Formulation of Field Theories of Composite Particles

K. NISHIJIMA*
Institute for Advanced Study, Princeton, New Jersey
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Based upon Feynman's over-all-space-time point of view, a general method for dealing with bound-state problems is presented. In this paper we are mainly interested in the properties of Green's functions that should be satisfied in the field theories with bound states. First the Chew-Low or Lehmann-Symanzik-Zimmermann equations are generalized so as to include composite particles. Then we examine the possibility of distinguishing between elementary and composite particles. Finally, an investigation is made of how the S matrix elements for processes involving composite particles are related to those involving no composite particles. This problem is illustrated by the relation between $p+p \rightarrow n+p+\pi^+$ and $p+p \rightarrow d+\pi^+$.

1. INTRODUCTION

THE scattering problems of composite particles in quantum field theory involve serious complications as compared with those in particle mechanics. Even the scattering problems of elementary particles already bare essentially these same complications. In particle mechanics the asymptotic form of the scattering amplitude is governed by the free part of the Hamiltonian which is obtained from the total Hamiltonian by dropping the potential acting between particles separated in distance in the remote future. In field theory the interaction between a particle and its self-field should be retained even when the particle is separated far from other particles, and when composite particles are involved in the final state the nuclear forces responsible for the formation of the composite particles should also be retained, whereas the interaction between two separated particles should be switched off in the asymptotic form. Thus the question is raised as to whether it is possible or not to decompose the whole interaction into two parts, one to be retained and the other to be switched off in the asymptotic form. In field theory, however, this is in general not possible in contradistinction to the case of particle mechanics. Since a creation or a destruction operator in the interaction Hamiltonian specifies only the momentum, spin direction, and similar quantities of the created or destroyed quantum but nothing about its history, we do not know beforehand whether it contributes to the self-interaction or to the nuclear force acting between two separated particles. From this standpoint the renormalization problem is essentially of the same nature as the bound-state problem. To be more precise, we shall illustrate this situation by nucleon-nucleon scattering. Suppose that the nucleon "1" emitted a virtual meson. If this meson is absorbed by the nucleon "2" it gives rise to the nuclear force which is to be switched off in the asymptotic form. On the other hand, if it is reabsorbed by the first nucleon it contributes to the self-interaction and is to be retained even when the two nucleons are separated far from one another. When the two nucleons form a composite particle in the final state, then even the nuclear force has to be retained in the asymptotic form. One does not know beforehand which of these alternatives will be realized, but quantum-mechanically all these three modes can take place with certain probabilities. For this reason the Hamiltonian formulation describing the temporal development of the system by means of the Schrödinger equation cannot give a clear-cut solution to this problem. It is clear that the time-independent formulation is also not suitable for the present purpose. In order to overcome this difficulty we shall employ Feynman's over-all-space-time point of view.[1] In this description, one sums up the amplitudes over all possible histories in order to calculate the transition amplitude. Since in each history the role played by a virtual meson is definite, it is clear whether one has to switch on or off the corresponding interaction in the asymptotic form. This is one of the possible reasons why the Feynman method was so successful in dealing with renormalization problems and we might hope that this is also the case for bound-states problems.

There are, however, some exceptional cases in which the time-independent description succeeds in consistently renormalizing the theory, for example, in Chew's extended-source meson theory.[2] This example does not contradict our general argument since it results from the special assumption that there is only one nucleon and hence all virtual mesons are self-field mesons. Indeed when there is another nucleon or an additional antinucleon, one has to refer to the over-all-space-time point of view in order to give a clear-cut renormalization procedure.[3]

Based on this viewpoint, we shall give an intuitive derivation of the S matrix in terms of Feynman amplitudes in Sec. 2. Then in Sec. 3 we shall examine the self-consistency of the formulas obtained in Sec. 2. In Sec. 4 various possible forms of the S matrix are given in connection with the description of composite particles. Finally in Sec. 4 the relation of the S-matrix elements between the two reactions $p+p \rightarrow n+p+\pi^+$

* On leave of absence from Osaka City University, Osaka, Japan.

[1] R. P. Feynman, Phys. Rev. 76, 749, 769 (1949).
[2] G. F. Chew, Phys. Rev. 94, 1748, 1755 (1954); 95, 1669 (1954).
[3] K. Nishijima, Suppl. Progr. Theoret. Phys. Japan No. 3, 138 (1956).

and $p+p \to d+\pi^+$ is discussed as an application of the results of Sec. 4.

2. FEYNMAN AMPLITUDES AND THE S MATRIX

Following the prescriptions given in the previous section, we discuss an intuitive derivation of the scattering matrix elements for processes involving composite particles. As for the mathematical consistency of this method we refer to Sec. 3.

We first introduce two complete orthonormal sets of state vectors $\{\Phi_\alpha^{(+)}\}$ and $\{\Phi_\alpha^{(-)}\}$. The superscripts $(+)$ and $(-)$ refer to the outgoing-wave and incoming-wave boundary conditions, respectively. Furthermore, each element of the two sets can be written as

$$\Phi_\alpha^{(\pm)}=\Phi_{a_1\times a_2\times\cdots\times a_n}{}^{(\pm)},\text{ or simply }\Phi_{a_1a_2\cdots a_n}{}^{(\pm)}. \quad (2.1)$$

By this notation is meant that the stationary scattering state $\Phi_\alpha^{(+)}$ is formed by stable particles $a_1, a_2, \cdots, a_n$ coming into collision, and $\Phi_\alpha^{(-)}$ has a similar interpretation. Needless to say, they are physically meaningful only when all individual particles $a_1, a_2, \cdots, a_n$ form wave packets.

The S matrix is defined as the transformation matrix between these two sets:

$$S_{\beta\alpha}=\langle\Phi_\beta^{(-)},\Phi_\alpha^{(+)}\rangle \quad\text{or}\quad \Phi_\alpha^{(+)}=\sum_\beta S_{\beta\alpha}\Phi_\beta^{(-)}. \quad (2.2)$$

The Feynman amplitude of a given state Φ_α is defined by

$$\langle\Omega, T[\varphi(x_1)\cdots\varphi(x_n)]\Phi_\alpha\rangle, \quad (2.3)$$

where T is Wick's time-ordering symbol and Ω is the vacuum state. $\varphi(x)$ is the field operator of a neutral spinless field in the Heisenberg representation.

By combining (2.2) and (2.3) we get

$$\langle\Omega, T[\varphi(x_1)\cdots\varphi(x_n)]\Phi_\alpha^{(+)}\rangle$$
$$=\sum_\beta\langle\Omega, T[\varphi(x_1)\cdots\varphi(x_n)]\Phi_\beta^{(-)}\rangle S_{\beta\alpha}, \quad (2.4)$$

which serves as the basic formula in the derivation of the S matrix. In order to derive the explicit form of the S matrix, it is necessary to introduce the asymptotic forms of both sides of the above equation. The asymptotic form of the Feynman amplitude is obtained by letting the particles propagate to the remote future and then bringing them back to the present after switching off the interaction between dressed particles. In general, there can occur various kinds of final states as a result of the creation and destruction of particles in the course of collision processes, and in order to derive the asymptotic form of a Feynman amplitude it is necessary to specify the constituents in the final state. For example, in neutron-deuteron collisions at low energies, there are two possible final states, a state consisting of a neutron and a deuteron and the other consisting of two neutrons and a proton, and we have to evaluate the asymptotic forms separately for these two components.

We denote the asymptotic form of the Feynman amplitude corresponding to a component γ by

$$\lim{}^\gamma\langle\Omega, T[\varphi(x_1)\cdots\varphi(x_n)]\Phi_\alpha^{(+)}\rangle. \quad (2.5)$$

When the momenta and spin directions of the constituents are not specified, but only the species of the constituents are specified, we shall denote the component by (γ). Since we know the boundary condition at $t\to\infty$ for an incoming state $\Phi_\beta^{(-)}$, we can readily evaluate the expression

$$\lim{}^\gamma\langle\Omega, T[\varphi(x_1)\cdots\varphi(x_n)]\Phi_\beta^{(-)}\rangle. \quad (2.6)$$

Namely, this state turns out in the remote future to be a component which consists of freely propagating wave packets $b_1, b_2, \cdots, b_m$ $(\beta=b_1b_2\cdots b_m)$, and the above expression survives only when $\gamma=\beta$. Hence we get from (2.4), (2.5), and (2.6) the formula

$$\lim{}^\beta\langle\Omega, T[\varphi(x_1)\cdots\varphi(x_n)]\Phi_\alpha^{(+)}\rangle$$
$$=\lim{}^\beta\langle\Omega, T[\varphi(x_1)\cdots\varphi(x_n)]\Phi_\beta^{(-)}\rangle S_{\beta\alpha}. \quad (2.7)$$

Since n is arbitrary we shall choose the most convenient n for the derivation of the S matrix.

(1) When β consists only of m elementary particles we shall choose $n=m$; then we get

$$\lim{}^\beta\langle\Omega, T[\varphi(x_1)\cdots\varphi(x_m)]\Phi_{b_1\cdots b_m}{}^{(-)}\rangle$$
$$=\sum_{\text{perm.}}\langle\Omega, \varphi(x_1)\Phi_{b_1}\rangle\cdots\langle\Omega, \varphi(x_m)\Phi_{b_m}\rangle. \quad (2.8)$$

Since $\Phi_b^{(+)}=\Phi_b^{(-)}$ for a stable single-particle state b, we need not write the superscript $(+)$ or $(-)$.

