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Meson Exchange Effects in Two-Nucleon States

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It is suggested that the most fruitful study of meson exchange effects requires a unified treatment of all processes in which they occur. A new representation of the eigenstates of the two-nucleon system, based on the Heitler-London method, is proposed as being particularly useful for this purpose. Heitler-London states have the property that they become equal to the exact eigenstates when the two nucleons are far apart. Matrix elements between Heitler-London states can be expressed in terms of the properties of isolated nucleons, by means of expansions in the number of exchanged mesons. The Heitler-London method affords a mathematically precise method of incorporating phenomenological isobar effects into a field-theoretical model. Most of the discussion of the details of the formalism is confined to the fixed-source model, but in the final part of the paper the application to a more general model is discussed briefly.

I. INTRODUCTION

THE problem of obtaining an understanding of nuclear interactions, starting from meson theory, has considerable general interest and has been the subject of intensive study. The particular problem which most earlier investigators have attempted to solve is that of deriving an equivalent potential, which could be used in a Schrödinger equation for calculating the binding and scattering of two nucleons. A general survey of many of the different methods which have been applied to this problem may be found in the review paper of Nishijima,[1] along with a discussion of certain limitations of this approach to the problem of nuclear forces. The general point is that the two-nucleon system can be represented by a two-body Schrödinger equation with an energy-independent, Hermitian potential only when the configuration of the meson field around the two nucleons is ignorable, as in the familiar adiabatic approximation. For a more generally valid treatment, it is necessary to pay more attention to the state of the meson field, and to take account explicitly of the extra degrees of freedom which are associated with it.

When the degrees of freedom of the meson field are not eliminated completely from the Schrödinger equation for the two-nucleon system, it is natural to think of treating simultaneously such problems as nucleon-nucleon scattering and the production of mesons. There is in fact great practical advantage in studying, in a systematic way, all properties of the two-nucleon system which involve the exchange of π mesons. Since our understanding of the interaction between mesons and single nucleons is still somewhat fragmentary, and since the interaction between nucleons at very short distances will involve in addition phenomena which are as yet unknown, any discussion which can be made at the present time of the effects which exchanges of π mesons have in any one problem will necessarily involve considerable guessing. It is to be expected, however, that in a comparison of several different processes, some of the uncertainties may be eliminated, and eventually understood.

A complete discussion of meson exchange effects in two-nucleon states must encompass the following problems: (1) the long-range part of the nuclear potential[1–5]; (2) the static electromagnetic properties of the deuteron[6,7]; (3) the photodisintegration of the deuteron, especially at high energies[8–11]; (4) the production of mesons in nucleon-nucleon collisions[12–16]; (5) the photoproduction of mesons from deuterium[17–19]; (6) meson-deuteron scattering.[20–22] While problem (1) is the most studied, much work has also been done on problems (2)–(6), leading to much valuable information, but since a unified method has not been used, it is difficult to correlate the studies of the various processes. The purpose of this paper is to suggest a method which may be used in a comprehensive study of the meson cloud around two interacting nucleons.

A useful method for treating any of the problems listed above should satisfy two criteria. The first is that it should be well suited for treating *all* of the problems, and for exhibiting their common features. That is, a particular problem, such as the calculation of the nuclear

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potential, should be considered not as an end in itself but as a part of the general problem of understanding meson exchange effects. The second criterion is that only physical particles should enter directly into the formalism, so that only the experimental mass, coupling constant, and other properties of a physical nucleon would appear in equations. This would make it possible to relate the structure of two-nucleon states to properties of single nucleons which had a direct physical significance.

Most of the previous investigators of the problems set forth above have used some type of perturbation method, such as the Tamm-Dancoff method. Such methods satisfy the first criterion, but only in a clumsy way. It is important to keep in mind the distinction between the *state vector* Ψ of the two-nucleon system and the "wave function" $\psi(x)$, which in the Tamm-Dancoff method gives merely the probability of finding two *bare* particles at given positions; it is only a very limited class of problems, such as the calculation of the scattering phase shifts, in which knowledge of $\psi(x)$ alone suffices. In all other physical problems, one must first calculate the rest of Ψ. The renormalization of the mass and coupling constant, and the elimination of other "bare" quantities, must all be done explicitly, and by perturbation methods. The defects of the Tamm-Dancoff method are all directly related to the fact that the second criterion is not satisfied.

The question of defining and constructing nuclear potentials has also been approached through the use of various general properties of that portion of the S matrix which describes the elastic scattering of one nucleon by another.[2–5] In these papers, direct use is made of the properties of physical particles, and hence the second criterion is satisfied completely. However, the first criterion is not satisfied; the "nuclear potentials" and the associated "wave functions" $\psi(x)$ are merely mathematical artifices, introduced as auxiliary quantities to aid in the discussion of the S matrix, and have no independent physical meaning. The state vector Ψ and this $\psi(x)$ are unrelated except that $\psi(x)$ gives the right phase shifts, and it is improper to use such a $\psi(x)$ to calculate, e.g., the charge distribution of the deuteron. A treatment of the two-nucleon problem, in the general sense, would require an analysis of the remainder of the S matrix, which refers to more than two incident or emergent particles.

One general way of treating the above-mentioned problems, in which both criteria would be automatically satisfied, would be to construct the actual state vectors of the system, in a representation in which the properties of physical particles were used explicitly. It would be desirable to imitate much of the recent work on the one-nucleon system, in which use is made of the one-nucleon eigenstates and various identities which these satisfy.[23–25]

[23] F. Low, Phys. Rev. **97**, 1392 (1955).
[24] G. C. Wick, Revs. Modern Phys. **27**, 339 (1955).
[25] G. F. Chew and F. Low, Phys. Rev. **101**, 1570, 1579 (1956).

In the body of this paper, a method which has these features will be described. It is based on a representation which is closely related to the method Heitler and London used to discuss the hydrogen molecule.[26] It is remarkable that this famous method has not been used hitherto in field-theoretical problems, especially since it appears to be even better suited to nuclear than to molecular physics.

Most of this paper will be devoted to a discussion of the fixed-source meson theory. The simplicity of this model will enable us to present the basic formalism with as little obscurity as possible, and its study will acquaint us with some of the mathematical features of the real two-nucleon problem. One should not expect to obtain from the fixed-source model any quantitatively reliable information about real two-nucleon states; nevertheless, one may expect to be able to obtain interesting qualitative information about nuclear interactions.

II. HEITLER-LONDON REPRESENTATION

The usual fixed-source meson theory has two simplifying features: the nucleons do not move, and the mesons do not interact with each other, only with the fixed nucleons. Thus the a_k^* and a_k, the creation and annihilation operators for *bare* mesons, also create and destroy the physical particles in this theory. For convenience in later calculations, we introduce the notation $\varphi_k = a_k + a_{-k}^*$, $\pi_k = -\frac{1}{2}i(a_k - a_{-k}^*)$.

Creation and annihilation operators for the bare nucleons will be denoted by α_x^* and α_x, x being the position of the nucleon. The spin and charge indices are suppressed from the notation.

The vacuum state will be denoted by $\rangle$. Eigenstates with one physical nucleon and zero, one, two, etc. mesons will be denoted by $|x\rangle$, $|x,K\rangle$, $|x,KL\rangle$, etc. We shall not generally specify whether the states have ingoing or outgoing scattered waves; it will generally be sufficient to assume that all the states are defined in the same way. Similarly, we shall denote two-nucleon eigenstates by $|xy\rangle$, $|xy,K\rangle$, etc.; sometimes we shall also use the symbols Ψ_{xy} and $\Psi_{xy,K}$, etc., for the same states.

The Hamiltonian is

$$H = \sum_p \omega_p(\pi_p^*\pi_p + \tfrac{1}{4}\varphi_p^*\varphi_p) + H'. \tag{1}$$

(Natural units are used here and throughout.) We shall assume in the formal development of this section that H' is some arbitrary function of the φ_k (and is bilinear in the nucleon operators). It is assumed that appropriate counter terms (independent of the φ_k and π_k) are included in H', so that $H\rangle = 0$ and $H|x\rangle = 0$.

In the development of Wick[24] and Chew and Low,[25] use is made of an expression for the one-meson scattering state which has the form

$$|x,K\rangle = a_K^*|x\rangle + \chi_{xK}^{\pm}. \tag{2}$$

The first term contains the incident plane wave; the

[26] W. Heitler and F. London, Z. Physik **44**, 455 (1927).

second, the scattered wave. The second term can also be formally expressed in terms of the first, as is shown in references 24 and 25. It should be noted that the simplicity of Eq. (2) depends on the fact that the meson operator a_K^* creates a *physical* meson. Thus it would not be reasonable for us to imitate Eq. (2) directly, and write

$$|xy\rangle=\alpha_y^*|x\rangle+\chi_{xy}. \quad (2')$$

This would treat the two nucleons unsymmetrically, and destroy whatever advantages result from including directly the meson cloud of the physical nucleon at x. A more suitable approach would be to define an operator $\mathfrak{F}_x^*$, which had the property that

$$\mathfrak{F}_x^*\rangle\equiv|x\rangle; \quad (3)$$

then we could use the state

$$\Phi_{xy}=\mathfrak{F}_x^*\mathfrak{F}_y^*\rangle, \quad (4)$$

as a first approximation to the eigenstate Ψ_{xy}.

