi lP\left[\frac{\cos(Q^2R^2-l^2)^{\frac{1}{2}}}{(Q^2R^2-l^2)^{\frac{1}{2}}}\right], \quad (6)$$

where

$$P\equiv\int_R^\infty rdrp(r).$$

In the event that $QR<l$, the active cylinder misses the nuclear core altogether, and merely passes through the fringes of the decreasing nuclear density distribution. Combining these results, the angular distribution is seen to be of the form[5]

$$\sigma(\theta)\propto\frac{\cos^2(Q^2R^2-l^2)^{\frac{1}{2}}}{Q^2R^2-l^2},\quad QR>l; \quad (7a)$$

$$\sigma(\theta)\rightarrow 0,\quad \text{for}\quad QR<1. \quad (7b)$$

The behavior of the differential cross section predicted by (7a) and (7b) is determined by the fact that Q increases monotonically with scattering angle, and also by the fact that Q frequently is quite small for forward angles of scattering. For a reaction with $l>0$, then, it can well happen for small angles of scattering that $QR<l$, and that the differential cross section will be very small, according to (7b). This is then a "forbidden region" in the angular distribution due to the fact that the "active cylinder" does not cut the nucleus—or in other words to the fact that there is no point within the nucleus at which a direct reaction can produce the required orbital angular momentum change.

As the angle of scattering increases, the cross section will rise as QR approaches l, and will reach a maximum at the point $QR=l$. Just at this point Eq. (6) is a poor approximation to Eq. (5), so Eq. (7a) is not accurate. Towards somewhat larger angles the accuracy becomes adequate, however, and it is seen that the angular distribution will describe oscillations according to the factor $\cos^2(Q^2R^2-l^2)^{\frac{1}{2}}$, the amplitude of which will decrease as $(Q^2R^2-l^2)^{-1}$. These oscillations stem solely from the factor $\cos(Q^2R^2-l^2)^{\frac{1}{2}}$ in the integrand of Eq. (5), which arises as a result of the coherent addition of the amplitudes from two circles around the opposite ends of the active cylinder.

The behavior of the direct-reaction angular distribution given by (7a) and (7b) is very similar to that obtained from quantum-mechanical calculations. The similarity may be brought out even more closely. Under the approximations mentioned at the beginning of this discussion, the quantum-mechanical result for the amplitude for a surface direct reaction takes the standard form[2,3]

$$T_{\text{Q.M.}}=K\int_R^\infty r^2dr\,y(r)j_l(Qr). \quad (8)$$

Here K is a constant (we are not at present concerned with absolute magnitudes), $y(r)$ is a function dependent on the bound-state wave functions, and $j_l(Qr)$ is the usual spherical Bessel function.[6]

For a (d,p) reaction, for example, $y(r)$ is the radial wave function of the captured neutron in its final nuclear state; i.e., for $r\gtrsim R$ one has for a (d,p) reaction

$$y(r)\propto h_l(i\kappa r),$$

where $\hbar^2\kappa^2/2m$ is the neutron binding energy (m being the nucleon mass) and h_l is the spherical Hankel function. Alternatively, for a reaction of the form (p,p') or (α,α') in which the incident particle is considered to be scattered by a nucleon in a given state,

[5] Strictly we also should allow the function $p(r)$ to depend on Q, since the probability amplitude for a direct scatter can well depend on the magnitude of the momentum transfer. Thus the quantity $P=\int_R^\infty rdr\,p(r)$ can be Q dependent, and provide an angle-dependent form factor to multiply (7). Such a function will always be smoothly varying with angle, and cannot affect significantly the positions of the maxima and minima of the angular distribution. It can, however, influence the relative heights of different peaks in the distribution.

[6] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), Chap. IV, p. 77.

thereby changing the state of this nucleon, the function $y(r)$ is a product of the initial and final radial wave functions of the nucleon. If the initial state of this nucleon has orbital angular momentum l_1, and the final state has orbital angular momentum l_2, and if the radial wave functions for these states be $u_{l_1}(r)$ and $u_{l_2}(r)$, respectively, we have in this case

$$y(r) \propto u_{l_1}(r)u_{l_2}(r) \propto h_{l_1}(i\kappa_1 r)h_{l_2}(i\kappa_2 r), \quad \text{for} \quad r \gtrsim R.$$

Here $\hbar^2\kappa_1^2/2m$ and $\hbar^2\kappa_2^2/2m$ are the binding energies of the nucleon before and after the collision.

Because of the fact that the above function $y(r)$ decreases fairly rapidly as r increases above R, the factor $j_l(Qr)$ can as a first approximation be replaced by $j_l(QR)$ in the integral of (8). Thus the angular distribution for a direct surface reaction, derived in the usual manner, is determined essentially by the factor $\{j_l(QR)\}^2$. Indeed, on comparing Eqs. (5) and (8) we see that the function $p(r)$ of the semiclassical method is to be identified with the wave function $ry(r)$ of the wave-mechanical matrix element, and that in addition the semiclassical method makes the following replacement:

$$j_l(Qr) \xrightarrow{\text{semiclassical}} \begin{cases} 0, & \text{if} \quad Qr<l \\ \dfrac{\cos(Q^2r^2-l^2)^{\frac{1}{2}}}{(Q^2r^2-l^2)^{\frac{1}{2}}}, & \text{if} \quad Qr \gtrsim l. \end{cases} \quad (9)$$

It may readily be verified that the right-hand side of (9) provides a quite good approximation to the spherical Bessel function, starting beyond the first maximum and going out through several oscillations.

Apart from the fact that it yields angular distributions very like those of quantum-mechanical calculations, the semiclassical method is of interest in showing how the familiar oscillations of these angular distributions arise. They arise as a result of interference between outgoing rays which have been produced in a direct scatter at different points of the nucleus, the same sort of effect as produces the well-known interference fringes of physical optics.