(2) When β consists of m elementary particles and of a composite particle B satisfying

$$\langle\Omega, \varphi(x)\Phi_B\rangle=0, \quad \langle\Omega, T[\varphi(x_1)\varphi(x_2)]\Phi_B\rangle\neq 0,$$

then we choose $n=m+2$. In this case we get

$$\lim{}^\beta\langle\Omega, T[\varphi(x_1)\cdots\varphi(x_{m+2})]\Phi_\beta^{(-)}\rangle$$
$$=\sum_{\text{perm.}}\langle\Omega, \varphi(x_1)\Phi_{b_1}\rangle\cdots\langle\Omega, \varphi(x_m)\Phi_{b_m}\rangle$$
$$\times\langle\Omega, T[\varphi(x_{m+1})\varphi(x_{m+2})]\Phi_B\rangle. \quad (2.9)$$

The generalization of the above formulas to more general cases is clear.

If we choose n smaller than those given above, both sides of Eq. (2.7) vanish. If on the contrary we choose n greater than those given above, both sides survive but it leads to an unnecessary complication. For the details on this point we refer to the discussions in Sec. 4.

It is worthwhile to notice that the asymptotic forms satisfy the Klein-Gordon equation with respect to the elementary-particle coordinates and the Bethe-Salpeter equation with respect to the composite-particle coordinates. This property shows that the self-interactions, like the nuclear force holding a composite particle together, are not switched off in the asymptotic form.

Our next task is to find out the procedure for evaluating the asymptotic form of the Feynman amplitude for an outgoing wave state $\Phi_\alpha^{(+)}$. To begin with, we shall first rewrite (2.7) in the following form:

$$\lim{}^{(\beta)}\langle\Omega, T[\varphi(x_1)\cdots\varphi(x_n)]\Phi_\alpha^{(+)}\rangle$$
$$=\sum_{(\beta)}\lim{}^{\beta}\langle\Omega, T[\varphi(x_1)\cdots\varphi(x_n)]\Phi_\beta^{(-)}\rangle S_{\beta\alpha}. \quad (2.10)$$

In order to evaluate the left-hand side of the above equation, we may use the graphical method. The expression on the left-hand side represents in terms of Feynman graphs the amplitude for a class of histories in which the wave packets $a_1, a_2, \cdots, a_l$ $(\alpha=a_1\cdots a_l)$ collide and then the collision products are annihilated at points $x_1, \cdots, x_n$.

Let us first consider the case in which the component (β) consists of elementary particles alone and choose $n=m$. Choose arbitrarily one point from $x_1, \cdots, x_n$, say x_1, and trace back the graph starting from x_1. Then there are three possible cases.

In the first case, the line ending at x_1 will begin at one of the incident particles, say at a_1, without being connected with any other of the x's and a's. Then the line connecting x_1 with a_1 clearly represents a self-energy type graph and the Feynman amplitude corresponding to this graph can be factored to give

$$\langle\Omega, \varphi(x_1)\Phi_{a1}\rangle\times\text{function of } x_2, \cdots, x_n.$$

Since one can factor the operation $\lim^{(\beta)}$ as

$$\lim{}^{(\beta)}=\prod_{i=1}^{n}\lim_{x_i\sim\infty}{}^{(b)},$$

we have only to establish the operation $\lim^{(b)}$ for an elementary-particle component b. In the present case one may write

$$\lim_{x\sim\infty}{}^{(b)}\langle\Omega, \varphi(x)\Phi_a\rangle=\langle\Omega, \varphi(x)\Phi_a\rangle. \quad (2.11)$$

Since the interaction of an elementary particle with its self-field should be retained, there is no difference between the true Feynman amplitude and its asymptotic form.

In the second case the line originating from x_1 will be connected to more than one of the a's and at least one other of the x's, and this graph generally involves a self-energy part starting from x_1 and ending at x_1'. Therefore the corresponding Feynman amplitude will involve a factor $\langle\Omega, T[\varphi(x_1)\varphi(x_1')]\Omega\rangle$ representing free propagation from x_1' to x_1 after suffering a true interaction with other particles. In order to get the asymptotic form in this case we shall refer to the following formula:

$$\lim_{x_1\sim\infty}{}^{(b)}\langle\Omega, T[\varphi(x_1)\varphi(x_1')]\Omega\rangle$$
$$=\sum_{(\beta)}\langle\Omega, \varphi(x_1)\Phi_b\rangle\langle\Phi_b, \varphi(x_1')\Omega\rangle. \quad (2.12)$$

This formula is derived as follows. First, letting t_1 tend to $+\infty$ we may drop the T symbol, and because of the destructive interference among continuous states we have to retain only the discrete levels in the summation appearing in the right-hand side of (2.12). This dropping of continuum intermediate states is also justified by the requirement that the asymptotic form should satisfy the Klein-Gordon equation with respect to the coordinates x_1.

In the third case, the x_1 may be connected in different ways from what we gave above. Generally they do not contribute to the asymptotic forms because of the conservation of energy and momentum.

(2) Next let us consider a case in which a composite particle B participates in the final state. By an argument similar to that given in the previous case, one can derive the following relations:

$$\lim_{(x_1,x_2)\sim\infty}{}^{(b)}\ \langle\Omega, T[\varphi(x_1)\varphi(x_2)]\Phi_B\rangle$$
$$=\langle\Omega, T[\varphi(x_1)\varphi(x_2)]\Phi_B\rangle, \quad (2.13)$$

$$\lim_{(x_1,x_2)\sim\infty}{}^{(B)}\ \langle\Omega, T[\varphi(x_1)\varphi(x_2)\varphi(x_1')\varphi(x_2')]\Omega\rangle$$
$$=\sum_{(B)}\langle\Omega, T[\varphi(x_1)\varphi(x_2)]\Phi_B\rangle$$
$$\times\langle\Phi_B, T[\varphi(x_1')\varphi(x_2')]\Omega\rangle. \quad (2.14)$$

These relations have already been derived and utilized in previous papers.[4] The generalization of the above method to other cases is clear.

The above rules are general enough to calculate the asymptotic form of an arbitrary Feynman amplitude. However, there is one thing about which one has to be careful. The operation $\lim^{(\beta)}$ is expressed by the product of $\lim^{(b)}$ operations, and in order to get a unique result the latter operations should be commutative with each other. This is, however, not the case in general, and a simple example is given by

$$\lim_{x_1\sim\infty}{}^{(b)}\lim_{x_2\sim\infty}{}^{(b)}\langle\Omega, T[\varphi(x_1)\varphi(x_2)]\Omega\rangle$$
$$=\sum_{(b)}\langle\Omega, \varphi(x_2)\Phi_b\rangle\langle\Phi_b, \varphi(x_1)\Omega\rangle,$$

$$\lim_{x_2\sim\infty}{}^{(b)}\lim_{x_1\sim\infty}{}^{(b)}\langle\Omega, T[\varphi(x_1)\varphi(x_2)]\Omega\rangle$$
$$=\sum_{(b)}\langle\Omega, \varphi(x_1)\Phi_b\rangle\langle\Phi_b, \varphi(x_2)\Omega\rangle.$$

In order to overcome this difficulty, one has to define the Feynman amplitude after subtracting such disconnected parts from the T product of field operators.[4] For this purpose we redefine the Feynman amplitude in terms of the normal product of field operators, namely by

$$\langle\Omega, N[\varphi(x_1)\cdots\varphi(x_n)]\Phi\rangle. \quad (2.15)$$

Then the operation $\lim^{(\beta)}$ is unique when applied to the new Feynman amplitude.

For the sake of later convenience we shall give the

[4] K. Nishijima, Progr. Theoret. Phys. Japan **10**, 549 (1953); **12**, 279 (1954); **13**, 305 (1955).

definition of the normal product.[5] Let $Q(x)$ be a c-number source and define the functional U by

$$U=T\exp\left[-i\int_{-\infty}^{\infty}(dx)\varphi(x)Q(x)\right], \quad (2.16)$$

where $(dx)\equiv d^4x$. Then the T product of field operators is given by

$$T[\varphi(x_1)\cdots\varphi(x_n)]=i^n\left[\frac{\delta^n U}{\delta Q(x_1)\cdots\delta Q(x_n)}\right]_{Q=0}. \quad (2.17)$$

Next put $W=U/\langle\Omega, U\Omega\rangle$; then the N product is given by

$$N[\varphi(x_1)\cdots\varphi(x_n)]=i^n\left[\frac{\delta^n W}{\delta Q(x_1)\cdots\delta Q(x_n)}\right]_{Q=0}. \quad (2.18)$$

Thus the S matrix is uniquely determined by

$$\lim{}^{(\beta)}\langle\Omega, N[\varphi(x_1)\cdots\varphi(x_n)]\Phi_\alpha{}^{(+)}\rangle =\sum_{(\beta)}\lim{}^{\beta}\langle\Omega, N[\varphi(x_1)\cdots\varphi(x_n)]\Phi_\beta{}^{(-)}\rangle S_{\beta\alpha}. \quad (2.19)$$

In previous papers[4] the applications of the above rules to the Bethe-Salpeter equations were discussed in detail and in this paper we do not enter into this question.