It is obviously possible to find infinitely many operators $\mathfrak{F}_x^*$ with the property of Eq. (3).[27,28] One way of suitably restricting the form of $\mathfrak{F}_x^*$, so that Eq. (3) defines a unique operator, is by requiring also that $\mathfrak{F}_x^*$ depend only on some specified combination of the operators φ_k and π_k ($\mathfrak{F}_x^*$ must also contain as a factor the operator α_x^*, and depend on the spin and isotopic spin matrices). That is, if we require that $\mathfrak{F}_x^*=\mathfrak{F}_x^*(a_k^*)$, or $\mathfrak{F}_x^*(\varphi_k)$, we obtain in either case a unique operator. In the first case, the coefficients in a power series expansion of $\mathfrak{F}_x^*$ in the a_k^* are the same as the amplitudes in the Fock-space representation; in the second case, the coefficients in an expansion in the φ_k can also be related to the Fock amplitudes, or recursion formulas may be derived directly with the aid of the Hamiltonian. There appears to be no simple way to decide which of the many ways of defining $\mathfrak{F}_x^*$ is the best. We shall develop the theory using both methods mentioned above. If the operators are required to be functions only of the φ_k, formulas which are comparatively simple and elegant in appearance result, and it is easier to examine certain general convergence properties, but it seems easier to apply the method to the linear-coupling model when the operators are chosen to be functions of the a_k^*. We shall distinguish operators which are functions of the φ_k by a prime: $\mathfrak{F}_x'^*(\varphi_k)$, and shall similarily denote the states formed by a product of two such operators: $\Phi_{xy}'=\mathfrak{F}_x'^*\mathfrak{F}_y'^*\rangle$. Operators and states not distinguished by a prime may be assumed general in the next few paragraphs, but in the latter part of this paper will refer to functions of the a_k^*. There is, of course, no reason why the operator $\mathfrak{F}_x^*$ should not depend on some canonical variables other than those we have chosen; we shall limit ourselves to a discussion of these two cases for simplicity.

We define $\mathfrak{F}_x$ to be the Hermitian conjugate of $\mathfrak{F}_x^*$; note that $\langle\mathfrak{F}_x\mathfrak{F}_x^*\rangle=1$. Since $\mathfrak{F}_x'^*$ is independent of the π_k, we find that if x and y are distinct points, $\mathfrak{F}_x'^*$ will anticommute with both $\mathfrak{F}_y'^*$ and $\mathfrak{F}_y'$, because α_x^* is assumed to anticommute with α_y^* and α_y. In general, the operator $\mathfrak{F}_x^*$ will anticommute with $\mathfrak{F}_y^*$, but not with $\mathfrak{F}_y$.

It is evident that when the two nucleons are widely separated ($r=|x-y|\rightarrow\infty$) Eq. (4) gives the exact ground state of the two-nucleon system ($\Phi_{xy}\rightarrow\Psi_{xy}$). This is a general characteristic of the Heitler-London method. Some insight into the nature of the approximation that is made when the exact state Ψ_{xy} is replaced by $\Phi_{xy}=\mathfrak{F}_x^*\mathfrak{F}_y^*\rangle$ can be obtained from Appendix A, which is devoted to the theory with neutral, scalar mesons.

In general, it will not be sufficient to consider only the state Φ_{xy}. In order to define additional Heitler-London states, we define additional operators, which also are to depend only on the φ_k or a_k^*, as the case may be.

$$\begin{aligned}\mathfrak{F}_{x,K}^*\rangle&=|x,K\rangle,\\ \mathfrak{F}_{x,KL}^*\rangle&=|x,KL\rangle,\text{ etc.}\end{aligned} \quad (5)$$

We remark that having chosen to define a unique $\mathfrak{F}_x^*$ by requiring it to be a function of particular variables, it is obviously desirable to use those same variables throughout. Then we can define as follows another state $\Phi_{xy,K}$, which we shall interpret as a Heitler-London state in which one of the nucleons is in its ground state and the other is in an excited state (scattering a meson):

$$\Phi_{xy,K}'=\mathfrak{F}_{x,K}'^*\mathfrak{F}_y'^*\rangle+\mathfrak{F}_x'^*\mathfrak{F}_{y,K}'^*\rangle-\varphi_K^*\mathfrak{F}_x'^*\mathfrak{F}_y'^*\rangle, \quad (6)$$

or

$$\Phi_{xy,K}=\mathfrak{F}_{x,K}^*\mathfrak{F}_y^*\rangle+\mathfrak{F}_x^*\mathfrak{F}_{y,K}^*\rangle-a_K^*\mathfrak{F}_x^*\mathfrak{F}_y^*\rangle.$$

The third term is required to make the total amplitude of the meson plane wave equal to unity. The states defined by Eq. (6) are asymptotically equal to the eigenstate $\Psi_{xy,K}$ when $r\rightarrow\infty$; $\Phi_{xy,K}$ gives automatically the impulse approximation to the scattering from the two nucleons. However, because of the multiple scattering effect, the difference $(\Psi_{xy,K}-\Phi_{xy,K})$ is of order r^{-1}, while $(\Psi_{xy}-\Phi_{xy})$ is of order $\exp(-r)$.

Before proceeding with the construction of the most general Heitler-London state $\Phi_{xy,K_1\cdots K_n}$, it will be useful to examine more closely the ones we already have. It will be observed that there are several differences from the usual Heitler-London method. First, we have defined the Heitler-London states by multiplying together two *operators*, rather than two wave functions. Secondly, the states are not uniquely defined, and require an additional assumption (that the operators depend only on certain canonical variables) to make them so. Thirdly, the definition of the state $\Phi_{xy,K}$ is a

[27] O. W. Greenberg and S. S. Schweber (to be published); operators with this property, but defined differently than in the present paper, are extensively discussed.

[28] Iu. V. Novozhilov, J. Exptl. Theoret. Phys. U.S.S.R. **32**, 1262 (1957) [translation: Soviet Phys. JETP **5**, 1030 (1958)]. Operators similar to those constructed in this paper are defined, and used as in Eq. (4). The author is grateful to H. Ekstein for calling attention to this paper. See also Iu. V. Novozhilov, J. Exptl. Theoret. Phys. U.S.S.R. **33**, 901 (1957) [translation: Soviet Phys. JETP **6**, 692 (1958)].

little more complicated than one might have anticipated. The origin of these differences, which are really superficial, can be easily understood.

If we were to use the second-quantized theory to discuss the hydrogen molecule, we would also multiply together two operators, each of which created an electron in a state centered around one of the protons. The resulting state would be automatically antisymmetric in the two electrons. Thus the representation of Heitler-London states by the product of operators is the natural concomitant of a field theory, and has the advantage that the statistics of the particles need not be considered explicitly.

The uniqueness of the Heitler-London approximation to the ground state of the hydrogen molecule is a result of the simplicity of that molecule. If it were important to a discussion of molecular binding to include the states with a positron and an extra electron, the incorporation of such states into the Heitler-London approximation would also be somewhat arbitrary. That is, in a situation in which no single set of Fock states (with a definite number of particles) forms the principal part of the eigenstates, there is in general no unique way to combine two states to form a Heitler-London state.

The extra complications of the definition (7) of $\Phi_{xy,K}$ (and of the definitions given in Appendix B for the remaining states) result from the fact that we consider the *continuum* states associated with each nucleon. The incident plane wave necessarily strikes both nucleons, so we must use an expression which allows for scattering by either of them. In molecular theory it is not usual to include the continuum states, so such complications are not met.

It is of course not sufficient to *define* a representation —we must also show that it can be used in a convenient way. Supposing that we define a complete set of states $\Phi_{xy,K_1\cdots K_n}$, we can represent the ground state of the two-nucleon system by the expansion

$$\Psi_{xy}=\chi_0\Phi_{xy}+\sum_K \chi(K)\Phi_{xy,K}+\sum_{KL}\chi(K,L)\Phi_{xy,KL}+\cdots. \quad (7)$$

An equation for the χ's can then be obtained from the variational principle

$$\delta(\Psi_{xy},[H-E_{xy}]\Psi_{xy})=0, \quad (8)$$

by varying the χ's. Evidently we shall need to know the matrix elements

$$(\Phi_{xy},H\Phi_{xy}),\ (\Phi_{xy},H\Phi_{xy,K}),\ \text{etc.},$$

as well as the matrix elements

$$(\Phi_{xy},\Phi_{xy}),\ (\Phi_{xy},\Phi_{xy,K}),\ \text{etc.}$$

(The $\Phi_{xy,K_1\cdots K_n}$ will not, in general, be orthonormal when r is finite, even though the original states $|x,K_1\cdots K_n\rangle$ are orthonormal.) The necessary matrix elements can all be expressed very simply by expansions in the number of mesons exchanged between the two nucleons. Of course, if explicit expressions for the $\mathfrak{F}$'s were known, the matrix elements could also be evaluated directly.