Of course the simple approximations employed in this section are not essential for application of the semiclassical method. They were employed solely to facilitate a comparison between the semiclassical and quantum-mechanical results in one particular case. Actually these simple approximations are liable to be quite inaccurate in certain cases. For example, if compound-nucleus absorption of the incident and outgoing rays be taken into account realistically, and if the mean free path for such absorption be quite short for at least one of these rays, then not all points in any shell of radius r within the nucleus will be equally accessible for a direct reaction. In particular the two circles around the active cylinder corresponding to a certain radial distance r may really contribute with quite different magnitudes. Thus it may be very much less likely for an incident ray to penetrate to a circle which is largely towards the rear "shaded" side of the nucleus than to the corresponding one on the near "light" side. In this case the interference between the outgoing rays emanating from these circles can never be complete. One would now expect different weights for the respective phase factors, so that in adding the contributions of these two circles one has a sum

$$\begin{aligned} e^{-i(\mathbf{Q}\cdot\mathbf{r})}+ae^{+i(\mathbf{Q}\cdot\mathbf{r})} &= (1-a)e^{-i(\mathbf{Q}\cdot\mathbf{r})}+2a\cos(\mathbf{Q}\cdot\mathbf{r}) \\ &= (1-a)e^{-iQz}+2a\cos Qz, \quad a<1 \end{aligned} \quad (10)$$

to replace what was a pure $\cos Qz$ term in the previous treatment.

Hence we would expect a more realistic account of compound nucleus absorption to fill in the valleys between the peaks in the angular distribution. However the peaks will still occur at the same places.

In the next section the semiclassical method is employed to explore this question in more detail, and it will be seen that the above qualitative expectations are borne out.

III. ABSORPTION AND REFRACTION EFFECTS

The semiclassical model can be applied very easily, in the examination of any particular experiment, by sketching the active cylinders for a variety of scattering angles. Then the paths which the incident and emerging rays must take to reach these cylinders may suggest the effects which are influencing the angular distribution.

A few typical effects will now be discussed.

(a) *Absorption*

We shall simplify this discussion of absorption by taking the function $p(r)$ of Eq. (4)—i.e., the probability amplitude that a direct scatter can occur at a point distant r from the center of the nucleus—to be a step function, i.e.,

$$\begin{aligned} p(r)&=p_0, \quad r \lesssim R \\ p(r)&=0, \quad r \gtrsim R. \end{aligned} \quad (11)$$

Thus we are ignoring the decreasing density distribution outside the surface.

We make allowance for compound-nucleus absorption by saying that the probability that the incident ray can reach a given point S (see Fig. 3) without being absorbed is given by a factor $\exp(-\Gamma_i R_i)$, where R_i is the distance the ray has had to penetrate through the nucleus in order to reach S. In other words we assume the ray has been attenuated by the amount $\exp(-\Gamma_i R_i)$. Similarly the probability of the outgoing ray actually emerging is taken to be a factor $\exp(-\Gamma_f R_f)$ where R_f is the distance this ray must traverse before escaping from the nucleus.

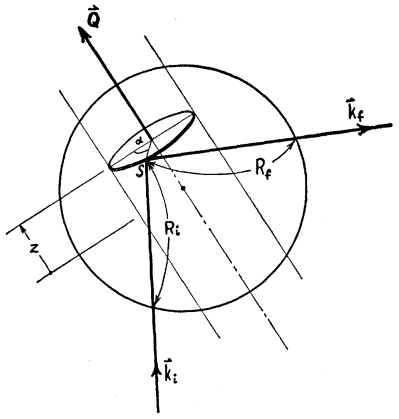

FIG. 3. A direct scattering event occurring on the surface of the active cylinder and within the nucleus. The distances over which absorption can occur for the incident and outgoing rays are the distances R_i and R_f, respectively.

If we neglect refraction effects and the corresponding change in wavelength of a ray as it penetrates into the nuclear matter, the following expression for the amplitude T for the direct reaction replaces Eq. (4):

$$T=\int_{\text{active cylinder}} d\mathbf{r}\, p(r)e^{i(\mathbf{Q}\cdot\mathbf{r})}e^{-\Gamma_i R_i}e^{-\Gamma_f R_f}. \quad (12)$$

In the event that $\Gamma_i=\Gamma_f=0$, no absorption, this amplitude may be evaluated, and is simply

$$T=4\pi l p_0 Q^{-2}\sin(Q^2R^2-l^2)^{\frac{1}{2}}, \quad \text{if} \quad QR>l.$$

In this case the angular distribution is

$$\sigma\propto Q^{-4}\sin^2(Q^2R^2-l^2)^{\frac{1}{2}}, \quad QR>l \quad (13a)$$

$$\sigma=0, \quad QR<l. \quad (13b)$$

Thus once again the angular distribution exhibits the familiar oscillations, although they now are damped towards large Q by the factor Q^{-4}.

To evaluate the integral of (12) for specific nonzero values of Γ_i and Γ_f it is once more convenient to employ cylindrical coordinates, in which the point of direct scatter is specified by its distance z along the cylinder, and by a polar angle α as indicated in Fig. 3. The angle α is taken to be the azimuthal angle of the radius vector $\mathbf{r}$ with respect to the plane of $\mathbf{k}_i$ and $\mathbf{k}_f$. If we let θ_1 be the angle between the vectors $\mathbf{k}_i$ and $\mathbf{Q}$, and θ be the angle of scattering, the distances R_i and R_f are expressed in terms of our integration variables as follows:

$$R_i=z\cos\theta_1+(l/Q)\sin\theta_1\cos\alpha+[\{z\cos\theta_1+(l/Q)\sin\theta_1\cos\alpha\}^2+R^2-z^2-l^2/Q^2]^{\frac{1}{2}}, \quad (14)$$

$$R_f=-z\cos(\theta+\theta_1)-(l/Q)\sin(\theta+\theta_1)\cos\alpha+[\{z\cos(\theta+\theta_1)+(l/Q)\sin(\theta+\theta_1\cos\alpha\}^2+R^2-z^2-l^2/Q^2]^{\frac{1}{2}}. \quad (15)$$

The amplitude (12) becomes

$$T=p_0\int_{-(R^2-l^2/Q^2)^{\frac{1}{2}}}^{(R^2-l^2/Q^2)^{\frac{1}{2}}} dz\, e^{iQz}\int_0^{2\pi} d\alpha \times\exp\{-\Gamma_i R_i(z,\alpha)-\Gamma_f R_f(z,\alpha)\}, \quad (16)$$

in which $R_i(z,\alpha)$ and $R_f(z,\alpha)$ are given by Eqs. (14) and (15), respectively. Of course for $QR<l$ (the "forbidden region" of the angular distribution) the active cylinder does not intersect the nucleus and the amplitude T is zero.