In the case of the Bethe-Salpeter approach the above rules were quite useful, but this is no longer the case for the general description of field theories. In order to apply formulas like (2.12) and (2.14), one has to factor the Green's functions from the Feynman amplitude, but this is generally not an easy task. Therefore in what follows we shall improve the rules so that one can apply them without factorization of the Feynman amplitudes. For this purpose the following formulas[6] are quite useful:

$$\int(dx')\Delta^{(+)}(x-x':m)K_{x'}{}^m\langle\Omega, T[\varphi(x')\varphi(y)]\Omega\rangle =\sum_{(b)}\langle\Omega, \varphi(x)\Phi_b\rangle\langle\Phi_b, \varphi(y)\Omega\rangle, \quad (2.20a)$$

$$\int(dx')\Delta^{(+)}(x-x':M) \times K_{x'}{}^M\langle\Omega, T[\varphi(x'+\tfrac{1}{2}\xi)\varphi(x'-\tfrac{1}{2}\xi)\varphi(y_1)\varphi(y_2)]\Omega\rangle =\sum_{(B)}\langle\Omega, T[\varphi(x+\tfrac{1}{2}\xi)\varphi(x-\tfrac{1}{2}\xi)]\Phi_B\rangle \times\langle\Phi_B, T[\varphi(y_1)\varphi(y_2)]\Omega\rangle, \quad (2.20b)$$

where the normalization of the Feynman amplitudes for single particle states is given by

$$\sum_{(b)}\langle\Omega, \varphi(x)\Phi_b\rangle\langle\Phi_b, \varphi(y)\Omega\rangle=i\Delta^{(+)}(x-y:m) =(2\pi)^{-3}\int(dp)e^{ip\cdot(x-y)}\theta(p_0)\delta(p^2+m^2).$$

[5] E. Freese, Nuovo cimento **11**, 312 (1954); P. Kristensen, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **28**, No. 12 (1954); Z. Maki, Progr. Theoret. Phys. Japan **15**, 237 (1956).

[6] K. Nishijima, Progr. Theoret. Phys. Japan **17**, 765 (1957).

Hence we get from (2.12) and (2.14)

$$\lim_{x\sim\infty}{}^{(b)}\langle\Omega, T[\varphi(x)\varphi(y)]\Omega\rangle=\int(dx')\Delta^{(+)}(x-x':m) \times K_{x'}{}^m\langle\Omega, T[\varphi(x')\varphi(y)]\Omega\rangle, \quad (2.21a)$$

$$\lim_{x\sim\infty}{}^{(B)}\langle\Omega, T[\varphi(x+\tfrac{1}{2}\xi)\varphi(x-\tfrac{1}{2}\xi)\varphi(y_1)\varphi(y_2)]\Omega\rangle =\int(dx')\Delta^{(+)}(x-x':M)K_{x'}{}^M \times\langle\Omega, T[\varphi(x'+\tfrac{1}{2}\xi)\varphi(x'-\tfrac{1}{2}\xi)\varphi(y_1)\varphi(y_2)]\Omega\rangle. \quad (2.21b)$$

By making use of these formulas, one can express the asymptotic operation in a compact form. However, the above operation is not a complete substitute for the $\lim^{(\beta)}$ operation, since we get

$$\int(dx')\Delta^{(+)}(x-x':m)K_{x'}{}^m\langle\Omega, \varphi(x')\Phi_b\rangle=0,$$

as a result of the Klein-Gordon equation for single-particle states, whereas the "lim" operation gives, when applied to single-particle Feynman amplitudes,

$$\lim_{x\sim\infty}{}^{(b)}\langle\Omega, \varphi(x)\Phi_b\rangle=\langle\Omega, \varphi(x)\Phi_b\rangle.$$

Hence one can substitute the above operation for lim only for contributions from such Feynman graphs that do not involve freely propagating particles. It is also worth noticing that the new operation always gives a unique result when applied to T-product Feynman amplitudes in contradistinction to the "lim" operations, since the nonunique parts vanish by this operation.

In this way we get the following expression for the S matrix:

$$\int(dx_1')(dx_2')\cdots\Delta^{(+)}(x_1-x_1':m)\Delta^{(+)}(x_2-x_2':m)\cdots \times K_{x_1'}{}^m K_{x_2'}{}^m\cdots\langle\Omega, T[\varphi(x_1')\varphi(x_2')\cdots]\Phi_\alpha{}^{(+)}\rangle =\sum_{(\beta)}S_{\beta\alpha}'F_\beta(x_1,x_2,\cdots), \quad (2.22a)$$

where

$$F_\beta(x_1,x_2,\cdots) =\sum_{\text{perm.}}\langle\Omega, \varphi(x_1)\Phi_{b_1}\rangle\langle\Omega, \varphi(x_2)\Phi_{b_2}\rangle\cdots. \quad (2.22b)$$

In the above formula we considered only elementary particles in the final state, but composite particles can also be included without difficulty. $S_{\beta\alpha}'$ is obtained from $S_{\beta\alpha}$ by dropping the contributions from graphs involving freely propagating particles, either elementary or composite. The relation between S and S' is given by

$$S_{\beta\alpha}'=\sum_\lambda(-1)^\lambda S_{\beta/\lambda,\,\alpha/\lambda} \quad\text{or}\quad S_{\beta\alpha}=\sum_\lambda S_{\beta/\lambda,\,\alpha/\lambda}'. \quad (2.23)$$

The division of a state by a single-particle state is defined generally by

$$\Phi_{a_1\cdots a_n/b}{}^{(\pm)}=\sum_{i=1}^{n}\delta(a_ib)\Phi_{a_1\cdots a_{i-1}a_{i+1}\cdots a_n}{}^{(\pm)},$$

and

$$S_{\beta/\lambda,\,\alpha/\lambda}=\langle\Phi_{\beta/\lambda}{}^{(-)},\,\Phi_{\alpha/\lambda}{}^{(+)}\rangle.$$

As an example, for the process $p+p\rightarrow d+\pi^+$, one has

$$\begin{aligned}\int(dx')(dz')\Delta^{(+)}(x-x':M)\Delta^{(+)}(z-z':\mu)\\ \times K_{x'}{}^{M}K_{z'}{}^{\mu}\langle\Omega,\,T[\psi_n(x'+\tfrac{1}{2}\xi)\psi_p(x'-\tfrac{1}{2}\xi)\varphi(z')]\Phi_{pp}{}^{(+)}\rangle\\ =\sum_{(d),(\pi^+)}\langle\Omega,\,T[\psi_n(x+\tfrac{1}{2}\xi)\psi_p(x-\tfrac{1}{2}\xi)]\Phi_d\rangle\\ \times\langle\Omega,\,\varphi(z)\Phi_{\pi^+}\rangle S(p+p\rightarrow d+\pi^+).\quad(2.24)\end{aligned}$$

The formula (2.22a) refers to a special class of matrix elements, but one can readily generalize this formula to a wider class of matrix elements. For instance, we get

$$\begin{aligned}\int(dz')\Delta^{(+)}(z-z':m)K_{z'}{}^{m}\langle\Phi_\beta{}^{(-)},\,T(x_1\cdots x_nz')\Phi_\alpha{}^{(+)}\rangle\\ =\textstyle\sum_{(b)}\langle\Omega,\,\varphi(z)\Phi_b\rangle\{\langle\Phi_{\beta b}{}^{(-)},\,T(x_1\cdots x_n)\Phi_\alpha{}^{(+)}\rangle\\ -\langle\Phi_\beta{}^{(-)},\,T(x_1\cdots x_n)\Phi_{\alpha/b}{}^{(+)}\rangle\},\quad(2.25a)\end{aligned}$$

where b is an elementary particle and $T(x_1,\cdots,x_n)$ is the abbreviation of $T[\varphi(x_1)\cdots\varphi(x_n)]$. This formula was first derived by LSZ[7] using the asymptotic condition. When B is a composite particle, we get

$$\begin{aligned}\int(dz')\Delta^{(+)}(z-z':M)K_{z'}{}^{M}\\ \times\langle\Phi_\beta{}^{(-)},\,T(x_1\cdots x_n,\,z'+\tfrac{1}{2}\xi,\,z'-\tfrac{1}{2}\xi)\Phi_\alpha{}^{(+)}\rangle\\ =\textstyle\sum_{(B)}\langle\Omega,\,T(z+\tfrac{1}{2}\xi,\,z-\tfrac{1}{2}\xi)\Phi_B\rangle\\ \times\{\langle\Phi_{\beta B}{}^{(-)},\,T(x_1\cdots x_n)\Phi_\alpha{}^{(+)}\rangle\\ -\langle\Phi_\beta{}^{(-)},\,T(x_1\cdots x_n)\Phi_{\alpha/B}{}^{(+)}\rangle\},\quad(2.25b)\end{aligned}$$

where M is the rest mass of B. The proof of these formulas will be given in Appendix A.