Let us consider first the normalization of the state Φ_{xy}'; we define

$$(\Phi_{xy}',\Phi_{xy}')=\langle\mathfrak{F}_y'\mathfrak{F}_x'\mathfrak{F}_x'^*\mathfrak{F}_y'^*\rangle=1+A_{xy}'. \quad (9)$$

Commuting the $\mathfrak{F}'$'s, and then using a closure expansion, we find

$$\langle\mathfrak{F}_y'\mathfrak{F}_x'\mathfrak{F}_x'^*\mathfrak{F}_y'^*\rangle=\langle\mathfrak{F}_x'\mathfrak{F}_x'^*\mathfrak{F}_y'\mathfrak{F}_y'^*\rangle=\sum_{n=0}^{\infty}\sum_{k_1\cdots k_n}\frac{1}{n!}\langle\mathfrak{F}_x'\mathfrak{F}_x'^*a_1^*\cdots a_n^*\rangle\times\langle a_1\cdots a_n\mathfrak{F}_y'\mathfrak{F}_y'^*\rangle. \quad (10)$$

Since $\mathfrak{F}_x'^*$ and $\mathfrak{F}_y'$ are independent of the π_k, we have, e.g.,

$$\mathfrak{F}_x'^*a_k^*=a_k^*\mathfrak{F}_x'^*+a_{-k}\mathfrak{F}_x'^*-\mathfrak{F}_x'^*a_{-k};$$

by repeated use of this rather trivial identity, we obtain

$$1+A_{xy}'=\sum_{n=0}^{\infty}\sum_{k_1\cdots k_n}(n!)^{-1}\langle x|:\varphi_1^*\cdots\varphi_n^*:|x\rangle\times\langle y|:\varphi_1\cdots\varphi_n:|y\rangle. \quad (11)$$

The colons denote Wick's normal product.[29] Note that in the proof of (11) we used only the facts that the $\mathfrak{F}'$ operators are independent of the canonical variables π_k, and that $\mathfrak{F}_x'^*\rangle=|x\rangle$, and did not require knowledge of an explicit expression for $\mathfrak{F}_x'^*$; hence we can use the same kind of expansion for the other matrix elements.

The zeroth term of (11) is $\langle x|x\rangle\langle y|y\rangle\equiv1$. The next term involves the matrix element $\langle x|\varphi_k^*|x\rangle$, which can be evaluated directly in terms of the *renormalized* coupling constant. Similarly, the second term involves the matrix element $\langle x|:\varphi_k^*\varphi_{k'}^*:|x\rangle$, which can be expressed in terms of the S-matrix element for elastic scattering of a meson (*off* the energy shell—see reference 2), and the higher terms can be related to the matrix elements for inelastic meson scattering. Hence, each term of (11) can be expressed—at least in principle—in terms of the physical properties of an isolated nucleon. (See also Appendix A.)

Before discussing the matrix elements of the Hamiltonian, we first derive some identities. Let $\mathfrak{F}_A'^*$ and $\mathfrak{F}_B'^*$ be operators that create states with energy E_A and E_B. We first note that

$$[H,\mathfrak{F}_A'^*]=\sum_p\omega_p[\pi_p,[\pi_p^*,\mathfrak{F}_A'^*]]+\sum_p 2\omega_p[\pi_p^*,\mathfrak{F}_A'^*]\pi_p+[H',\mathfrak{F}_A'^*]. \quad (12)$$

Since H' and $\mathfrak{F}_A'^*$ both depend only on the nucleon operators and on the φ_k, we have $[H',\mathfrak{F}_A'^*]\mathfrak{F}_B'^*$

[29] G. C. Wick, Phys. Rev. 80, 268 (1950).

$=-\mathfrak{F}_B'^*[H',\mathfrak{F}_A'^*]$. Hence

$$H\mathfrak{F}_A'^*\mathfrak{F}_B'^*\rangle=\mathfrak{F}_A'^*H\mathfrak{F}_B'^*\rangle-\mathfrak{F}_B'^*[H,\mathfrak{F}_A'^*]\rangle +\sum_p 2\omega_p[\pi_p^*,\mathfrak{F}_A'^*][\pi_p,\mathfrak{F}_B'^*]\rangle =(E_A+E_B)\mathfrak{F}_A'^*\mathfrak{F}_B'^*\rangle +\sum_p 2\omega_p[\pi_p^*,\mathfrak{F}_A'^*][\pi_p,\mathfrak{F}_B'^*]\rangle. \quad (13)$$

We may also write

$$H\mathfrak{F}_A'^*\mathfrak{F}_B'^*\rangle=(E_A+E_B)\mathfrak{F}_A'^*\mathfrak{F}_B'^*\rangle -\sum_p(2\omega_p)[a_{-p},\mathfrak{F}_A'^*][a_p,\mathfrak{F}_B'^*]\rangle. \quad (14)$$

For our second identity, we let G denote any function of the φ_k. Using the fact that $\pi_k\rangle=\frac{1}{2}i\varphi_k\rangle$, we obtain

$$H\varphi_K^*G\rangle=\varphi_K^*(H+\omega_K)G\rangle-2i\omega_K[\pi_K^*,G]\rangle. \quad (15)$$

Since $[a_p,\mathfrak{F}_A'^*]$ depends only on the variables φ_k, we can again use our previous expansion theorem, Eq. (11), to evaluate matrix elements of the Hamiltonian. Thus we find for the "potential"[30]

$V_{xy}'\equiv(\Phi_{xy}',H\Phi_{xy}')$:

$$V_{xy}'=-\sum_p(2\omega_p)\sum_{n=0}^{\infty}\sum_{k_1\cdots k_n}\langle x|:\varphi_1^*\cdots\varphi_n^*:a_{-p}|x\rangle \times(n!)^{-1}\langle y|:\varphi_1\cdots\varphi_n:a_p|y\rangle. \quad (16)$$

The first term of (16) is:

$$V_{xy}'^{(1)}=-\sum_p 2\omega_p\langle x|a_{-p}|x\rangle\langle y|a_p|y\rangle. \quad (17)$$

It is easy to show that this gives the usual second-order potential, with the *renormalized* coupling constant.

The two examples above will suffice for an illustration of how matrix elements may be expressed in terms of mesons exchanged between physical nucleons. However, the matrix elements which involve the other states have certain complicating features, which we shall now describe. Let us first consider $A_{xy}'(;K)\equiv(\Phi_{xy}',\Phi_{xy,K}')$. Using the same proof as for Eq. (11), we find

$$\begin{aligned}A_{xy}'(;K)=&\sum_{n=0}^{\infty}\sum_{k_1\cdots k_n}(n!)^{-1}\\&\times\{\langle x|:\varphi_1^*\cdots\varphi_n^*:|x,K\rangle\langle y|:\varphi_1\cdots\varphi_n:|y\rangle\\&+\langle x|:\varphi_1^*\cdots\varphi_n^*:|x\rangle\langle y|:\varphi_1\cdots\varphi_n:|y,K\rangle\\&-\langle x|:\varphi_1^*\cdots\varphi_n^*:\varphi_K^*|x\rangle\langle y|:\varphi_1\cdots\varphi_n:|y\rangle\}\\=&\sum_{n=0}^{\infty}\sum_{k_1\cdots k_n}(n!)^{-1}\\&\times\{[\langle x|:\varphi_1^*\cdots\varphi_n^*:|x,K\rangle\\&-\langle x|:\varphi_1^*\cdots\varphi_n^*\varphi_K^*:|x\rangle]\\&\times\langle y|:\varphi_1\cdots\varphi_n:|y\rangle+\langle x|:\varphi_1^*\cdots\varphi_n^*:|x\rangle\\&\times[\langle y:|:\varphi_1\cdots\varphi_n:|y,K\rangle\\&-\langle y|:\varphi_1\cdots\varphi_n\varphi_K^*:|y\rangle]\}. \quad (18)\end{aligned}$$

[30] If the interaction Lagrangian depends linearly on the $\dot{\varphi}$, Eq. (16) holds with the destruction operators replaced by the positive-frequency part of φ_p.

In some of the matrix elements of (18), certain delta-function singularities arise; it can be seen from the relation[24]

$$a_k|x,K\rangle=\delta_{k,K}|x\rangle-(H-\omega_K+\omega_k\pm i\eta)^{-1}V_k^*|x,K\rangle,$$

that in the matrix element $\langle x|:\varphi_1\cdots\varphi_n:|x,K\rangle$ terms like $\delta_{K,k_i}\langle x|:\varphi_1\cdots\varphi_{i-1}\varphi_{i+1}\cdots\varphi_n:|x\rangle$ (where $i=1\cdots n$) appear, and furthermore, after these singular terms are subtracted from the matrix element $\langle x|:\varphi_1\cdots\varphi_n:|x,K\rangle$, the remainder is free of terms which contain the delta function of two momenta. We shall call the nonsingular remainder the *irreducible part* of the matrix element, and denote it by the symbol $\langle x|:\varphi_1\cdots\varphi_n:|x,K\rangle_I$. Since the difference between $\langle x|\varphi_1\cdots\varphi_n|x,K\rangle$ and $\langle x|:\varphi_1\cdots\varphi_n:|x,K\rangle$ also lies in terms proportional to delta functions like $\delta_{k_i,-k_j}$, which are contained in the first matrix element but not the second, by interpreting the expression "irreducible part" and the subscript I to mean the discarding of *all* such singular terms, we can omit the colons from the notation. Separating the matrix elements in (18) into singular and irreducible parts, we find that the contribution of the singular parts is exactly cancelled by the contribution of the correction term $-\varphi_K^*\mathfrak{F}_x'^*\mathfrak{F}_y'^*\rangle$; hence we obtain

$$\begin{aligned}A_{xy}'(;K)=&\sum_{n=0}^{\infty}\sum_{k_1\cdots k_n}(n!)^{-1}\\&\times\{\langle x|\varphi_1^*\cdots\varphi_n^*|x,K\rangle_I\langle y|\varphi_1\cdots\varphi_n|y\rangle_I\\&+\langle x|\varphi_1^*\cdots\varphi_n^*|x\rangle_I\langle y|\varphi_1\cdots\varphi_n|y,K\rangle_I\}. \quad (19)\end{aligned}$$

The zeroth term of (19) is zero, since $\langle x|x,K\rangle=0$. The significance of the irreducible matrix elements is that by a suitable definition of $\Phi_{xy,K}'$ the contribution of the incident plane wave to the overlap of the two states has been made to vanish.