We have evaluated (16) numerically for a number of cases using the University of Sydney electronic computer (SILLIAC), some typical results being shown in Figs. 4, 5, and 6. These figures pertain to reactions of the type (p,p'), (d,p) or (d,n), and (α,p), respectively. In each case a variety of different values of the absorption is used, but always with the fixed ratios,

Fig. 4, (p,p'): $\Gamma_i/\Gamma_f=1$;
Fig. 5, (d,p): $\Gamma_i/\Gamma_f=2$;
Fig. 6, (α,p): $\Gamma_i/\Gamma_f=4$.

An approximate evaluation of the double integral of Eq. (16) also can be made quite readily, particularly for cases of strong absorption (large Γ_i and Γ_f).

The expectations of the previous section are seen to be realized. As the absorption is increased the valleys between oscillations are more and more filled in. The positions of the peaks and valleys remain essentially the same irrespective of absorption, and the first maximum corresponding to $QR\approx l$ is always in evidence.

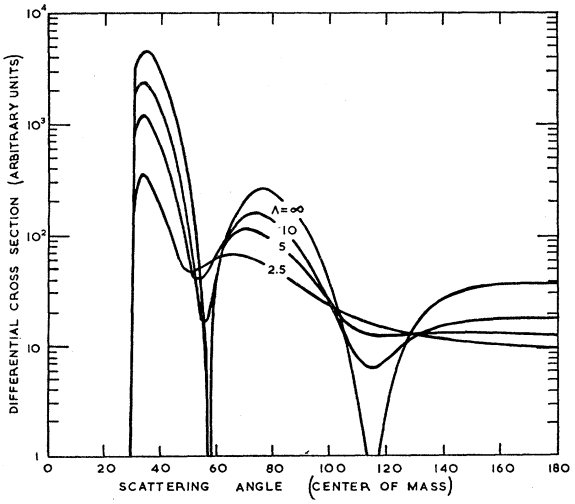

FIG 4. Angular distributions for a (p,p') reaction as computed with the semiclassical model, for a variety of absorptivities, and using the step function form of the nuclear density $p(r)$. For these curves $l=2$, $R=5.0\times10^{-13}$ cm, and the incident and outgoing energies (c.m.) are 14 Mev and 11.7 Mev, respectively. Also $\Gamma_i=\Gamma_f$ is assumed, with $\Gamma=\Lambda^{-1}$, the reciprocal of the mean free path. The values of Λ, in units 10^{-13} cm, are indicated on each curve.

Subsequent peaks can however be smoothed out entirely in cases of very strong absorption.

The shortcomings of the simple plane-wave approximations are now evident. These approximations start from the assumption of strong compound-nucleus absorption (large values of Γ), but on the other hand allow equal contributions to the direct reaction from the entire outer shell of the nucleus; it is this latter assumption which then gives rise to the pronounced subsidiary oscillations subsequent to the first maximum. We now see that it is just in the case of strong absorption that these subsidiary maxima and minima tend to be smoothed out.

Such effects have certainly been observed experimentally. We will, however, not attempt detailed comparisons with experiment here, as we view the present investigation as preliminary only. In the first instance the form of (11) for the function $p(r)$ is clearly an idealization, and more general shapes should be allowed.

A second important factor is that as soon as any degree of penetration into the nucleus is permitted, refraction (and reflection) effects will certainly be of importance. Although we have as yet made no detailed calculations of these effects, some very interesting consequences of them can be discussed qualitatively on the semiclassical picture.

(*b*) *Refraction*

One of the most significant effects of refraction is that it is capable of producing a violation of a rule that often is considered quite basic for direct reactions, that the differential cross section is small at forward angles of scattering for which $QR<l$ (the "forbidden region").

All that need happen to violate this rule is that Q^*R [see Eq. (2)] be greater than l for scattering, even through QR is less than l. This is illustrated in Fig. 7, in which we see how refraction at the nuclear surface can produce an outgoing ray *parallel* to the incoming ray, but with a *quite different impact parameter*. Thus an appreciable angular momentum transfer can be achieved, even for zero angle of scattering. In such cases

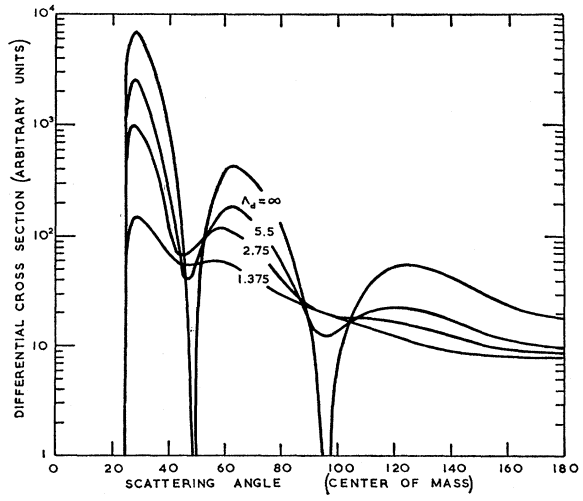

FIG. 5. Angular distributions for a (d,n) reaction, computed as described in the caption of Fig. 3. Parameters are $l=1$, $R=5.5\times10^{-13}$ cm, 8 Mev incident, 11 Mev outgoing, $\Gamma_i/\Gamma_f=2$.