When some of the factor states a_1, a_2, $\cdots$ are identical, $\Phi_{a_1a_2\cdots/b}$ is generally not properly normalized, but the formula (2.25) holds for the unnormalized state vector. Now making use of the formula

$$i\Delta^{(+)}(z-z':m)=\textstyle\sum_{(b)}\langle\Omega,\,\varphi(z)\Phi_b\rangle\langle\Phi_b,\,\varphi(z')\Omega\rangle,\quad(2.26)$$

one can simplify (2.25a) by division to obtain

$$\begin{aligned}\langle\Phi_{\beta b}{}^{(-)},\,T(x_1\cdots x_n)\Phi_\alpha{}^{(+)}\rangle-\langle\Phi_\beta{}^{(-)},\,T(x_1\cdots x_n)\Phi_{\alpha/b}{}^{(+)}\rangle\\ =-i\int(dz)\langle\Phi_b,\,\varphi(z)\Omega\rangle\\ \times K_z{}^{m}\langle\Phi_\beta{}^{(-)},\,T(x_1\cdots x_nz)\Phi_\alpha{}^{(+)}\rangle.\quad(2.27a)\end{aligned}$$

For (2.25b) one cannot apply such a simple division, but instead one needs a special device. Inserting the relation

$$\begin{aligned}\Delta^{(+)}(z:M)\\ =\frac{-i}{2(2\pi)^3}\int\frac{d\mathbf{p}}{p_0}e^{ip\cdot z}=\frac{-i}{2(2\pi)^3}\sum_{(B)}e^{ip_B\cdot z},\quad(p^2+M^2=0)\end{aligned}$$

into (2.25b), we get

$$\begin{aligned}\frac{-i}{2(2\pi)^3}e^{ip_B\cdot z}\int(dz')e^{-ip_B\cdot z'}K_{z'}{}^{M}\\ \times\langle\Phi_\beta{}^{(-)},\,T(x_1\cdots x_n,\,z'+\tfrac{1}{2}\xi,\,z'-\tfrac{1}{2}\xi)\Phi_\alpha{}^{(+)}\rangle\\ =\langle\Omega,\,T(z+\tfrac{1}{2}\xi,\,z-\tfrac{1}{2}\xi)\Phi_B\rangle\{\langle\Phi_{\beta B}{}^{(-)},\,T(x_1\cdots x_n)\Phi_\alpha{}^{(+)}\rangle\\ -\langle\Phi_\beta{}^{(-)},\,T(x_1\cdots x_n)\Phi_{\alpha/B}{}^{(+)}\rangle\}.\quad(2.25\text{b}')\end{aligned}$$

From the translational invariance of the theory, the z-dependence of the Feynman amplitude for the bound state B is given by

$$\langle\Omega,\,T(z+\tfrac{1}{2}\xi,\,z-\tfrac{1}{2}\xi)\Phi_B\rangle=e^{ip_Bz}F_B(\xi).$$

Now choose a function $\tilde{f}_B(z,\xi)$ of the form

$$\tilde{f}_B(z,\xi)=e^{-ip_Bz}\tilde{g}_B(\xi),$$

which is quite arbitrary except for the normalization condition

$$\int(d\xi)\tilde{g}_B(\xi)F_B(\xi)=1/2(2\pi)^3,$$

then integrate (2.25b′) with respect to ξ after multiplying by $\tilde{f}_B(z,\xi)$ on both sides. Then one finds

$$\begin{aligned}\langle\Phi_{\beta B}{}^{(-)},\,T(x_1\cdots x_n)\Phi_\alpha{}^{(+)}\rangle-\langle\Phi_\beta{}^{(-)},\,T(x_1\cdots x_n)\Phi_{\alpha/B}{}^{(+)}\rangle\\ =-i\int(dz)(d\xi)\tilde{f}_B(z,\xi)K_z{}^{M}\\ \times\langle\Phi_\beta{}^{(-)},\,T(x_1\cdots x_n,\,z+\tfrac{1}{2}\xi,\,z-\tfrac{1}{2}\xi)\Phi_\alpha{}^{(+)}\rangle,\quad(2.27\text{b})\end{aligned}$$

for an arbitrary function $\tilde{f}_B(z,\xi)$ which satisfies

$$\int\tilde{f}_B(z,\xi)\langle\Omega,\,T(z+\tfrac{1}{2}\xi,\,z-\tfrac{1}{2}\xi)\Phi_B\rangle(d\xi)=1/2(2\pi)^3.\quad(2.28)$$

The recursion formula of the form (2.27b) has already been discussed before,[6] and it is has been proved that the normalization condition (2.28) is an inevitable consequence of Eq. (2.27b). What is essentially new in the present paper is the recognition that the function $\tilde{f}_B(z,\xi)$ can be quite arbitrary except for the normalization condition.

Corresponding to (2.28) we introduce another function $f_B(z,\xi)$ satisfying

$$\int f_B(z,\xi)\langle\Phi_B,\,T(z+\tfrac{1}{2}\xi,\,z-\tfrac{1}{2}\xi)\Omega\rangle(d\xi)=1/2(2\pi)^3.\quad(2.29)$$

[7] Lehmann, Symanzik, and Zimmermann, Nuovo cimento 1, 205 (1955); G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570, 1579 (1956).

If the CPT theorem holds, we get for spinless composite particle B the relation

$$\tilde{f}_B(z,\xi)=f_B(-z,\xi). \tag{2.30}$$

3. SOME THEOREMS ON THE RECURSION FORMULAS

The main results obtained in the previous section are the recursion formulas (2.27a) and (2.27b) which correspond to the asymptotic conditions in the LSZ theory.[7] In connection with the problem of meson-nucleon scattering, these formulas were also utilized by Chew and Low.[7] In the previous section these formulas were derived in an intuitive manner, and it seems to be worth while to examine their self-consistency. Without loss of generality we consider only a neutral spinless field. We shall give here some basic assumptions which form the basis of the following discussions.

Assumption I.—The theory is invariant under the proper inhomogeneous Lorentz transformations.

Hence we may introduce the energy-momentum four-vector P_μ. The vacuum state Ω satisfies the equation

$$P_\mu\Omega=0. \tag{3.1}$$

Let a single particle state a, either elementary or composite, satisfy the eigenvalue equation

$$P_\mu\Phi_a=(p_a)_\mu\Phi_a; \tag{3.2}$$

then a general state $\Phi_{a_1\cdots a_n}{}^{(\pm)}$ satisfies

$$P_\mu\Phi_{a_1\cdots a_n}{}^{(\pm)}=(p_{a_1}+\cdots+p_{a_n})_\mu\Phi_{a_1\cdots a_n}{}^{(\pm)}. \tag{3.3}$$

From the completeness of the set $\{\Phi^{(+)}\}$ or $\{\Phi^{(-)}\}$, the positive energy condition is always satisfied.

Assumption II.—The local commutativity condition

$$[\varphi(x),\varphi(y)]=0 \quad \text{for} \quad (x-y)^2>0 \tag{3.4}$$

is satisfied.

Assumption III.—The asymptotic conditions are satisfied. By asymptotic conditions we mean the recursion formulas (2.27).

Assumption IV.—If an operator O satisfies

$$[O,\varphi(x)]=0 \tag{3.5}$$

at any space-time point x, then O is a c-number. We shall call this property the irreducibility condition.

For later convenience we shall recapitulate the asymptotic conditions.

$$\langle\Phi_\beta{}^{(-)},T(x_1\cdots x_n)\Phi_{\alpha b}{}^{(+)}\rangle-\langle\Phi_{\beta/b}{}^{(-)},T(x_1\cdots x_n)\Phi_\alpha{}^{(+)}\rangle$$
$$=-i\int(dy)\langle\Omega,\varphi(y)\Phi_b\rangle$$
$$\times K_y\langle\Phi_\beta{}^{(-)},T(x_1\cdots x_ny)\Phi_\alpha{}^{(+)}\rangle, \tag{3.6a}$$

$$\langle\Phi_{\beta b}{}^{(-)},T(x_1\cdots x_n)\Phi_\alpha{}^{(+)}\rangle-\langle\Phi_\beta{}^{(-)},T(x_1\cdots x_n)\Phi_{\alpha/b}{}^{(+)}\rangle$$
$$=-i\int(dz)\langle\Phi_b,\varphi(z)\Omega\rangle$$
$$\times K_z\langle\Phi_\beta{}^{(-)},T(x_1\cdots x_nz)\Phi_\alpha{}^{(+)}\rangle. \tag{3.6b}$$

Equation (3.6b) is the same as (2.27a), and (3.6a) is derived in a similar way to (3.6b) but by making use of the rules for obtaining the asymptotic forms for $t\to-\infty$. For a composite particle B, we get similar formulas. In the formula corresponding to (3.6a) one has to use f_B, whereas in the one corresponding to (3.6b) one needs $\tilde{f}_B$. In the latter case the precise form has been given by (2.27b), and so we do not explicitly write them down.