The matrix element $(\Phi_{xy,K'}',\Phi_{xy,K}')$ obviously contains a term proportional to $\delta_{K,K'}$, and it is clear that the coefficient of the delta function can only be $1+A_{xy}'$. Let us define the *irreducible part* of $(\Phi_{xy,K'}',\Phi_{xy,K}')$ to be $A_{xy}'(K';K)$:

$$(\Phi_{xy,K'}',\Phi_{xy,K}')=\delta_{K,K'}(A_{xy}'+1)+A_{xy}'(K';K). \quad (20)$$

It is a matter of straightforward algebra to verify the form of Eq. (20) and show that

$$\begin{aligned}A_{xy}'(K',K)=&\sum_{n=1}^{\infty}\sum_{k_1\cdots k_n}(n!)^{-1}\\&\times\{\langle x,K'|\varphi_1^*\cdots\varphi_n^*|x,K\rangle_I\langle y|\varphi_1\cdots\varphi_n|y\rangle_I\\&+\langle x,K'|\varphi_1^*\cdots\varphi_n^*|x\rangle_I\langle y|\varphi_1\cdots\varphi_n|y,K\rangle_I\\&+\langle x|\varphi_1^*\cdots\varphi_n^*|x,K\rangle_I\langle y,K'|\varphi_1\cdots\varphi_n|y\rangle_I\\&+\langle x|\varphi_1^*\cdots\varphi_n^*|x\rangle_I\\&\times\langle y,K'|\varphi_1\cdots\varphi_n|y,K\rangle_I\}. \quad (21)\end{aligned}$$

Similar calculations are used for the matrix elements of the Hamiltonian. Let $V_{xy}'(;K)=(\Phi_{xy}',H\Phi_{xy,K}')$; we obtain

$$V_{xy}'(;K)=-\sum_{p}\sum_{n=0}^{\infty}\sum_{k_1\cdots k_n}2\omega_p(n!)^{-1}\times\{\langle x|a_p^*\varphi_1^*\cdots\varphi_n^*|x,K\rangle_I\times\langle y|a_{-p}^*\varphi_1\cdots\varphi_n|y\rangle_I+\langle x|a_p^*\varphi_1^*\cdots\varphi_n^*|x\rangle_I\times\langle y|a_{-p}^*\varphi_1\cdots\varphi_n|y,K\rangle_I\}. \quad (22)$$

In discussing the matrix element $(\Phi_{xy,K'}',H\Phi_{xy,K}')$ we proceed as follows. Let

$$H\Phi_{xy,K}'=\omega_K\Phi_{xy,K}'+X_{xy,K}', \quad (23)$$

where $X_{xy,K}'$ is calculated by using Eqs. (14) and (15). It is only the contribution of $X_{xy,K}'$ to the matrix element that we can consider as the "potential." In order to make Hermitian the irreducible potentials which are defined, we symmetrize explicitly:

$$(\Phi_{xy,K'}',H\Phi_{xy,K}')=\tfrac{1}{2}(\omega_K+\omega_{K'})(\Phi_{xy,K'}',\Phi_{xy,K}')+\tfrac{1}{2}(X_{xy,K'}',\Phi_{xy,K}')+\tfrac{1}{2}(\Phi_{xy,K'}',X_{xy,K}'). \quad (24)$$

As in Eq. (20), we must separate out the irreducible potential:

$$\tfrac{1}{2}(X_{xy,K'}',\Phi_{xy,K}')+\tfrac{1}{2}(\Phi_{xy,K'}',X_{xy,K}')=\delta_{K,K'}V_{xy}'+V_{xy}'(K';K). \quad (25)$$

In Appendix B, the general state $\Phi_{xy,K_1\cdots K_n}'$ is defined, and the derivation of general expansions similar to those given above is sketched.

When the operators are chosen to be functions of the a_k^*, slight modifications in the previous derivations must be made. We use an identity for the anticommutator of $\mathfrak{F}_y(a_k)$ and $\mathfrak{F}_x^*(a_k^*)$,

$$\{\mathfrak{F}_y,\mathfrak{F}_x^*\}+\sum_{N=1}^{\infty}\frac{1}{N!}\sum_{K_1\cdots K_n}[a_1,[\cdots[a_N,\mathfrak{F}_x^*]\cdots]]\times[[\cdots[\mathfrak{F}_y,a_N^*]\cdots],a_1^*]=0, \quad (26)$$

and then use a closure expansion to prove that

$$(\Phi_{xy},\Phi_{xy})=1+A_{xy}=\langle\mathfrak{F}_y\mathfrak{F}_x\mathfrak{F}_x^*\mathfrak{F}_y^*\rangle=\sum_{N=0}^{\infty}\sum_{\nu=0}^{N}\frac{1}{\nu!(N-\nu)!}\times\sum_{k_1\cdots k_N}\langle x|a_1^*\cdots a_\nu^*a_{\nu+1}\cdots a_N|x\rangle\times\langle y|a_N^*\cdots a_{\nu+1}^*a_\nu\cdots a_1|y\rangle. \quad (11')$$

The matrix elements of the Hamiltonian involve the commutators

$$V_{|k}=[H',a_k^*],\quad V_{k|}=[a_k,H'], \quad (27)$$

if the interaction is linear; more generally, the commutators

$$V_{1\cdots r|1\cdots s}=\frac{1}{s}[V_{1\cdots r|1\cdots s-1},a_s^*]=\frac{1}{r}[a_r,V_{1\cdots r-1|1\cdots s}] \quad (27')$$

are also required. An identity analogous to Eq. (14) is not hard to prove:

$$H\mathfrak{F}_A^*\mathfrak{F}_B^*\rangle=(E_A+E_B)\mathfrak{F}_A^*\mathfrak{F}_B^*\rangle+\sum_{N=1}^{\infty}\sum_{k_1\cdots k_N}[a_1,[\cdots[a_N,\mathfrak{F}_A^*]]\cdots]V_{|1\cdots N}\mathfrak{F}_B^*\rangle-\sum_{N=1}^{\infty}\sum_{k_1\cdots k_N}\times[a_1,[\cdots[a_N,\mathfrak{F}_B^*]]\cdots]V_{|1\cdots N}\mathfrak{F}_A^*\rangle. \quad (14')$$

Using this identity, and proceeding as for Eq. (11′), one finds

$$(\Phi_{xy},H\Phi_{xy})=1+V_{xy}=\sum_{M=1}^{\infty}\sum_{\mu=0}^{M-1}\sum_{N=0}^{\infty}\sum_{\nu=0}^{N}\sum_{k_1\cdots k_N}\sum_{p_I\cdots p_M}\frac{1}{(N-\nu)!\nu!}\times\{\langle x|a_1^*\cdots a_\nu^*V_{I\cdots\mu|\mu+I\cdots M}a_{\nu+1}\cdots a_N|x\rangle\times\langle y|a_N^*\cdots a_{\nu+1}^*a_I^*\cdots a_\mu^*a_{\mu+I}\cdots a_Ma_\nu\cdots a_1|y\rangle+\text{symmetrical term in }(x,y)\}. \quad (16')$$

The general state $\Phi_{xy,K_1\cdots K_n}$, and the general expansions, are also given in Appendix B.

In previous discussions of the interaction energy, the energy has been expanded in powers of the coupling constant, or, in the more recent work, in the number of mesons which are exchanged between the nucleons during an "elementary" interaction.[2–5] The Heitler-London method leads to a completely different approach, as two expansions are used. The first is the expansion of the state vector of the system in the basic states; the second is an expansion of the "overlap functions" A_{xy} and the "potentials" V_{xy} in terms of the number of mesons exchanged between the clouds, or, more precisely, in terms of the number of mesons that are found in states which are common to the clouds of both nucleons. There is, of course, no direct relation between the number of exchanged mesons as it is used in the present work, and what previous writers have considered to be the number of exchanged mesons when looking at the net interaction energy.