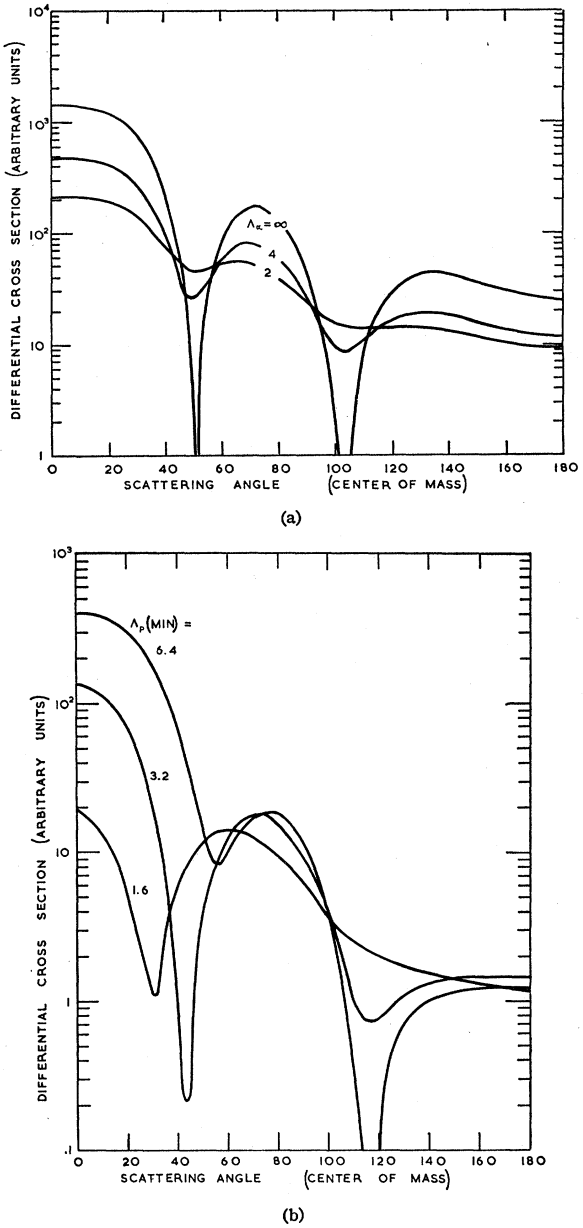

FIG. 6. (a) Angular distributions for an (α,p) reaction, computed as described in the caption of Fig. 3. Parameters are $l=1$, $R=4.0\times10^{-13}$ cm, 20.8 Mev incident, 16.8 Mev outgoing, $\Gamma_i/\Gamma_f=4$. (b) Same reaction as in Fig. 6(a), except with a diffuse-surface model for $p(r)$, and with the absorption concentrated near the surface. This $p(r)$ has Saxon form, a half-height radius $R=3.5\times10^{-13}$ cm, and a surface thickness parameter $a=0.65\times10^{-13}$ cm. The absorption has Gaussian form about the radius R, and with a half-breadth 0.98×10^{-13} cm. *Minimum* values of the nucleon mean free path are marked on the curves.

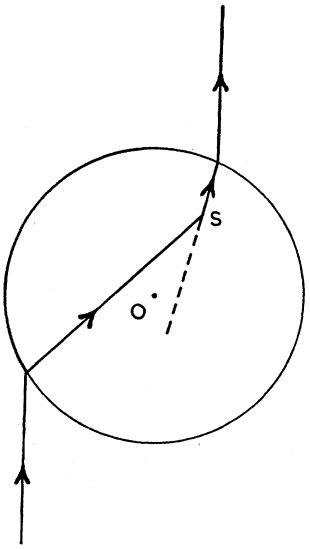

FIG. 7. Refraction can be responsible for forward emergent rays with nonzero angular momentum transfers.

the angular distributions may well peak in the forward direction even if a certain region of small angles of scatter is "forbidden" when refraction is ignored.

The above effect almost certainly occurs in the case of the reaction $C^{12}(p,p')C^{12*}(Q=-4.4$ Mev) which peaks in the forward direction for all bombarding energies in the range 11.8–96 Mev. This reaction has $l=2$, and it is known from angular correlation measurements[7] that it proceeds as a direct reaction. We therefore expect that for an appreciable portion of the energy range studied the angular distribution should have a "forbidden region" at small angles of scattering. The results of this particular reaction have been fitted by the detailed distorted waves calculations of Levinson and Banerjee.[3,4] They obtained the forward peaking.

In general one would expect that it is near the forward direction that refraction effects will be most important, since it is here that the greatest differences in the vectors $\mathbf{Q}$ and $\mathbf{Q}^*$ will arise. For angles of scattering much greater than $\theta=0°$ the vectors $\mathbf{Q}$ and $\mathbf{Q}^*$ are more nearly the same, so that refraction should not produce important changes. The shape and strength of a refracted forward peak are of course influenced by the range and depth of the optical potential. It is also of interest that many of the rays which enter into producing the refracted peak must pass through the deep interior of the target nucleus. Thus refracted forward peaks are to be expected only if the nucleus is fairly transparent to either the incident or outgoing particles.

(c) *Reflection*

Another interesting effect can occur when either the incident or outgoing particles see the nucleus as being sufficiently transparent. The possibility exists that a large peak in the angular distribution at or near the forward direction may be "imaged" in the backward direction.[8] To illustrate this point, suppose that it is the outgoing ray which suffers little absorption, and that the direct reaction is one which leads to a strong forward peaking. As each outgoing ray which can contribute to this forward peak strikes the nuclear surface, it splits into two rays. One is the transmitted ray which does contribute to the forward peak; the other is a reflected ray which then proceeds backwards and can penetrate through the back surface of the nucleus and contribute towards a backward peak.