Next, applying complex conjugation to (3.6a), one finds

$$\langle\Phi_{\alpha b}{}^{(+)},\tilde{T}(x_1\cdots x_n)\Phi_\beta{}^{(-)}\rangle-\langle\Phi_\alpha{}^{(+)},\tilde{T}(x_1\cdots x_n)\Phi_{\beta/b}{}^{(-)}\rangle$$
$$=i\int(dy)\langle\Phi_b,\varphi(y)\Omega\rangle$$
$$\times K_y\langle\Phi_\alpha{}^{(+)},\tilde{T}(x_1\cdots x_nz)\Phi_\beta{}^{(-)}\rangle, \tag{3.7}$$

and a similar one from (3.6b). $\tilde{T}$ is the antichronological ordering operator. We shall call (3.6) and (3.7) the recursion formulas for T products. In what follows we introduce recursion formulas for retarded products and for advanced products. The retarded or R-product is defined by[8]

$$R(x:x_1\cdots x_n)=R[\varphi(x):\varphi(x_1)\cdots\varphi(x_n)]$$
$$=\sum_{\text{perm.}}(-i)^n\theta(x-x_1)\theta(x_1-x_2)\cdots\theta(x_{n-1}-x_n)$$
$$\times[\cdots[[\varphi(x),\varphi(x_1)]\varphi(x_2)]\cdots\varphi(x_n)], \tag{3.8}$$

where $\theta(x)=1$ for $x_0>0$, $\theta(x)=\frac{1}{2}$ for $x_0=0$ and $\theta(x)=0$ for $x_0<0$, and the summation should be taken over all possible permutations of $x_1, x_2, \cdots, x_n$. We introduce a functional $\varphi_R(x)$ by

$$\varphi_R(x)=U^{-1}T[U\varphi(x)], \tag{3.9}$$

where U is the functional defined by (2.16). Then $\varphi_R(x)$ is the generating functional of R-products as given by

$$R(x:x_1\cdots x_n)=\left[\frac{\delta^n\varphi_R(x)}{\delta Q(x_1)\cdots\delta Q(x_n)}\right]_{Q=0}. \tag{3.10}$$

From (2.17), (3.9), and (3.10) one can readily derive

$$R(x:x_1\cdots x_n)=\sum_{\text{comb.}(x_i)}(-i)^n(-1)^k$$
$$\times\tilde{T}(x_1\cdots x_k)T(xx_{k+1}\cdots x_n). \tag{3.11}$$

[8] See reference 6. Also see Lehmann, Symanzik, and Zimmermann, Nuovo cimento 6, 319 (1957); Glaser, Lehmann, and Zimmermann, Nuovo cimento 6, 1122 (1957).

Hence one gets

$$\langle\Phi_\beta^{(+)}, R(x:x_1\cdots x_ny)\Phi_\alpha^{(+)}\rangle=\sum_\gamma\sum_{\text{comb.}}(-i)^{n+1}(-1)^k$$
$$\times[\langle\Phi_\beta^{(+)}, \tilde{T}(x_1\cdots x_k)\Phi_\gamma^{(-)}\rangle$$
$$\times\langle\Phi_\gamma^{(-)}, T(xx_{k+1}\cdots x_ny)\Phi_\alpha^{(+)}\rangle$$
$$-\langle\Phi_\beta^{(+)}, \tilde{T}(x_1\cdots x_ky)\Phi_\gamma^{(-)}\rangle$$
$$\times\langle\Phi_\gamma^{(-)}, T(xx_{k+1}\cdots x_n)\Phi_\alpha^{(+)}\rangle].$$

Now, applying the operation $\int(dy)\langle\Omega, \varphi(y)\Phi_b\rangle\cdot K_y$ on both sides of the above equation and making use of (3.6) and (3.7), one finds

$$\int(dy)\langle\Omega, \varphi(y)\Phi_b\rangle K_y\langle\Phi_\beta^{(+)}, R(x:x_1\cdots x_ny)\Phi_\alpha^{(+)}\rangle$$
$$=\langle\Phi_\beta^{(+)}, R(x:x_1\cdots x_n)\Phi_{\alpha b}^{(+)}\rangle$$
$$-\langle\Phi_{\beta/b}^{(+)}, R(x:x_1\cdots x_n)\Phi_\alpha^{(+)}\rangle, \quad (3.12\text{a})$$

and similarly

$$\int(dz)\langle\Phi_b, \varphi(z)\Omega\rangle K_z\langle\Phi_\beta^{(+)}, R(x:x_1\cdots x_nz)\Phi_\alpha^{(+)}\rangle$$
$$=\langle\Phi_{\beta b}^{(+)}, R(x:x_1\cdots x_n)\Phi_\alpha^{(+)}\rangle$$
$$-\langle\Phi_\beta^{(+)}, R(x:x_1\cdots x_n)\Phi_{\alpha/b}^{(+)}\rangle. \quad (3.12\text{b})$$

We call these the recursion formulas for R-products. One can also easily derive similar formulas for composite particles.

The advanced or A-product is defined by

$$A[\varphi(x):\varphi(x_1)\cdots\varphi(x_n)]$$
$$=\sum_{\text{perm.}}(-i)^n\theta(x_1-x)\theta(x_2-x_1)\cdots\theta(x_n-x_{n-1})$$
$$\times[\cdots[[\varphi(x), \varphi(x_1)]\varphi(x_2)]\cdots\varphi(x_n)]. \quad (3.13)$$

In an analogous way to the case of R-products, one can derive the recursion formulas for A-products.

$$\langle\Phi_\beta^{(-)}, A(x:x_1\cdots x_n)\Phi_{\alpha b}^{(-)}\rangle$$
$$-\langle\Phi_{\beta/b}^{(-)}, A(x:x_1\cdots x_n)\Phi_\alpha^{(-)}\rangle$$
$$=-\int(dy)\langle\Omega, \varphi(y)\Phi_b\rangle$$
$$\times K_y\langle\Phi_\beta^{(-)}, A(x:x_1\cdots x_ny)\Phi_\alpha^{(+)}\rangle, \quad (3.14\text{a})$$

$$\langle\Phi_{\beta b}^{(-)}, A(x:x_1\cdots x_n)\Phi_\alpha^{(-)}\rangle$$
$$-\langle\Phi_\beta^{(-)}, A(x:x_1\cdots x_n)\Phi_{\alpha/b}^{(-)}\rangle$$
$$=-\int(dz)\langle\Phi_b, \varphi(z)\Omega\rangle$$
$$\times K_z\langle\Phi_\beta^{(-)}, A(x:x_1\cdots x_nz)\Phi_\alpha^{(-)}\rangle. \quad (3.14\text{b})$$

It must be noticed that (3.12b) and (3.14b) are obtained by applying complex conjugation to (3.12a) and (3.14a). This is in general the case even when composite particles participate in the theory, but this property in turn imposes a restriction on the possible choice of the function $f_B(z,\xi)$. Indeed the recursion formulas (3.12) and (3.14) cannot be proved for an arbitrary choice of the function $f_B(z,\xi)$, but one has to choose a function $f_B(z,\xi)$ such that it satisfies

$$\tilde{f}_B(z,\xi)=f_B^*(z,\xi).$$

Because of the local commutativity condition the above relation can be satisfied if we assume a function of the following type:

$$f_B(z,\xi)=0 \quad \text{for} \quad \xi^2<0.$$

Hence we arrive at the following theorem:

Theorem I.—The recursion formulas for R- and A-products follow from those for T-products if one chooses a function $f_B(z,\xi)$ such that it vanishes for time-like ξ.

The theorem that states the inverse of Theorem I is as follows:

Theorem II.—The recursion formulas for the T-products follow from those for R- and A-products provided that the irreducibility condition is satisfied. In this case too, the function $f_B(z,\xi)$ should be chosen under the condition given in Theorem I.

The proof of this theorem is elementary but rather tedious, and is given in Appendix B.

Whenever we are concerned with the relations between different kinds of products, we have to choose a special type of $f_B(z,\xi)$. We shall call this special choice the space-like gauge, since the function $f_B(z,\xi)$ does not vanish only for space-like ξ.

Next we shall examine the self-consistency of the recursion formulas, and we begin with the following theorem:

Theorem III.—If the recursion formulas for R-products hold, then the vacuum expectation values of the R-products of field operators should satisfy the following set of integral equations:

$$r(x:yx_1\cdots x_n)-r(y:xx_1\cdots x_n)+i\sum_{\text{comb. }(x_i)}\sum_{l,m}\frac{i^{l+m}}{l!m!}\prod_i^l\int(du_i)(dv_i)\prod_j^m\int(dX_j)(dY_j)(d\xi_j)(d\eta_j)$$
$$\times[K_{u_1}\cdots K_{u_l}K_{X_1}\cdots K_{X_m}r(x:x_1\cdots x_ku_1\cdots u_l, X_1+\tfrac{1}{2}\xi_1, X_1-\tfrac{1}{2}\xi_1, \cdots, X_m+\tfrac{1}{2}\xi_m, X_m-\tfrac{1}{2}\xi_m)$$
$$\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)\Delta^{(+)}(X_1-Y_1:\xi_1\eta_1)\cdots\Delta^{(+)}(X_m-Y_m:\xi_m\eta_m)$$
$$\times K_{v_1}\cdots K_{v_l}K_{Y_1}\cdots K_{Y_m}r(y:x_{k+1}\cdots x_nv_1\cdots v_l, Y_1+\tfrac{1}{2}\eta_1, Y_1-\tfrac{1}{2}\eta_1, \cdots, Y_m+\tfrac{1}{2}\eta_m, Y_m-\tfrac{1}{2}\eta_m)-(x\rightleftarrows y)]=0, \quad (3.15)$$

where $r(x:x_1\cdots x_n)=\langle\Omega, R(x:x_1\cdots x_n)\Omega\rangle$, and the Klein-Gordon operators K_u and K_v refer to the elementary particle and K_X and K_Y refer to the composite particle. The function $\Delta^{(+)}(u-v)$ also refers to the elementary particle, and $\Delta^{(+)}(X-Y:\xi)$ is defined by

$$i\Delta^{(+)}(X-Y:\xi\eta)=\sum_{(B)} f_B(X,\xi)f_B{}^*(Y,\eta).$$

The boundary conditions to be imposed upon the above equations are

(1) $r(x:x_1\cdots x_n)$ is symmetric in $x_1,\cdots,x_n$.
(2) $r(x:x_1\cdots x_n)$ is a real Lorentz-invariant function.
(3) $r(x:x_1\cdots x_n)=0$, if $x<x_i$ for any x_i.