It is generally believed that the conventional expansion of the interaction energy in the number of exchanged mesons converges when r is sufficiently large. But it is possible that when $r<r_0$, where r_0 is some critical distance, such an expansion does not converge—there is at present no information about this point. In other words, the interaction energy calculated in such a way is not known to be reliable, even qualitatively, at distances smaller than some unknown distance. The double expansion provided by the Heitler-London method fortunately allows us to investigate this mathematical question more fully. As is shown in Appendix B, provided the *one-nucleon* states satisfy certain conditions, the Heitler-London method is a convergent method for calculating the energy of two-nucleon states, however close the two nucleons may be. These conditions are expected to be fulfilled in a large class of simple models, including those of greatest physical interest. The double expansion of the Heitler-London method leads most naturally to *nonrational* approximations to the energy, and is accordingly not equivalent to a simple perturbative type of expansion. It is therefore reasonable that in many cases in which the usual types of expansions diverged, the Heitler-London method would still give a variety of convergent sequences of approximations to the energy.

The rate of convergence of the expansion of the state vector in Heitler-London states is difficult to assess, and is perhaps best studied with reference to particular examples; the existence of a simple model (Appendix A) in which a single term suffices, suggests that even in more complicated models a few terms might give a good approximation. It is important to notice, however, that for calculating the energy, we may use *variational* methods, and thereby not only increase the accuracy, but obtain reliable upper bounds even with crude assumptions about the state vector. It is essential to the estimation of the interaction energy by a variational method that the self-energy of the particles automatically be eliminated exactly, and this is achieved by the use of Heitler-London states.

When we use the method described in the previous paragraphs to calculate the properties of a two-nucleon system, we achieve both a more direct relation to the measured properties of a single nucleon, and a greater confidence in the reliability of the results, than is possible with any formalism hitherto applied. A further advantage of the Heitler-London method is that it enables us to discuss the structure of a two-nucleon state in a simple, natural, physical way. For instance, we wish to understand to what extent, and in what way, the structure of a physical nucleon is altered, and its properties changed, when it approaches another. The meson clouds which make up the outer part of the nucleons will certainly be distorted in some way when the nucleons come close together. The Heitler-London method allows us to make a convenient and important distinction between two kinds of distortion effects, each of which has its counterpart in the structure of molecules.

The first distortion effect is due to the Bose-Einstein statistics of the mesons, and is similar to the effect of the Pauli exclusion principle on the electronic clouds in molecules. This mesonic "exchange effect" arises from the stimulated emission and absorption of mesons by a nucleon which is caused by the presence of the meson cloud of the other nucleon. The density of mesons is therefore generally different from the sum of the densities associated with isolated nucleons. Another picture of this effect is gotten when we realize that the expression "meson cloud" gives only a meager description of the properties of the meson wave field; when considering the state of the meson field, we must take into account the interference between the meson waves that circulate around each of the nucleons. In the Heitler-London method, this exchange polarization effect is taken into account automatically, even when only a single basic state is used.

The other polarization effect, which is related to the dynamical properties of the meson cloud around a nucleon, is similar to the polarization of atoms which is associated with the Van der Waals interaction between them. It arises from the necessity for adding extra Heitler-London states to the basic state Φ_{xy}, and hence depends explicitly on the properties of the excited states of nucleons. The incorporation of these excited states into the formalism affords a mathematically precise way of treating isobar effects, which hitherto have been introduced only in a purely phenomenological manner.[8,15,16] When the nucleons are far apart ($r\gtrsim 1$) this polarization effect is the least important; in the atomic case, the situation is reversed, because there the Van der Waals interaction depends on the long-range Coulomb force. The separation of dynamical and statistical polarization effects is of course not unique; each way of defining the Heitler-London representation leads to a slightly different separation.

The distortion of the meson clouds around the nucleons makes every property of the two-nucleon system, such as the charge density, a nonadditive function of the corresponding property of single nucleons. In any method which centers around the explicit construction of the states, all the properties of the system may be obtained directly; in the Heitler-London method all expectation values and matrix elements can be expressed by expansions similar to those obtained for the energy. As an illustration, we shall conclude this section by writing down the expansions for the average number of mesons and for the charge density, in the state Φ_{xy} (it must be remembered that this state is normalized to $1+A_{xy}$). The general method of deriving these expansions is to commute all the operators of the form a_k^* or a_k through the $\mathfrak{F}^*$ or $\mathfrak{F}$ operators—then the basic expression (11′) can be used.

The number of mesons in the field is $\mathfrak{N}=\sum_p a_p^*a_p$.

The average number in the state Φ_{xy} is obtained from

$$(\Phi_{xy},\mathfrak{N}\Phi_{xy})=\sum_{N=0}^{\infty}\sum_{\nu=0}^{N}\sum_{k_1\cdots k_N}\frac{1}{\nu!(N-\nu)!}$$
$$\times\{\langle x|a_1^*\cdots a_\nu^*\mathfrak{N}a_{\nu+1}\cdots a_N|x\rangle$$
$$\times\langle y|a_N^*\cdots a_{\nu+1}^*a_\nu\cdots a_1|y\rangle$$
$$+\langle x|a_1^*\cdots a_\nu^*a_{\nu+1}\cdots a_N|x\rangle$$
$$\times\langle y|a_N^*\cdots a_{\nu+1}^*\mathfrak{N}a_\nu\cdots a_1|y\rangle$$
$$+\sum_p[\langle x|a_1^*\cdots a_\nu^*a_p^*a_{\nu+1}\cdots a_N|x\rangle$$
$$\times\langle y|a_N^*\cdots a_{\nu+1}^*a_pa_\nu\cdots a_1|y\rangle$$
$$+\langle x|a_1^*\cdots a_\nu^*a_pa_{\nu+1}\cdots a_N|x\rangle$$
$$\times\langle y|a_N^*\cdots a_{\nu+1}^*a_p^*a_\nu\cdots a_1|y\rangle]\}. \quad (28)$$

The charge density at the point $\mathbf{z}$, $\rho(\mathbf{z})$, consists of two parts: $\rho(\mathbf{z})=\rho_n(\mathbf{z})+\rho_m(\mathbf{z})$, where $\rho_n(\mathbf{z})$ is the charge density of the bare nucleons. The meson charge density is

$$\rho_m(\mathbf{z})=-e\epsilon_{3\alpha\beta}\varphi_\alpha(\mathbf{z})\pi_\beta(\mathbf{z})=\sum_{qp}\rho_{qp}(\mathbf{z})\varphi_q\pi_p,$$

where

$$\rho_{q\alpha,\,p\beta}(\mathbf{z})=-e\epsilon_{3\alpha\beta}\exp[i\mathbf{z}\cdot(\mathbf{q}+\mathbf{p})]\omega_p^{\frac{1}{2}}\omega_q^{-\frac{1}{2}}.$$

With this notation, we have

$$(\Phi_{xy},\rho(\mathbf{z})\Phi_{xy})=\sum_{N=0}^{\infty}\sum_{\nu=0}^{N}\frac{1}{\nu!(N-\nu)!}\sum_{k_1\cdots k_N}$$
$$\times\{\langle x|a_1^*\cdots a_\nu^*\rho(\mathbf{z})a_{\nu+1}\cdots a_N|x\rangle$$
$$\times\langle y|a_N^*\cdots a_{\nu+1}^*a_\nu\cdots a_1|y\rangle$$
$$+\text{symmetrical term in } x,\, y$$
$$+\sum_{q,p}\rho_{qp}(\mathbf{z})[\langle x|a_1^*\cdots a_\nu^*\pi_pa_{\nu+1}\cdots a_N|x\rangle$$
$$\times\langle y|a_N^*\cdots a_{\nu+1}^*\varphi_qa_\nu\cdots a_1|y\rangle$$
$$+\text{symmetrical term in } x,\, y]\}. \quad (29)$$

An application of the results of this section to the linear-coupling model will be presented in a subsequent paper.

III. A GENERALIZED MODEL

In this section the method introduced above will be applied to a more general model, for the purpose of describing briefly how various additional effects might be treated.

There is no reason in principle why the Heitler-London method (with operators defined as functions of interaction representation variables) should not be applied directly to the usual covariant theories. However, the operators $\mathfrak{F}^*$ would then be functions of operators which did not create *physical* mesons, hence the expansions for matrix elements would involve unrenormalized quantities, which, because it would require that the meson propagation be treated by perturbation theory, it would be desirable to avoid. The main difficulties lie in the treatment of virtual nucleon pairs; it is reasonable to assume that at low energies such effects could be treated phenomenologically. This may be done by an extension of the method of Sec. II.