This is depicted in Fig. 8, in which refraction and reflection at the nuclear surface is illustrated for a number of rays which would contribute to a forward peak. Twice-reflected rays are ignored. If the nuclear surface be assumed sharp, and if the ratio of the wave number of the outgoing *outside* the nucleus to that *inside* the nucleus be n, then the reflection coefficient $R(i)$ for the ray striking the surface at point A with angle of incidence i is given by

$$R(i)=\left[\frac{\cos i-(n^2-\sin^2 i)^{\frac{1}{2}}}{\cos i+(n^2-\sin^2 i)^{\frac{1}{2}}}\right]^2. \tag{17}$$

Thus the ray reflected at A has its intensity reduced by the factor $R(i)$. The probability $P(i)$ that the initial ray incident on the surface at A produces an outgoing backward ray at A' is equal to

$$P(i)=R(i)[1-R(i)]. \tag{18}$$

It may readily be ascertained that the angle between the two rays—the one emerging at A and the other at A'—is $(\pi-2i)$. If the first ray proceeds exactly forward, for example, the second ray emerges in the backward hemisphere and proceeds in a direction making an angle of $2i$ with the true backward direction.

The probability $P(i)$ is a fairly flat function of the

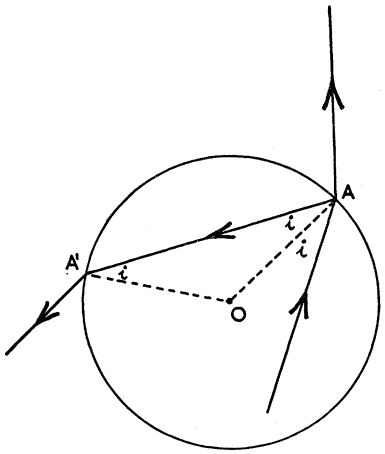

FIG. 8. Reflection tends to image forward peaks in the backward direction.

[7] R. Sherr and W. F. Hornyak, Bull. Am. Phys. Soc. Ser. II, **1**, 197 (1956); R. Sherr, Conference on Nuclear Structure, University of Pittsburgh, 1957 (unpublished).

[8] One of us (N.A.) wishes to acknowledge useful discussions with Professor K. J. Le Couteur and Professor D. C. Peaslee.

angle i, increasing slightly as i increases from zero, until finally it abruptly drops to zero at the critical angle of incidence i_c for which $\sin i_c = n$. All rays with angles of incidence greater than this will be totally reflected, and ultimately completely absorbed.

We thus see that those internal rays which contribute to exactly forward emergent rays will also produce outgoing rays proceeding within a backward cone making an angle of at most $2i_c$ with the backward direction.[9] Of course the same is also true of any outgoing direction; all those internal rays which produce rays emerging at an angle θ to the incident direction also produce rays emerging in a cone around the direction $\pi-\theta$, within at most an angle $2i_c$ from this direction.

As one example, if n be taken as $\frac{1}{2}$ (which is roughly the appropriate value if the outgoing particles be say 10-Mev protons or neutrons), then the angle i_c is 30°. The average value of the probability $P(i)$ in the range of 0° to 30° is about 0.13. Thus emerging forward rays are accompanied by less intense backward rays spread out in a cone making an angle of $<60°$ with the backward direction.

It is seen that this backward imaging effect is probably incapable of reproducing any detailed forward oscillations. On the other hand if an angular distribution has a very strong forward peak, followed by subsidiary oscillations of smaller magnitude, the tendency of the reflection effect is to produce a smeared out backward peak which resembles the main forward peak.

Such an effect could perhaps be the cause of the backward peaking observed by Rickey and Sherr[10] for the reaction $C^{12}(\alpha,p)N^{15}$, similarly for several (He^3,p) reactions,[11] and also observed for some low-energy (d,p) reactions.[12] When the outgoing proton energy is low, the protons have long mean free paths in nuclear matter and the parameter n of the above discussion can be sufficiently small that the backward imaging becomes particularly precise.

"Heavy-particle stripping"[13] competes with the reflection effect in producing backwards peaks, although it cannot produce any sharp peaks.

[9] The spread may actually be less than this, since the above discussion ignores the limitations imposed on the angles of incidence i by the fact that the rays must have originated at the active surface. If the permitted angles of incidence i are all less than i_c, the spread in the backward direction will be correspondingly smaller.

[10] M. Rickey and R. Sherr (to be published).

[11] E. H. Geer *et al.*, Bull. Am. Phys. Soc. Ser. II, **1**, 211 (1956); R. L. Johnston *et al.*, Bull. Am. Phys. Soc. Ser. II, **1**, 197 (1956).

[12] T. W. Bonner *et al.*, Phys. Rev. **101**, 209 (1956); J. B. Marion and G. Weber, Phys. Rev. **103**, 167 (1956); E. G. Illsley *et al.*, Phys. Rev. **107**, 538 (1957); C. E. Dickerman, Phys. Rev. **109**, 443 (1958).

[13] G. E. Owen and L. Madansky, Phys. Rev. **99**, 1608 (1957); **105**, 1766 (1957). T. Fulton and G. E. Owen, Phys. Rev. **108**, 789 (1957).

IV. DISCUSSION OF RELIABILITY OF SEMICLASSICAL APPROXIMATION

In this section we wish to inquire as to how the semiclassical method may be derived from a quantum-mechanical calculation. As yet we have been able to carry out such a derivation only for the case that the amplitude for the direct reaction is given by a quantum-mechanical matrix element of the form

$$T_{\text{Q.M.}} = C\int d\mathbf{r}\, F(r) e^{i(\cdot \mathbf{Q}\mathbf{r})} P_l(\cos\chi), \tag{19}$$

in which χ is the angle between the vectors $\mathbf{r}$ and $\mathbf{Q}$, and C is an angle-independent constant. This is, for example, the general form for a direct surface reaction amplitude derived under the approximations of Sec. II, in which case the function $F(r)$ would be zero for $r \lesssim R$ and otherwise to be identified with the function $y(r)$ of Eq. (8). Alternatively the form (19) could be the quantum-mechanical amplitude for a direct reaction in which little compound-nucleus absorption occurs and in which refraction of the incident and outgoing particles is ignored. In further discussion of this section we shall however specifically bear in mind the surface reaction.