If there are no bound states, the equations are reduced into much simpler ones.[8]

$$\begin{aligned}&r(x:yx_1\cdots x_n)-r(y:xx_1\cdots x_n)\\&+i\sum_{\text{comb.}}\sum_l\frac{i^l}{l!}\prod_j^l\int(du_j)(dv_j)\\&\times[K_{u1}\cdots K_{ul}r(x:x_1\cdots x_ku_1\cdots u_l)\\&\times\Delta^{(+)}(u_1-v_1)\cdots\Delta^{(+)}(u_l-v_l)K_{v1}\cdots K_{vl}r\\&\qquad\times(y:x_{k+1}\cdots x_nv_1\cdots v_l)-(x\rightleftarrows y)]=0.\end{aligned}\quad(3.16)$$

Since this theorem has been proved for the simpler case (3.16), we shall briefly sketch the proof. We start from the recursion formula:

$$\begin{aligned}&R(x:yx_1\cdots x_n)-R(y:xx_1\cdots x_n)\\&+i\sum_{\text{comb.}}[R(x:x_1\cdots x_k),R(y:x_{k+1}\cdots x_n)]=0,\end{aligned}\quad(3.17a)$$

$$R(x:yx_1\cdots x_n)=0\quad\text{if}\quad x<y.\quad(3.17b)$$

This is equivalent to the definition of the R-product provided that it is symmetric in the variables standing to the right of the colon. Take the vacuum-vacuum matrix element of (3.17a); then the second term involves a summation over all possible outgoing-wave stationary states $\{\Phi_\alpha{}^{(+)}\}$. By the repeated use of the recursion formulas, one can express the matrix element of an R-product between vacuum and an outgoing wave state $\Phi_\alpha{}^{(+)}$ in terms of the vacuum expectation value of the R-product.

$$\begin{aligned}&\langle\Omega,R(x:x_1\cdots x_n)\Phi_{a_1\cdots a_m}{}^{(+)}\rangle\\&=\int(dy_1)\cdots(dy_m)K_{y1}\cdots K_{ym}\\&\times\langle\Omega,R(x:x_1\cdots x_ny_1\cdots y_m)\Omega\rangle\\&\qquad\times\langle\Omega,\varphi(y_1)\Phi_{a_1}\rangle\cdots\langle\Omega,\varphi(y_m)\Phi_{a_m}\rangle,\end{aligned}\quad(3.18a)$$

$$\begin{aligned}&\langle\Phi_{a_1\cdots a_m}{}^{(+)},R(x:x_1\cdots x_n)\Omega\rangle\\&=\int(dz_1)\cdots(dz_m)K_{z1}\cdots K_{zm}\\&\times\langle\Omega,R(x:x_1\cdots x_nz_1\cdots z_m)\Omega\rangle\\&\qquad\times\langle\Phi_{a_1},\varphi(z_1)\Omega\rangle\cdots\langle\Phi_{a_m},\varphi(z_m)\Omega\rangle.\end{aligned}\quad(3.18b)$$

Inserting the above expressions into the vacuum-vacuum matrix element of (3.17a), one arrives at (3.15). The extension of (3.18) to composite particles is clear.

What is important in the present formulation is the inverse of Theorem III.

Theorem IV.—If a set of functions $\{r(x:x_1\cdots x_n)\}$ satisfying Eqs. (3.15) as well as the three boundary conditions be given, then one can construct a field operator $\varphi(x)$ so as to satisfy the recursion formulas for R-products. We shall again only briefly sketch the proof. Introduce the following notations:

$$D_z{}^b=\int(dz)\langle\Omega,\varphi(z)\Phi_b\rangle K_z,\quad\bar{D}_z{}^b=\int(dz)\langle\Phi_b,\varphi(z)\Omega\rangle K_z,$$

$$D_{z,\xi}{}^B=\int(dz)(d\xi)f_B(z,\xi)K_z{}^M,$$

$$\bar{D}_{z,\xi}{}^B=\int(dz)(d\xi)f_B{}^*(z,\xi)K_z{}^M,$$

and further

$$D^\beta=D^{b_1}\cdots D^{b_m},\quad\bar{D}^\beta=\bar{D}^{b_1}\cdots\bar{D}^{b_m},$$

when $\beta=b_1\cdots b_m$. Unless there might be some confusion we shall spare the subscript z or z,ξ. Then with this notation, Eq. (3.18) assumes the form

$$\langle\Omega,R(x:x_1\cdots x_n)\Phi_\alpha{}^{(+)}\rangle=D^\alpha\langle\Omega,R(x:x_1\cdots x_n,\cdots)\Omega\rangle.$$

By the repeated use of the recursion formula, we get

$$\langle\Phi_\beta{}^{(+)},\varphi(x)\Phi_\alpha{}^{(+)}\rangle=\textstyle\sum_\lambda D^{\alpha/\lambda}\bar{D}^{\beta/\lambda}\langle\Omega,R(x:\cdots)\Omega\rangle.\quad(3.19)$$

Now when Eqs. (3.15) are satisfied by $\{r(x:x_1\cdots x_n)\}$, we shall define the field operator $\varphi(x)$ through

$$\langle\Phi_\beta{}^{(+)},\varphi(x)\Phi_\alpha{}^{(+)}\rangle=\textstyle\sum_\lambda D^{\alpha/\lambda}\bar{D}^{\beta/\lambda}r(x:\cdots);\quad(3.20)$$

then with the help of mathematical induction one can prove in a straightforward manner that the recursion formulas for R-products are satisfied. In this connection we shall notice that the theorem is true independently of the form of the function $f_B(z,\xi)$ provided that the normalization condition is satisfied. As for the details on this point we shall refer to Sec. 4.

Thus the self-consistency of the recursion formulas is reduced to the existence of the solution of the integral equations (3.15).

Theorem V.—The recursion formulas for A-products are equivalent to those for R-products provided that the CPT theorem holds.

Proof: As has been proved in the previous two theorems, the recursion formulas for R-products are equivalent to the integral equations (3.15). Similarly one can prove the equivalence between the recursion formulas for A-products and corresponding integral equations for the functions $\{a(x:x_1\cdots x_n)\}=\{\langle\Omega, A(x:x_1\cdots x_n)\Omega\rangle\}$. It is important that the functions a satisfy exactly the same equations and boundary conditions (1) and (2). The third boundary condition turns out to be

$$a(x:x_1\cdots x_n)=0 \quad \text{if} \quad x>x_i \text{ for any } x_i.$$

Now the CPT theorem in Jost form is given by

$$\langle\Omega, \varphi(x_1)\cdots\varphi(x_n)\Omega\rangle=\langle\Omega, \varphi(-x_n)\cdots\varphi(-x_1)\Omega\rangle, \quad (3.21)$$

from which the following relation is derived:

$$a(x:x_1\cdots x_n)=(-1)^n r(-x:-x_1,\cdots,-x_n). \quad (3.22)$$

One can directly check that if the functions $\{r(x:x_1\cdots x_n)\}$ satisfy the integral equations (3.15), then so do the functions $\{(-1)^n r(-x:-x_1\cdots -x_n)\}$. It is necessary, however, to choose the space-like gauge for $f_B(x,\xi)$ so that one can utilize

$$\Delta^{(+)}(X-Y:\xi\eta)=\Delta^{(+)}(X-Y:\eta\xi),$$

which results from the choice that $g_B(\xi)=f_B(0,\xi)=e^{-ip_B\cdot z}f_B(z,\xi)$ is a real even function of ξ. As has been proved by Jost,[9] (3.21) follows from the assumptions I, II, and the positive energy condition.

Hence, combining Theorems I, II, and V, we get:

Theorem VI.—Under the assumptions I, II, and IV all three kinds of recursion formulas for T-, R-, and A- products are equivalent to each other, so that Eqs. (3.15) guarantee the existence of two sets $\{\Phi^{(+)}\}$ and $\{\Phi^{(-)}\}$ and consequently the existence of a unitary S matrix.

4. DEVIATION OF THE S MATRIX

The explicit form of the S matrix follows from the recursion formulas for T-products. Before entering into the question how one can express the S matrix we shall first discuss related problems. We shall first discuss the arbitrariness of the function $f_B(z,\xi)$, and for this purpose we generalize Eqs. (2.20).

Let a be a stable particle either elementary or composite; then generally we get

$$\begin{aligned}\int(dz')\Delta^{(+)}(z-z':M_a)K_{z'}{}^a\\ \times\langle\Omega, T(z'+\xi_1,\cdots,z'+\xi_m,y_1\cdots y_n)\Omega\rangle\\ =\textstyle\sum_{(a)}\langle\Omega, T(z+\xi_1,\cdots z+\xi_m)\Phi_a\rangle\\ \times\langle\Phi_a, T(y_1\cdots y_n)\Omega\rangle, \quad (4.1)\end{aligned}$$

where

$$\sum\xi_i=0.$$

[9] R. Jost, Helv. Phys. Acta 30, 40 (1957).