We introduce operators F_k^*, which have the property that $F_k^*\rangle=|,k\rangle$, where $|,k\rangle$ denotes a state with a physical meson of momentum k, which are Hermitian $(F_k^*=F_{-k})$, and which satisfy $[\mathbf{P},F_k^*]=\mathbf{k}F_k^*$, $\mathbf{P}$ being the momentum operator. We may use as annihilation and creation operators the quantities $b_k=(2\omega_k)^{-1}(\omega_kF_k-[H,F_k])$, and b_k^*. Operators which create the inner structure of a nucleon are also considered. We shall not discuss how definite operators might be constructed, but use the operators in a phenomenological theory. The commutators of the b_k^* and b_k will, in general, be much more complicated than the commutators of the a_k^* and a_k, but the vacuum expectations of these commutators have the usual forms. When the operators b_k^* and b_k are used to describe the meson field, the Hamiltonian may contain terms which lead to an interaction between mesons, but the self-energy of a meson is automatically taken into account. The failure of the b_k^* and the b_k to satisfy the usual commutation rules introduces an additional, kinematical, interaction between mesons.[31]

Meson-nucleon scattering may be discussed in exactly the same way as in the static model, by using the operators which create physical mesons. We let $|p,\rangle$ denote the state with a physical nucleon of momentum p and energy E_p, and let $|p,\,k\pm\rangle$ denote a scattering state. Then

$$|p,\,k\pm\rangle=b_k^*|p,\rangle-(H-\omega_k-E_p\mp i\epsilon)^{-1}V_k|p\rangle, \quad (30)$$

where $V_k=[H,b_k^*]-\omega_kb_k^*$. A Low equation for the scattering amplitude $\langle p',\,k'-|V_k|p,\rangle$ may be derived in the usual way. The meson-meson interaction does not appear explicitly in the Low equation for the elastic scattering amplitude; it is hidden in the prescription for extrapolating matrix elements such as $\langle p',\,k'-|V_k|p,\rangle$ off of the energy shell. It should be noted that quantities such as $\langle p',\,k'-|V_k|p,\rangle$ do not have simple Lorentz transformation properties (except on the energy shell), because the definition of the b_k^* is not covariant; information about the transformation of general matrix elements may be deduced by considering the Low equation in various coordinate systems.

The Heitler-London method is now formulated as before, using the operators b_k and b_k^*, and the nucleon operators, to construct operators $\mathfrak{F}_p^*$ such that $\mathfrak{F}_p^*\rangle=|p,\rangle$, etc., and using these operators to construct the states Φ_{pq}, $\Phi_{pq,\,k}$, etc. As before, it is necessary to specify the operators of which the $\mathfrak{F}^*$ are supposed to be functions; now it is also necessary to specify their order. Since the b_k^* create physical mesons (when acting upon

[31] F. J. Dyson, Phys. Rev. **102**, 1217 (1956).

the vacuum), the effects of virtual nucleon pairs in the outer part of the nucleon will be accounted for in the meson-meson interaction; remaining effects will arise in several ways: the nucleon operators may not appear linearly in the $\mathfrak{F}^*$ and they might not commute with the F_k, the closure expansion by which the matrix elements are to be evaluated must include states with real nucleon pairs, and Heitler-London states which contain nucleon pairs must be included in the expansion for the eigenstate. These effects are expected to be important only at small distances, and might be treated phenomenologically.

The expansions previously obtained will be modified by the meson-meson interaction: the meson operators do not commute, the Hamiltonian contains additional interaction terms, and in the closure expansion it is not possible to use a product of n "creation" operators b_k^* to represent an n-meson state. If some assumed meson-meson interaction were used, these effects would be calculable; for the present it is simpler to neglect meson-meson interactions (since they are not known to be important) and use the previous expansions with a_k and a_k^* replaced by b_k and b_k^*. The form of the matrix elements is slightly modified, because they now refer to nucleons of given momenta; for instance:

$$(\Phi_{p'q'},\Phi_{pq})=\delta_{p'p}\delta_{q'q}-\delta_{p'q}\delta_{q'p}+A(p'q';pq), \quad (31)$$

and

$$(\Phi_{p'q'},H\Phi_{pq})=\tfrac{1}{2}(E_p+E_q+E_{p'}+E_{q'})(\Phi_{p'q'},\Phi_{pq}) +V(p'q';pq), \quad (32)$$

where A and V are antisymmetric in p and q, and in p' and q'.

The Heitler-London method, being based on a Schrödinger representation, is not manifestly covariant, but this introduces no insurmountable difficulties. There are no renormalization problems, because only renormalized masses and charges, and other physical quantities, appear in any equation. While it is not possible to use invariance properties directly to simplify the dependence on the momenta of such matrix elements as $\langle p'|V_k|p\rangle$, the transformation properties are related to those of the Low equation.

A two-nucleon eigenstate (in the c.m. system) may be represented as a sum of the form

$$\Psi_\alpha=\sum_p \chi(p)\Phi_{p,-p}+\sum_{p,k}\chi(p,k)\Phi_{p-\frac{1}{2}k,-p-\frac{1}{2}k,k}+\cdots, \quad (33)$$

and a Schrödinger equation for the amplitudes χ obtained by the variational method. An interesting question is the relation of the static model with an adiabatic assumption, that is, with use of the energy E_{xy} as the potential energy in a simple Schrödinger equation, to a more general model such as discussed here. Velocity-dependent and other corrections to the interaction energy appear in several ways. The matrix elements $A(p';p)$ and $V(p';p)$ ($\mathbf{q}=-\mathbf{p}$, and the exchange term is omitted) can be approximated by functions of the difference $\mathbf{p}'-\mathbf{p}$, which could be identified with the Fourier transforms of the A_{xy} and V_{xy} of the static model; velocity-dependent effects appear as dependences on the variable $\mathbf{p}'+\mathbf{p}$. Since the velocity-dependent effects which arise in this way depend on the static properties of the meson clouds of the nucleons, they are not to be interpreted as nonadiabatic corrections. Much more complicated effects are associated with the amplitudes $\chi(p,k)$, $\chi(p,k_1k_2)$, etc., which refer to excited configurations of the meson field. Inclusion of the amplitudes for these excited configurations in the Schrödinger equation makes it possible to describe such phenomena as meson production with the same equation that describes the deuteron. It is obvious that the contribution of such states to the interaction energy E_{xy} only very imperfectly represents their true role, and may be considered as a measure of the inadequacy of any method which represents two-nucleon states by a single amplitude $\chi(p)$.

In the deuteron, when the two nucleons are separated by a relatively large distance, the additional amplitudes should be small, and we might expect the static matrix elements to be adequate; assuming this to be true, and treating the nucleons nonrelativistically, we are led to the equation

$$\left(-\frac{1}{M}\nabla^2+V(x)\right)\chi(x)-\frac{1}{2M}\{\nabla^2,A(x)\}\chi(x) =E[1+A(x)]\chi(x), \quad (34a)$$

which is not of the same form as is given by the adiabatic assumption. The relation of the adiabatic equation to an equation such as (34) has been the subject of much discussion in connection with the Tamm-Dancoff method; the proper analysis for that method is appropriate, and more directly applied, here.[1,32,33] The correct normalization of the amplitude $\chi(x)$ is obtained from the expansion (33) for Ψ_α; if we define $\Gamma=(1+A)^{-\frac{1}{2}}$, $\chi(x)=\Gamma(x)\phi(x)$, then the proper normalization is $\int\phi^+\phi dV=1$. The normalized amplitude $\phi(x)$ satisfies the equation

$$E\phi=-(1/M)\nabla^2\phi+U\phi, \quad (34b)$$

where

$$U(x)=\frac{V(x)}{1+A(x)}-\frac{1}{2M}\{G_i,\Gamma G_i\Gamma\}-\frac{1}{2M}\{\nabla_i,[\Gamma,G_i]\}, \quad (\mathbf{G}=\nabla\Gamma^{-1}). \quad (35)$$

The only true velocity dependence contained in Eq. (34) is thus the spin-orbit interaction given by the last term of Eq. (35). The first term in $U(x)$ is the adiabatic interaction energy; the two corrections do not have

[32] K. A. Brueckner and K. M. Watson, Phys. Rev. 92, 1023 (1953). The erroneous numerical estimate arises from neglect of the tensor interaction.

[33] D. Feldman, Phys. Rev. 98, 1456 (1955).

great significance. We see that the static model, with the adiabatic assumption, gives very nearly (but not exactly) the correct potential, when the conditions stated in the first part of this paragraph are satisfied.

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APPENDIX A. NEUTRAL SCALAR MESONS

The model in which neutral scalar mesons interact linearly with fixed sources is well understood; the eigenstates can be found exactly by very simple methods. This makes it a good example for illustrating some of the properties of the Heitler-London method.