Quite apart from the reasonableness of the approximations which lead to the form (19) in a quantum-mechanical calculation, we know that *under the same approximations* the semiclassical method leads to an amplitude of the following form:

$$T = C'\int_{\text{active cylinder}} d\mathbf{r}\, F(r) e^{i(\mathbf{Q}\cdot\mathbf{r})}. \tag{20}$$

Here C' is again an angle-independent constant. The question thus arises as to the nature of the approximation by which we can derive (20) from (19).

We note that the integral of (20) is a two-dimensional one because of the restriction of the region of integration to the surface of the active cylinder. The integral of (19) is on the other hand a three-dimensional one, and to transform it into the form (20) we should perform one of the integrals. We will therefore perform the integral over the angle χ of the radius vector $\mathbf{r}$ of (19). Of course this can be done exactly, but would lead immediately to the form given in Eq. (8); we are interested rather in approximately evaluating this integral in such a way as to obtain the form (20).

We thus consider the integral

$$\Theta_l(Qr) = \int_0^\pi \sin\chi d\chi\, e^{iQr\cos\chi} P_l(\cos\chi). \tag{21}$$

We note that if this were to receive most of its contribution from the vicinity of some angle χ_l, determined by

$$Qr\sin\chi_l = l, \tag{22}$$

we would have achieved the semiclassical result (20).

To investigate this point it is convenient to employ the following sum for the Legendre polynomial $P_l(\cos\chi)$:

$$P_l(\cos\chi)=\sum_{n(\text{odd or even})}^{l} A_n\cos n\chi$$

$$=\tfrac{1}{2}\sum_{n(\text{odd or even})=-l}^{l} A_n e^{in\chi}. \qquad (23)$$

The ratios of the coefficients A_n obey the recursion relationship

$$A_n=A_{n+2}\left[\frac{l(l+1)-(n+1)(n+2)}{l(l+1)-n(n+1)}\right].$$

In particular, for example,

$$A_{l-2}=A_l(l/2l-1). \qquad (24)$$

If we substitute (23) into (21) and change the variable of integration by letting $\zeta=\cos\chi$ we obtain

$$\Theta_l(Qr)=\tfrac{1}{2}\sum_{n=-l(\text{odd or even})}^{l} A_n\int_1^1 d\zeta \times\exp\{i(Qr\zeta+n\cos^{-1}\zeta)\}. \qquad (25)$$

We note also that in direct reactions one in general has $l<4$ so that at most two groups of terms in (25) are required, *viz*, $n=\pm l$ and $n=\pm(l-2)$.

We now evaluate each term of (25) by the method of stationary phase.[14] The point of stationary phase in the integral of (25), call it $\zeta=\zeta_n$, is given as the solution of the equation

$$Qr(1-\zeta_n^2)^{\frac{1}{2}}=|n|. \qquad (26)$$

The equivalent condition in terms of the corresponding angle $\chi_n=\cos^{-1}\zeta_n$ is

$$Qr\sin\chi_n=n. \qquad (27)$$

For n positive, χ_n lies between 0 and $\pi/2$, and for n negative, between 0 and $-\pi/2$; thus $\zeta_n=\cos\chi_n$ is always positive. It is to be noted that for the terms $n=\pm l$, the condition (26) or (27) corresponds precisely to the condition (22) for the active cylinder.

If we expand the exponent in the integral of (25) to second order in the vicinity of the point of stationary phase, we obtain

$$\Theta_l(Qr)=\sum_{n=(0,1)}^{l} A_n\exp\{i(Qr+n\cos^{-1}\zeta_n)\}\times\int_{-1}^{1} d\zeta\cos\{a_n^2(\zeta-\zeta_n)^2\}, \qquad (28)$$

where

$$a_n^2=\tfrac{1}{2}n\zeta_n(1-\zeta_n^2)^{-\frac{3}{2}}=\tfrac{1}{2}(Qr/n)^2(Q^2r^2-n^2)^{\frac{1}{2}}. \qquad (29)$$

[14] This method was used by R. Huby for a related calculation—see Appendix of paper by H. C. Newns, Proc. Phys. Soc. (London) **A66**, 477 (1953).

We now assume—and will investigate this later—that a_n^2 is appreciably greater than unity, so that contributions to the integral over ζ will come from the vicinity of $\zeta=\zeta_n$. We therefore extend the limits of integration from $-\infty$ to $+\infty$ and write

$$\int_{-1}^{1} d\zeta\cos\{a_n^2(\zeta-\zeta_n)^2\}\approx(\pi/2a_n^2)^{\frac{1}{2}}.$$

Thus, if we write out explicitly the terms of (28) corresponding to $n=l$ and $n=l-2$, the result is

$$\Theta_l(Qr)=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{A_l}{a_l}\left[\exp\{i(l\chi_l+Qr\cos\chi_l)\}+\frac{A_{l-2}}{A_l}\frac{a_l}{a_{l-2}}\times\exp\{i((l-2)\chi_{l-2}+Qr\cos\chi_{l-2})\}+\cdots\right]. \qquad (30)$$

On substitution of (30) into (19), we obtain the following form for the amplitude $T_{\text{Q.M.}}$:

$$T_{\text{Q.M.}}=C\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left(\frac{A_l}{a_l}\right)e^{il\chi_l}\left[\int_{\text{active cylinder }(\chi=\chi_l)} d\mathbf{r}\times F(r)e^{i(\mathbf{Q}\cdot\mathbf{r})}+\frac{A_{l-2}}{A_l}\frac{a_l}{a_{l-2}}\exp\{i(l-2)\chi_{l-2}-l\chi_l\}\times\int_{\text{active cylinder }(\chi=\chi_{l-2})} d\mathbf{r}\,F(r)e^{i(\mathbf{Q}\cdot\mathbf{r})}+\cdots\right]. \qquad (31)$$