Or, taking a special Fourier component, one can derive

$$\begin{aligned}\frac{-i}{2(2\pi)^3}e^{\pm ip_a\cdot z}\int(dz')e^{\mp ip_a\cdot z'}K_{z'}{}^a\\ \times\langle\Omega, T(z'+\xi_1,\cdots z'+\xi_m,y_1\cdots y_n)\Omega\rangle\\ =\langle\Omega, T(z+\xi_1,\cdots z+\xi_m)\Phi_a\rangle\langle\Phi_a, T(y_1\cdots y_n)\Omega\rangle\\ \text{for the upper sign,}\\ =\langle\Omega, T(y_1\cdots y_n)\Phi_a\rangle\langle\Phi_a, T(z+\xi_1,\cdots z+\xi_m)\Omega\rangle\\ \text{for the lower sign.} \quad (4.1')\end{aligned}$$

The recursion formula for the composite particle B is given by

$$\begin{aligned}\langle\Phi_\beta{}^{(-)}, T(x_1\cdots x_n)\Phi_{\alpha B}{}^{(+)}\rangle-\langle\Phi_{\beta/B}{}^{(-)}, T(x_1\cdots x_n)\Phi_\alpha{}^{(+)}\rangle\\ =-i\int(dz)(d\xi)f_B(z,\xi)K_z{}^M\\ \times\langle\Phi_\beta{}^{(-)}, T(x_1\cdots x_n, z+\tfrac{1}{2}\xi, z-\tfrac{1}{2}\xi)\Phi_\alpha{}^{(+)}\rangle. \quad (4.2)\end{aligned}$$

Now we shall prove the arbitrariness of the function f_B, and without loss of generality we shall show this arbitrariness in the case $\Phi_\beta{}^{(-)}=\Omega$.

Take arbitrarily an f_B' that satisfies the normalization condition

$$\int(d\xi)f_B'(z,\xi)\langle\Phi_B, T(z+\tfrac{1}{2}\xi, z-\tfrac{1}{2}\xi)\Omega\rangle=1/2(2\pi)^3;$$

then what we have to prove is the relation

$$\begin{aligned}-i\int(dz)(d\xi)f_B'(z,\xi)K_z{}^M\\ \times\langle\Omega, T(x_1\cdots x_n, z+\tfrac{1}{2}\xi, z-\tfrac{1}{2}\xi)\Phi_\alpha{}^{(+)}\rangle\\ =\langle\Omega, T(x_1\cdots x_n)\Phi_{\alpha B}{}^{(+)}\rangle.\end{aligned}$$

With the help of the recursion formula and the notation D^α introduced in Sec. 3, we have

$$\begin{aligned}\langle\Omega, T(x_1\cdots x_n, z+\tfrac{1}{2}\xi, z-\tfrac{1}{2}\xi)\Phi_\alpha\rangle\\ =(-i)^\alpha D^\alpha\langle\Omega, T(x_1\cdots x_n, z+\tfrac{1}{2}\xi, z-\tfrac{1}{2}\xi, y_1\cdots y_\alpha)\Omega\rangle,\end{aligned}$$

where the y's are the arguments upon which D^α operates. Now applying $\int(dz)(d\xi)f_B'(z,\xi)K_z{}^M$ to the above equation, we have with the help of (4.1′)

$$\begin{aligned}(-i)^\alpha D^\alpha\cdot(-i)\int(dz)(d\xi)f_B'(z,\xi)K_z{}^M\\ \times\langle\Omega, T(x_1\cdots x_n, z+\tfrac{1}{2}\xi, z-\tfrac{1}{2}\xi, y_1\cdots y_\alpha)\Omega\rangle\\ =(-i)^\alpha D^\alpha\cdot\Big(2(2\pi)^3\int(d\xi)f_B'(z,\xi)\\ \times\langle\Phi_B, T(z+\tfrac{1}{2}\xi, z-\tfrac{1}{2}\xi)\Omega\rangle\Big)\langle\Omega, T(x_1\cdots x_n y_1\cdots y_\alpha)\Phi_B\rangle\\ =(-i)^\alpha D^\alpha\langle\Omega, T(x_1\cdots x_n y_1\cdots y_\alpha)\Phi_B\rangle\\ =\langle\Omega, T(x_1\cdots x_n)\Phi_{\alpha B}{}^{(+)}\rangle.\end{aligned}$$

Thus the arbitrariness of the function f_B is proved. Eventually f_B can also be an operator as we shall discuss in the next section. The arbitrariness of f_B can also be proved in the case of R- and A-products with the aid of relations like

$$\int (dz')\Delta^{(+)}(z-z':M_a)K_{z'}{}^a \times\langle\Omega, R(x:z'+\xi_1,\cdots,z'+\xi_m,y_1\cdots y_n)\Omega\rangle = i^m\sum_{(a)}\langle\Omega,\tilde{T}(z+\xi_1,\cdots,z+\xi_m)\Phi_a\rangle \times\langle\Phi_a, R(x:y_1\cdots y_n)\Omega\rangle,$$

or

$$\int (dz')\Delta^{(+)}(z'-z:M_a)K_{z'}{}^a \times\langle\Omega, R(x:z'+\xi_1,\cdots,z'+\xi_m,y_1\cdots y_n)\Omega\rangle = (-i)^m\sum_{(a)}\langle\Omega,R(x:y_1\cdots y_n)\Phi_a\rangle \times\langle\Phi_a, T(z+\xi_1,\cdots,z+\xi_m)\Omega\rangle.$$

Next let us consider two interacting neutral spinless fields φ and ψ. Suppose that we have

$$\langle\Omega, T[\psi(x)\psi(y)]\Phi_a\rangle\neq 0,$$

where Φ_a represents a one φ-quantum state. The usual recursion formula for the φ-field is given by

$$\langle\Phi_\beta{}^{(+)}, T(\cdots)\Phi_{\alpha a}{}^{(+)}\rangle-\langle\Phi_{\beta/a}{}^{(-)}, T(\cdots)\Phi_\alpha{}^{(+)}\rangle = -i\int(dz)\langle\Omega,\varphi(z)\Phi_a\rangle K_z{}^a \times\langle\Phi_\beta{}^{(-)}, T[\cdots\varphi(z)]\Phi_\alpha{}^{(+)}\rangle, \quad (4.3)$$

but in this case one can also derive

$$\langle\Phi_\beta{}^{(-)}, T(\cdots)\Phi_{\alpha a}{}^{(+)}\rangle-\langle\Phi_{\beta/a}{}^{(-)}, T(\cdots)\Phi_\alpha{}^{(+)}\rangle = -i\int(dz)(d\xi)f_a(z,\xi)K_z{}^a \times\langle\Phi_\beta{}^{(-)}, T[\cdots\psi(z+\tfrac{1}{2}\xi)\psi(z-\tfrac{1}{2}\xi)]\Phi_\alpha{}^{(+)}\rangle, \quad (4.4)$$

provided that the normalization condition is satisfied, i.e.,

$$\int(d\xi)f_a(z,\xi) \times\langle\Phi_a, T[\psi(z+\tfrac{1}{2}\xi)\psi(z-\tfrac{1}{2}\xi)]\Omega\rangle = 1/2(2\pi)^3. \quad (4.5)$$

Equation (4.4) can be proved using exactly the same technique as that used in the proof of the arbitrariness of f_B. The recursion formula (4.4) shows that within the framework of our formulation the φ-field quantum can be treated also as a composite particle consisting of ψ-field quanta. Therefore the integral equations (3.16) for two fields without bound states can be transformed into the integral equations (3.15) for a single field with bound states. What is interesting is the fact that the inverse is also true, namely a composite particle can always be described as the quantum of a new field. The problem is to prove (4.3) starting from (4.4), and since we have to prove it also for other products we shall choose the space-like gauge for the function f_B. Hence we have

$$f_a(z,\xi)=e^{ip_a\cdot z}g_a(\xi),$$

where $g_a(\xi)$ is a real even function of ξ and survives only for space-like ξ. Now in order to carry out this program we take a special form of $g_a(\xi)$:

$$g_a(\xi)=\delta(\xi)G_a(\xi),$$

where $G_a(\xi)$ is a Lorentz-invariant function of ξ and p_a. Then the normalization condition turns out to be

$$\int(d\xi)\delta(\xi)G_a(\xi)\langle\Phi_a, T[\psi(\tfrac{1}{2}\xi)\psi(-\tfrac{1}{2}\xi)]\Omega\rangle = \lim_{\xi\to 0}G_a(\xi)\langle\Phi_a, T[\psi(\tfrac{1}{2}\xi)\psi(-\tfrac{1}{2}\xi)]\Omega\rangle = 1/2(2\pi)^3.$$

The function $G_a(\xi)$ should be so chosen as to cancel the singularity of $\langle\Phi_a, T[\psi(\frac{1}{2}\xi)\psi(-\frac{1}{2}\xi)]\Omega\rangle$ at $\xi=0$, and the limit should be taken for space-like ξ in conformity with the space-like gauge. A possible choice of $G_a(\xi)$ is

$$G_a(\xi)=\theta(\xi^2)[2(2\pi)^3\langle\Phi_a, T[\psi(\tfrac{1}{2}\xi)\psi(-\tfrac{1}{2}\xi)]\Omega\rangle]^{-1}. \quad (4.6)$$

If we insert this function $G_a(\xi)$ into (4.4), we get (4.3) again with the help of (4.1′), provided that $\varphi(x)$ is defined by

$$\varphi(x)=\frac{1}{(2(2\pi)^3)^{\frac{1}{2}}}\lim_{\xi\to 0}\frac{N[\psi(x+\frac{1}{2}\xi)\psi(x-\frac{1}{2}\xi)]}{\langle\Phi_a, T[\psi(\frac{1}{2}\xi)\psi(-\frac{1}{2}\xi)]\Omega\rangle}, \quad (4.7)$$

where again the vector ξ whould approach to zero from a space-like direction. We took here the normal product of field operators in accordance with the relation $\langle\Omega, \varphi(x)\Omega\rangle=0$ required from the recursion formula. The independence of the field operator $\varphi(x)$ on the direction p_a is guaranteed if the limiting value (4.7) is independent of the direction in which ξ approaches to zero. Since we choose the space-like gauge, the same argument applies to R- and A-products. Similar results have been obtained also by Zimmermann[10] and by Haag.[10]

Irreducible Set of Fields

As shown above, there is no essential difference between elementary and composite particles, and it is to some extent a matter of convention to call a particle elementary or composite.[11] Therefore we shall look for a substitute for the concept of elementary particles.