The Hamiltonian is

$$H=H_0+g\sum_{\mathbf{k}}(2\omega_k)^{-\frac{1}{2}}v_k(a_{\mathbf{k}}+a_{-\mathbf{k}}^*)\sum_i e^{i\mathbf{k}\cdot\mathbf{x}_i}\alpha_{x_i}^*\alpha_{x_i} - E\sum_i \alpha_{x_i}^*\alpha_{x_i}, \quad \text{(A1)}$$

where v_k is the source function which is introduced to make the theory finite. The nucleon self-energy is eliminated by setting

$$E=-g^2\sum_{\mathbf{k}}\tfrac{1}{2}(\omega_k)^{-2}v_k^2.$$

The following expressions for the eigenstates are well known:

$$\begin{aligned}|x\rangle&=C_x'\exp\{-g\textstyle\sum_{\mathbf{k}} v_k\omega^{-1}e^{i\mathbf{k}\cdot\mathbf{x}}(2\omega)^{-\frac{1}{2}}\varphi_{\mathbf{k}}\}\alpha_x^*\rangle\\ &=C_x\exp\{-g\textstyle\sum_{\mathbf{k}} v_k\omega^{-1}e^{-i\mathbf{k}\cdot\mathbf{x}}(2\omega)^{-\frac{1}{2}}a_{\mathbf{k}}^*\}\alpha_x^*\rangle,\\ |xy\rangle&=C_{xy}'\exp\{-g\textstyle\sum_{\mathbf{k}} v_k\omega^{-1}(e^{i\mathbf{k}\cdot\mathbf{x}}+e^{i\mathbf{k}\cdot\mathbf{y}})\\ &\qquad\times(2\omega)^{-\frac{1}{2}}\varphi_{\mathbf{k}}\}\alpha_x^*\alpha_y^*\rangle,\\ &=C_{xy}\exp\{-g\textstyle\sum_{\mathbf{k}} v_k\omega^{-1}(e^{-i\mathbf{k}\cdot\mathbf{x}}+e^{-i\mathbf{k}\cdot\mathbf{y}})\\ &\qquad\times(2\omega)^{-\frac{1}{2}}a_{\mathbf{k}}^*\}\alpha_x^*\alpha_y^*\rangle. \quad \text{(A2)}\end{aligned}$$

In (A2), C_x, C_x', C_{xy}, and C_{xy}' are constants chosen to normalize the state vectors. The interaction energy is

$$E_{xy}=-g^2\sum_{\mathbf{k}}\omega^{-2}v^2e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}.$$

We see from (A2) that

$$\begin{aligned}\mathfrak{F}_x'^*(\varphi_{\mathbf{k}})&=C_x'\exp\{-g\textstyle\sum_{\mathbf{k}} v\omega^{-1}e^{i\mathbf{k}\cdot\mathbf{x}}(2\omega)^{-\frac{1}{2}}\varphi_{\mathbf{k}}\}\alpha_x^*,\\ \mathfrak{F}_x^*(a_{\mathbf{k}}^*)&=C_x\exp\{-g\textstyle\sum_{\mathbf{k}} v\omega^{-1}e^{-i\mathbf{k}\cdot\mathbf{x}}(2\omega)^{-\frac{1}{2}}a_{\mathbf{k}}^*\}\alpha_x^*. \quad \text{(A3)}\end{aligned}$$

In either case $\mathfrak{F}_x^*$ is an exponential function of the meson variables; hence it is obvious that $\Phi_{xy}'=\mathfrak{F}_x'^*(\varphi_{\mathbf{k}})\mathfrak{F}_y'^*(\varphi_{\mathbf{k}})\rangle$ and $\Phi_{xy}=\mathfrak{F}_x^*(a_{\mathbf{k}}^*)\mathfrak{F}_y^*(a_{\mathbf{k}}^*)\rangle$ are both exact eigenstates—both are proportional to $|xy\rangle$.

We may use our expansion theorem, Eq. (11), to calculate the normalization of Φ_{xy}'. We first consider

$$M_{\mathbf{p}\cdots\mathbf{q}}=\langle x|:(a_{\mathbf{p}}^*+a_{-\mathbf{p}})\cdots(a_{\mathbf{q}}^*+a_{-\mathbf{q}}):|x\rangle.$$

Since $V_{\mathbf{k}}^*=[a_{\mathbf{k}},H']$ commutes with H in this model, we have

$$\begin{aligned}a_{\mathbf{k}}|x\rangle&=-(H+\omega_k)^{-1}V_{\mathbf{k}}^*|x\rangle\\ &=-\omega_k^{-1}(2\omega_k)^{-\frac{1}{2}}gv_ke^{-i\mathbf{k}\cdot\mathbf{x}}|x\rangle; \quad \text{(A4)}\end{aligned}$$

therefore we find that

$$M_{\mathbf{p}\cdots\mathbf{q}}=\left(-\frac{2gv_pe^{i\mathbf{p}\cdot\mathbf{x}}}{\omega_p(2\omega_p)^{\frac{1}{2}}}\right)\cdots\left(-\frac{2gv_qe^{i\mathbf{q}\cdot\mathbf{x}}}{\omega_q(2\omega_q)^{\frac{1}{2}}}\right)\langle x|x\rangle. \quad \text{(A5)}$$

Hence with the notation

$$A(r)=g^2\sum_{\mathbf{k}}\omega_k^{-3}v_k^2e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})},$$

we obtain

$$(\Phi_{xy}',\Phi_{xy}')=1+A_{xy}'=e^{2A(r)}. \quad \text{(A6)}$$

If $v_k=1$, we find $A(r)=(g^2/4\pi)2K_0(r)/\pi$.

It is not hard to show, for instance by using the expansions of Appendix B, that the states $\Phi_{xy,\mathbf{q}_1\cdots\mathbf{q}_N}'$ are also eigenstates, and are also normalized to $\exp[2A(r)]$. The simple form of the n-meson eigenstates is of course related to the fact that the mesons are not actually scattered by the nucleons, in this model.

The expansion theorem (11′) may be used to calculate the norm of Φ_{xy}. It is found that $(\Phi_{xy},\Phi_{xy})=\exp[A(r)]$. In this example, the terms in the expansions decrease more rapidly if one uses functions of the $a_{\mathbf{k}}^*$ to define the representation, rather than functions of the $\varphi_{\mathbf{k}}^*$.

We have shown that in this special model, the Heitler-London method gives an exact solution to the two-nucleon problem. In this example it is natural to speak of the two nucleons as preserving their identity and remaining unexcited when they approach one another. It should be noted, however, that the density of mesons at any point is not the sum of the densities associated with isolated nucleons, so that the average number of mesons in the field increases when the nucleons are brought closer together. Thus the Heitler-London states describe correctly the stimulated emission and absorption of mesons by one nucleon which is caused by the presence of the meson cloud surrounding the other.

APPENDIX B. *N*-MESON STATES

We shall first give an inductive construction of the state $\Phi_{xy,q_1\cdots q_n}'$, which has n mesons impinging on two nucleons. Let $\{n\}$ denote the set of n meson variables, $\{n|\alpha\}$ one of the 2^n subsets, and $\{n|\alpha'\}$ the complementary subset. Let $(\alpha)=(ij\cdots)$ refer to the subset $\{n|\alpha\}$ obtained by removing q_i, q_j, $\cdots$ from $\{n\}$. Then we define

$$\begin{aligned}\Phi_{xy,\{n\}}'&=\sum_\alpha \mathfrak{F}_{x,\{n|\alpha\}}'^*\mathfrak{F}_{y,\{n|\alpha'\}}'^*\rangle\\ &\quad-\sum_i \varphi_i^*\Phi_{xy,\{n|i\}}'\\ &\quad-\sum_{i<j}:\varphi_i^*\varphi_j^*:\Phi_{xy,\{n|ij\}}'\\ &\quad-\cdots\\ &\quad-:\varphi_1^*\cdots\varphi_n^*:\Phi_{xy}'. \quad \text{(B1)}\end{aligned}$$

It is not hard to verify that when $r\rightarrow\infty$, $\Phi_{xy,\{n\}}'\rightarrow\Psi_{xy,\{n\}}$.

The following lemma is useful for separating the matrix elements of the $\Phi_{xy,\{n\}}'$ into irreducible parts:

$$a_{q'}\Phi_{xy,\{n\}}'=\sum_i \delta_{q'q_i}\Phi_{xy,\{n|i\}}'+(\text{nonsingular part}). \quad \text{(B2)}$$

The nonsingular part is free of delta functions of the momenta. We obtain for the matrix element $(\Phi_{xy,\{m\}}',\Phi_{xy,\{n\}}')$:

$$\begin{aligned}(\Phi_{xy,\{m\}}',\Phi_{xy,\{n\}}') &= A_{xy}'(\{m\};\{n\}) \\ &+\sum_i\sum_a \delta_{q_a'q_i}A_{xy}'(\{m|a\};\{n|i\}) \\ &+\sum_{i<j}\sum_{a<b}[\delta_{q_a'q_i}\delta_{q_b'q_j}+\delta_{q_b'q_i}\delta_{q_a'q_j}] \\ &\times A_{xy}'(\{m|ab\};\{n|ij\})+\cdots, \quad \text{(B3)}\end{aligned}$$

where

$$\begin{aligned}A_{xy}'(\{m\};\{n\}) &= \sum_\alpha\sum_\beta\sum_{N=0}^{\infty}\sum_{k_i\cdots k_N}(N!)^{-1} \\ &\times\langle x,\{m|\beta\}|\varphi_1^*\cdots\varphi_N^*|x,\{n|\alpha\}\rangle_I \\ &\times\langle y,\{m|\beta'\}|\varphi_1\cdots\varphi_N|y,\{n|\alpha'\}\rangle_I. \quad \text{(B4)}\end{aligned}$$

Unless $m=n=0$, the $N=0$ term vanishes.

The potentials are obtained by first using a simple extension of Eqs. (14) and (15):

$$H\Phi_{xy,\{n\}}'=\sum_i \omega_i\Phi_{xy,\{n\}}'+X_{xy,\{n\}}', \quad \text{(B5a)}$$

$$\begin{aligned}X_{xy,\{n\}}' &= \sum_p\sum_\alpha 2\omega_p\{[\pi_p,\mathfrak{F}_{x,\{n|\alpha\}}'^*][\pi_p^*,\mathfrak{F}_{y,\{n|\alpha'\}}'^*]\}_I\rangle \\ &-\sum_i \varphi_i^*X_{xy,\{n|i\}}'-\sum_{i<j}:\varphi_i^*\varphi_j^*:X_{xy,\{n|ij\}}' \\ &-\cdots. \quad \text{(B5b)}\end{aligned}$$

The *irreducible part* (denoted by $\{\cdots\}_I$) of the first term of (B5b) means the part left over after discarding the terms containing delta functions such as δ_{pq_i}.