The first term of (31) is precisely of the form of the semiclassical result (20), in which the integral is taken over the surface of the active cylinder defined by $|\mathbf{Q}\times\mathbf{r}|=l$. The second term however is a surface integral over the surface of the cylinder defined by $|\mathbf{Q}\times\mathbf{r}|=l-2$, and does not arise in the semiclassical treatment. Remembering that the magnitudes of the integrals involved are proportional to the circumferences of the appropriate cylinders of integration, we find that the actual ratio (say $\mathfrak{R}$) of the magnitudes of the second and first terms of (31) is given by the quantity

$$\mathfrak{R}=\frac{l-2}{l}\frac{A_{l-2}}{A_l}\frac{a_l}{a_{l-2}}=\left\{\frac{l-2}{2l-1}\right\}\frac{a_l}{a_{l-2}}=\frac{(l-2)^2}{l(2l-1)}\left[\frac{Q^2r^2-l^2}{Q^2r^2-(l-2)^2}\right]^{\frac{1}{2}}. \qquad (32)$$

Now once we are beyond the main first maximum in the angular distribution, i.e., $QR>l$, the fourth root factor of (32) becomes essentially unity; the ratio $\mathfrak{R}$ then

reduces to

$$\mathcal{R}\approx\frac{(l-2)^2}{l(2l-1)}. \tag{33}$$

For $l<2$, of course, the second term of (31) does not arise; for $l=2$ the ratio as given by (33) is zero (indicating that in reality the second term of (31) is very small compared with the first), and for $l=3$ the ratio is still only 1/15. Thus for the l values of importance in direct reactions the second term of (31) is small compared with the first, and its neglect is quite reasonable.[15] In effect, therefore, the result (31) is precisely that of the semiclassical method. The second term (if it exists) can at most affect somewhat the rise of the angular distribution (for $QR<l$) towards the first maximum.

Having "derived" the semiclassical result it remains to investigate the accuracy of the approximation used, and this simply amounts to a discussion of the accuracy of the method of stationary phase which was used to evaluate the integral of (25). This will be a good approximation so long as the values of $\zeta=\cos\chi$ which contribute in the integral of (28) are indeed well localized around the point ζ_n of stationary phase. The quantity which determined this spread is the quantity a_n^2 of Eq. (29) which is thus required to be substantially greater than unity. For the all-important term $n=l$ of (28), which leads to the semiclassical result, we thus have the requirement

$$a_l^2\gg 1.$$

For surface reactions, in which the contributions come from $r\approx R$, this condition is simply

$$\tfrac{1}{2}(QR/l)^2(Q^2R^2-l^2)^{\frac{1}{2}}\gg 1. \tag{34}$$

We see that right at the position of the first maximum, when $QR=l$, this condition is not fulfilled. Thus the semiclassical method is most inaccurate at and before the first maximum in the angular distribution. This is to be expected from the results of Sec. II, since the "forbidden" region in the angular distribution, predicted by the semiclassical method, differs radically from the gradual rise of the spherical Bessel function $j_l(QR)$. At the same time it is gratifying to recall that even when condition (34) is not fulfilled and quantitative accuracy not to be expected, the semiclassical result still reproduces the qualitative features of the angular distributions as discussed in Sec. II.

It is clear from (34) that, ideally, we require $QR\gg l$ in order that the spread in contributions around the active cylinder be really sharp; for a reasonable degree of sharpness (spread around the active cylinder small compared with the nuclear radius), we only require that QR be at least several times greater than l.

The results of this section thus lead to the conclusion that the semiclassical method can be quantitatively accurate for those parts of a differential cross section for which QR is at least several times greater than l. In addition, the comparisons made in Sec. II indicate that the semiclassical results are qualitatively reliable even when the above condition is not fulfilled.

ACKNOWLEDGMENTS

The authors are indebted to Dr. J. M. Blatt and Dr. M. R. Schafroth for valuable discussions on the subject of this paper.

APPENDIX

An alternative procedure for justifying the model makes use of wave-packet ideas.

Let it be *assumed* that the region of direct reaction is localized near the active cylinder, as the model requires. Then it is possible to ask what sort of wave function this assumption implies *for the internal motion of the product nucleus:* to what extent the final state really can have a well-defined angular momentum.

Evidently the final state which the model implies is a wave packet, having the dimensions of the localized region of interaction. Let us describe this wave packet in the cylindrical coordinates (ρ,α,z), and suppose it to be localized in Gaussian fashion about the active cylinder. Then if nuclear refraction and opacity effects are ignored, as in Sec. IV, the form of the wave packet is

$$\psi=\exp\{iQz-\beta^2z^2-\gamma^2(\rho-\rho_0)^2\}, \tag{A1}$$

where γ is a parameter to be determined, and $\rho_0=l/Q$. The parameter β is

$$\beta^2=Q^2/2(Q^2R^2-l^2), \tag{A2}$$

giving ψ a z dimension which equals the length of the active cylinder.