As with the $\Phi_{xy,\{n\}}'$, we find

$$a_{q'}X_{xy,\{n\}}'=\sum_i \delta_{q'q_i}X_{xy,\{n|i\}}' +(\text{nonsingular parts}). \quad \text{(B6)}$$

The matrix elements of H are very similar to (B3) and (B4). When $m>n$, we write

$$(\Phi_{xy,\{m\}}',X_{xy,\{n\}}')=V_{xy}'(\{m\};\{n\}) +\sum_{i,a}\delta_{q_iq_a'}V_{xy}'(\{m|a\};\{n|i\}) +\cdots, \quad \text{(B7)}$$

where

$$\begin{aligned}V_{xy}'(\{m\};\{n\}) &= -\sum_p\sum_\alpha\sum_\beta\sum_{N=0}^{\infty}\sum_{k_i\cdots k_N}2\omega_p \\ &\times\langle x,\{m|\beta\}|\varphi_1^*\cdots\varphi_N^*a_{-p}|x,\{n|\alpha\}\rangle_I \\ &\times\langle y,\{m|\beta'\}|\varphi_1\cdots\varphi_N a_p|y,\{n|\alpha'\}\rangle_I. \quad \text{(B8)}\end{aligned}$$

When $m=n$, the above expressions are symmetrized.

If the representation is defined by functions of the creation operators, the n-meson state is defined as follows:

$$\begin{aligned}\Phi_{xy,\{n\}} &= \sum_\alpha \mathfrak{F}_x{}^*{}_{\{n|\alpha\}}{}^*\mathfrak{F}_{y\{n|\alpha'\}}{}^*\rangle \\ &-\sum_i a_i^*\Phi_{xy\{n|i\}} \\ &-\sum_{i<j}a_i^*a_j^*\Phi_{xy\{n|ij\}} \\ &-\cdots \\ &-a_1^*\cdots a_n^*\Phi_{xy}. \quad \text{(B1}'\text{)}\end{aligned}$$

Equations (B2), (B3), (B5a), (B6), and (B7) apply also in this representation. Equation (B4) is replaced by

$$\begin{aligned}A_{xy}(\{m\};\{n\}) &= \sum_{\alpha,\beta}\sum_{N=0}^{\infty}\sum_{\nu=0}^{N}\sum_{k_1\cdots k_N}\frac{1}{\nu!(N-\nu)!} \\ &\times\langle x\{m|\beta\}|a_1^*\cdots a_\nu^*a_{\nu+1}\cdots a_N|x\{n|\alpha\}\rangle_I \\ &\times\langle y\{m|\beta'\}|a_N^*\cdots a_{\nu+1}^*a_\nu\cdots a_1|y\{n|\alpha'\}\rangle_I, \quad \text{(B4}'\text{)}\end{aligned}$$

and Eq. (B8) is replaced by

$$\begin{aligned}V_{xy}(\{m\};\{n\}) &= \sum_{\alpha,\beta}\sum_{N=0}^{\infty}\sum_{\nu=0}^{N}\sum_{M=0}^{\infty}\sum_{\mu=0}^{M-1}\sum_{k_I\cdots k_N}\sum_{p_I\cdots p_M}\frac{1}{\nu!(N-\nu)!} \\ &\times\{\langle x\{m|\beta\}|a_1^*\cdots a_\nu^*V_{I\cdots\mu|\mu+I\cdots M} \\ &\times a_{\nu+1}\cdots a_N|x\{n|\alpha\}\rangle_I \\ &\times\langle y\{m|\beta'\}|a_N^*\cdots a_{\nu+1}^*a_I^*\cdots a_\mu^*a_{\mu+I}\cdots \\ &\times a_Ma_\nu\cdots a_1|y\{n|\alpha'\}\rangle_I \\ &+\text{symmetrical term in }(x,y)\}. \quad \text{(B8}'\text{)}\end{aligned}$$

The states $\Phi_{xy,q_1\cdots q_n}$ have the property of being *asymptotically stationary* (in the sense of Van Hove) when $r\rightarrow\infty$, and are asymptotically orthonormal.[34] It is not necessary, however, to also remove the mesons very far from the nucleons in order to achieve this property,

[34] L. Van Hove, Physica 21, 901 (1955). The author is grateful to B. Zumino for pointing out the relation of this paper to the present work.

as is the case with the Van Hove states; if $r\to\infty$, $\Phi_{xy,q_1\cdots q_n}$ describes correctly the interaction of the incident mesons with one or the other of the nucleons. The $\Phi_{xy,q_1\cdots q_n}$ also form an *asymptotically complete* set when $r\to\infty$; we shall in addition assume that they form a complete set for all finite separations of the two nucleons. In applications, we must of course require that a *small* number of Heitler-London states give a good approximation to the exact states. In any case, it is always possible to form other states (by multiplying together various operators), which are not Heitler-London states; they do not approach eigenstates when $r\to\infty$. Such states are inconvenient, and it is to be hoped that they are redundant in physically interesting models, as well as in the neutral scalar model.

Provided the Heitler-London states are normable and complete, we may form expansions from them which will converge (in the mean) to the required eigenstates of H. It is obviously necessary to assume that the states $\Phi_{xy,q_1\cdots q_n}'$ are normalizable, and in order to insure the convergence of various matrix elements of physical interest, we shall further require that

$$\pi_p \prod_{i=1}^{S} \varphi_i \Phi_{xy,\{n\}}'$$

be normalizable, where S is any integer. (We restrict ourselves here to a discussion of states formed by operators which are functions of the φ_k; other cases seem to be more difficult to analyze.) This condition, which is not trivial, may be investigated most easily by using the representation in which the φ_k are diagonalized (standing waves are also used for this discussion). We shall assume that only mesons with a sufficiently low momentum interact with nucleons, and that the system is enclosed in a box, so there are only a finite number of degrees of freedom. Then it is sufficient to assume that the functions

$$\pi_p \prod_{i=1}^{S} \varphi_i \mathfrak{F}_{x,\{n\}}'^*(\varphi_k) \mathfrak{F}_{y,\{m\}}'^*(\varphi_k) \exp(-\tfrac{1}{4}\textstyle\sum_k \varphi_k^2)$$

be square integrable functions of the φ_k. (We shall call this condition A.) Condition A will be satisfied if the more restrictive condition B, that

$$\pi_p \prod_{i=1}^{S} \varphi_i \mathfrak{F}_{x,\{n\}}'^*(\varphi_k) \exp(-\tfrac{1}{8}\textstyle\sum_k \varphi_k^2)$$

be bounded and square integrable, is satisfied. The existence of discrete one-nucleon eigenstates, and the requirement that the same operators have finite matrix elements between these states, yields only the weaker condition that

$$\pi_p \prod_{i} \varphi_i \mathfrak{F}_{x,\{n\}}'^*(\varphi_k) \exp(-\tfrac{1}{4}\textstyle\sum_k \varphi_k^2)$$

be square integrable. This is equivalent to condition A when the two nucleons are very far apart, but not when their meson clouds overlap.

The expansions for the matrix elements were derived from a closure expansion, which converges, provided the operators involved satisfy certain boundedness conditions. These conditions are examined in the same way as in the preceding paragraph. A sufficient condition that expansions such as those in Eqs. (B4) and (B8) converge is that

$$\pi_p \prod_{i} \varphi_i \mathfrak{F}_{x,\{n\}}'(\varphi_k) \mathfrak{F}_{x,\{m\}}'^*(\varphi_k) \exp(-\tfrac{1}{4}\textstyle\sum_k \varphi_k^2)$$

be bounded and square integrable, for which condition B is sufficient. If the sufficient condition B is not satisfied, the matrix element might still exist and be equal to the sum of the series, if the sum is evaluated appropriately (perhaps by first summing over the $k_1\cdots k_n$ and then over n). If condition B is not satisfied, the sum might converge only when the nucleons are far enough apart.

The model with neutral, scalar mesons (Appendix A) obviously satisfies the sufficient conditions of this section. In general, if the interaction Hamiltonian is linear ($H'\sim\sum U_k\varphi_k$), then $\mathfrak{F}_x'^*(\varphi_k)$ has the asymptotic form

$$\mathfrak{F}_x'^*(\varphi_k) \sim C \exp(-\textstyle\sum \omega_k^{-1} U_k \varphi_k),$$

when the argument of the exponential is large. This asymptotic form is such that we might expect that condition B will be satisfied when the interaction is linear. It may be noted that the class of models in which condition B is not satisfied includes models in which discrete two-nucleon eigenstates do not exist when the nucleons are too close to one another.[35]

[35] This occurs in the pair-interaction model [G. Wentzel, Helv. Phys. Acta **15**, 111 (1942)] with a suitable, negative, interaction constant.