Straightforward calculation of the expectation value of the quantum-mechanical operator l^2 gives

$$\begin{aligned}\langle l^2\rangle&=Q^2\rho_0^2+\langle\Delta l^2\rangle\\&=Q^2\rho_0^2+\gamma^2/4\beta^2+\beta^2/4\gamma^2+\beta^2\rho_0^2+Q^2/4\gamma^2-\tfrac{1}{2}.\end{aligned} \tag{A3}$$

Then the radial breadth of the packet may be adjusted, by choice of γ, so as to minimize the departure of

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[15] For high l values one can employ the asymptotic (WKB) form for $P_l(\cos\chi)$, i.e., $P_l(\cos\chi)\propto(\sin\chi)^{-\frac{1}{2}}\cos\{(l+\frac{1}{2})\chi-\pi/4\}$. As long as χ is not very close to zero, the same treatment as above leads to the semiclassical result but with the active cylinder defined by $|\mathbf{Q}\times\mathbf{r}|=l+\frac{1}{2}$. The "sharpness" of the active cylinder is now determined by the quantity $\{a_{l+\frac{1}{2}}\}^2$—Eq. (33)—given as

$$\{a_{l+\frac{1}{2}}\}^2=\tfrac{1}{2}[Qr/(l+\tfrac{1}{2})]^2\{Q^2r^2-(l+\tfrac{1}{2})^2\}^{\frac{1}{2}}.$$

If we write this in terms of the classical quantities momentum and angular momentum, say $\Delta p=\hbar$, and $\mathcal{L}=\hbar l$, it becomes

$$\{a_{l+\frac{1}{2}}\}^2=\frac{1}{2\hbar}\left(\frac{r\Delta p}{\mathcal{L}+\hbar/2}\right)^2\{(r\Delta p)^2-(\mathcal{L}+\hbar/2)^2\}^{\frac{1}{2}},$$

and in the classical limit $\hbar\to 0$, keeping $\mathcal{L}$ and Δp constant, we see that the quantity $\{a_{l+\frac{1}{2}}\}^2$ goes to infinity as $1/\hbar$. Thus, in this classical limit, contributions to the direct reaction arise identically from the surface of the active cylinder.

$\langle l^2\rangle$ from the semiclassical value:

$$(\gamma^2)_{\min}=[\beta^2(\beta^2+Q^2)]^{\frac{1}{2}}, \quad \text{(A4)}$$

$$\langle\Delta l^2\rangle_{\min}=\tfrac{1}{2}[1+Q^2/\beta^2]^{\frac{1}{2}}+\beta^2\rho_0{}^2-\tfrac{1}{2}. \quad \text{(A5)}$$

It is found from these expressions that the wave packets are so large as to invalidate the model only if QR is very close to the classical cutoff, $QR=l$. The packets sharpen up very rapidly as (Q^2R^2/l^2) becomes very slightly greater than unity, reach their best breadths for $(Q^2R^2/l^2)\approx 2$, then slowly deteriorate as this parameter goes to much larger values. A typical value for $\langle\Delta l^2\rangle_{\min}$ is that for the sharpest packet,

$$\langle\Delta l^2\rangle_{\min}\approx\tfrac{1}{2}(1+2l^2)^{\frac{1}{2}}. \quad \text{(A6)}$$

In the same circumstance the radial thickness of the packet is found to be

$$(1/\gamma\sqrt{2})=(R/\sqrt{2})(1+2l^2)^{-\frac{1}{4}}. \quad \text{(A7)}$$

Evidently the model becomes better for the larger values of l. It already seems reasonable if $l=2$.

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Al^{26} Decay Scheme*

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The energies and intensities of the beta and gamma radiations from the long-lived ground level of Al^{26} were studied with scintillation spectrometers. The positron spectrum was obtained using a plastic scintillator with 4π geometry and was found to have a forbidden shape and an endpoint of 1.160 ± 0.008 Mev. The positrons are in coincidence with a gamma ray with an energy of 1.84 ± 0.01 Mev, which is presumably from the first excited level of Mg^{26}. There is also a weak gamma ray with an energy of 1.10 ± 0.05 Mev, in coincidence with the 184-Mev gamma ray, and with an intensity of 0.03 relative to the 1.84-Mev gamma ray. This would be from the second excited level of Mg^{26}, to which the Al^{26} decays weakly by electron capture. No other gamma rays are observed. It appears that a peak in the pulse-height spectrum at 700 kev is due to scattering effects rather than a gamma-ray photopeak.

THE beta and gamma radiations from the long-lived ground level of Al^{26} were studied using scintillation spectrometers. The measurements were made on a 0.01-microcurie source of Al^{26} recovered from several old magnesium cyclotron targets which had undergone a few thousand microampere hours of 15-Mev deuteron bombardment. A Kurie plot of the beta spectrum was obtained and two gamma rays were detected.

The beta spectrum was measured using a plastic scintillator designed to give 4π geometry. The scintillator is a rectangular block of plastic with a 2/100-in. slot cut in it. The source to be counted was deposited on a 0.25-mg/cm² sheet of rubber hydrochloride. The resolution (full width at half-height, divided by peak pulse height) of the Ba^{137} conversion electron peak (640 kev) was 13%. To check the linearity of the instrument the spectra of several known beta emitters (Na^{22}, P^{32}, Ca^{45}) were obtained and, after applying a resolution correction using the method of Owen and Primakoff,[1] Kurie plots were made. All were found to be linear down to about 150 kev.

Figure 1 shows a Kurie plot of the Al^{26} positron spectrum obtained with this instrument. The plot is linear when either the unique first forbidden or second forbidden correction factors[2] are added (Figs. 2 and 3). The endpoint is 1.160 ± 0.008 Mev. Coincidence experiments show that this spectrum is in coincidence with the annihilation radiation and the 1.84-Mev gamma ray.

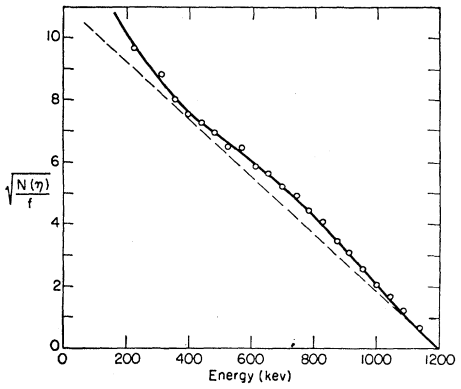

FIG. 1. Kurie plot of the Al^{26} positron spectrum, *not* corrected for forbiddenness.

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