[10] W. Zimmermann (private communication); R. Haag (private communication).

[11] The author is indebted to Professor W. Heisenberg for valuable discussions on this point.

Let us consider a set of fields $(\varphi_1,\varphi_2,\cdots,\varphi_n)$, then some of them might be treated as composite in the sense mentioned above. Now choose a subset $(\varphi_a,\varphi_b,\cdots)$ from the whole set, and if the equation

$$[O,\ \varphi_a(x)]=[O,\ \varphi_b(x)]=\cdots=0 \tag{4.8}$$

which holds at any space-time point x requires that O be a c-number, we shall call this subset an irreducible set. In general there are many ways to select an irreducible set. For instance, a set consisting of an irreducible set and some other fields is also an irreducible set. If the whole set is divided into two subsets S_1 and S_2, one can regard the quanta belonging to S_1 as elementary and those belonging to S_2 as composite only when S_1 is an irreducible set, since otherwise Assumption IV of our formulation is violated. For example, take the pion field φ and the nucleon field ψ. If there is no interaction between them, the only irreducible set is (φ,ψ), while if there is an interaction of the Yukawa type the possible irreducible sets are (φ,ψ) and (ψ). In the former case both the pion and the nucleon should be regarded as elementary, while in the latter case the pion might be regarded as composite.

Field Equations

In our formulation the equation of motion is given by (3.15), and once the field operators are decided on, one might hope to obtain the field equations of the usual type. It is, however, not clear if it is always possible, since local field equations involve products of field operators at the same space-time point and hence bring about divergent constants in the equations. In this respect our equations involving no divergent constant seem to have a great advantage over the field equations. The defect of our approach consists in the lack of knowledge of the detailed nature of interactions, but the author hopes that it might be overcome to some extent. Indeed it might be possible, at least in special cases, to determine the theory almost uniquely by imposing some additional restrictions upon the integral equations. For instance, in quantum electrodynamics it might suffice to assume the nonexistence of bound states and gauge invariance, as suggested by perturbation theory.[6]

Quantization

Finally it must be noticed that our theory is quantized through the recursion formulas which give relations between fields and particles. It is not yet clear if our quantization procedure leads to the usual one, namely

$$[\varphi(x),\ \dot{\varphi}(y)]=c\text{-number} \tag{4.9}$$

for equal time.

Now we are in a position to write down the explicit form of the S matrix. By the repeated use of the recursion formulas for T-products we get

$$S_{\beta\alpha}=\sum_\lambda(-i)^\alpha(-i)^\beta D^{\alpha/\lambda}\tilde{D}^{\beta/\lambda}\langle\Omega,\ T(\cdots)\Omega\rangle, \tag{4.10}$$

or

$$S_{\beta\alpha}=\sum_\lambda(-i)^{\beta/\lambda}\tilde{D}^{\beta/\lambda}\langle\Omega,\ T(\cdots)\Phi_{\alpha/\lambda}{}^{(+)}\rangle \tag{4.11a}$$

$$=\sum_\lambda(-i)^{\alpha/\lambda}D^{\alpha/\lambda}\langle\Phi_{\beta/\lambda}{}^{(-)},\ T(\cdots)\Omega\rangle. \tag{4.11b}$$

In (4.11a) and (4.11b) the T-products can be replaced by the normal products to achieve the same results as shown in Appendix B. The arguments in the T-product upon which D and $\tilde{D}$ operate are spared for the sake of simplicity. The operator D is defined in the proof of Theorem IV in Sec. 3, and the operator $\tilde{D}$ is not the same as $\bar{D}$ but the former is obtained from the latter by substituting $\tilde{f}_B$ for $f_B{}^*$.

In order to prove the unitarity of the S-matrix we start from the functional equation $U^\dagger U=1$. By differentiating this relation with respect to the external field $Q(x)$, we get

$$\sum_{\text{comb. }(x_i)}(-i)^n(-1)^k \times\tilde{T}(x_1\cdots x_k)T(x_{k+1}\cdots x_n)=0. \tag{4.12}$$

Apply $D^\alpha\tilde{D}^\beta$ on the vacuum-vacuum matrix element of the above equation, then we get with the help of the recursion formulas for T products the desired relation $S^\dagger S=1$. The conjugate equation $SS^\dagger=1$ follows from $UU^\dagger=1$.

5. RELATION TO THE S-MATRIX THEORY OF BOUND STATES

As we have seen in the previous section the function or eventually an operator f_B for bound states can be arbitrary except for the normalization condition. Making use of this property it seems to be possible to relate the S-matrix elements for reactions involving composite particles to those involving no composite particles. To illustrate the problem, we discuss the relation between the following two reactions:

$$p+p\rightarrow n+p+\pi^+, \tag{5.1}$$

$$p+p\rightarrow d+\pi^+. \tag{5.2}$$

The answer to this question is that if one knows the analytic forms of the S-matrix elements of (5.1) and of the scattering process

$$n+p\rightarrow p+n, \tag{5.3}$$

one can derive the S-matrix element for (5.2), provided that these S-matrix elements satisfy certain analytic properties. Generalizing this statement one may infer that the S-matrix elements for reactions involving only the quanta of an irreducible set would determine the entire S matrix.

For the sake of simplicity we shall consider that all particles participating in (5.1) and (5.2) are spinless.

The S-matrix elements for these reactions are given by

$$S(p+p\rightarrow n+p+\pi^+)=(-i)^3\int(dz_1)(dz_2)(dy)$$
$$\times\langle\Phi_n,\ \varphi_n^*(z_1)\Omega\rangle\langle\Phi_p,\ \varphi_p^*(z_2)\Omega\rangle\langle\Phi_\pi,\ \varphi_\pi^*(y)\Omega\rangle$$
$$\times K_{z_1}{}^n K_{z_2}{}^p K_y{}^\pi\langle\Omega,\ T[\varphi_n(z_1)\varphi_p(z_2)\varphi_\pi(y)]\Phi_{pp}{}^{(+)}\rangle, \quad (5.4)$$

$$S(p+p\rightarrow d+\pi^+)=(-i)^2\int(dz)(d\xi)(dy)$$
$$\times\tilde{f}_d(z,\xi)\langle\Phi_\pi,\ \varphi_\pi^*(y)\Omega\rangle K_z{}^d K_y{}^\pi$$
$$\times\langle\Omega,\ T[\varphi_n(z+\tfrac{1}{2}\xi)\varphi_p(z-\tfrac{1}{2}\xi)\varphi_\pi(y)]\Phi_{pp}{}^{(+)}\rangle. \quad (5.5)$$

Now choose such a $\tilde{f}_d$ as

$$\tilde{f}_d(z,\xi)=\tilde{g}_d(z,\xi)K_{z_1}{}^n K_{z_2}{}^p, \quad (5.6)$$

where $z_1=z+\frac{1}{2}\xi$, $z_2=z-\frac{1}{2}\xi$, namely

$$K_{z_1}{}^n=\left(\frac{1}{2}\frac{\partial}{\partial z}+\frac{\partial}{\partial\xi}\right)^2-m_n{}^2,\quad K_{z_2}{}^p=\left(\frac{1}{2}\frac{\partial}{\partial z}-\frac{\partial}{\partial\xi}\right)^2-m_p{}^2,$$

and $\tilde{g}_d$ is a suitable function, then (5.5) turns out to be

$$(-i)^2\int(dz_1)(dz_2)(dy)\tilde{g}_d(z_1,z_2)$$
$$\times\langle\Phi_\pi,\ \varphi^*(y)\Omega\rangle K_z{}^d K_{z_1}{}^n K_{z_2}{}^p K_y{}^\pi$$
$$\times\langle\Omega,\ T[\varphi_n(z_1)\varphi_p(z_2)\varphi_\pi(y)]\Phi_{pp}{}^{(+)}\rangle. \quad (5.7)$$

This expression is very similar to (5.4) and if we had put

$$\tilde{g}_d(z_1,z_2)=c\langle\Phi_n,\ \varphi_n^*(z_1)\Omega\rangle\langle\Phi_p,\ \varphi_p^*(z_2)\Omega\rangle, \quad (5.8)$$

the similarity would have been complete. In order to interpret (5.8) let us denote the momenta of neutron and proton by p_n and p_p, respectively; then they satisfy

$$p_n{}^2+m^2=p_p{}^2+m^2=0, \quad (5.9)$$

where m is the nucleon rest mass. Put

$$P=p_n+p_p,\quad k=(p_n-p_p)/2, \quad (5.10)$$

then P has to satisfy $P^2+M^2=0$, M being the rest mass of the deuteron. Equation (5.9) is expressed in terms of P and k by

$$\tfrac{1}{4}P^2+k^2+m^2=0,\quad k\cdot P=0. \quad (5.11)$$

The second equation shows that k is a space-like vector and the first equation implies

$$k^2=(M/2)^2-m^2<0.$$

Hence k cannot be a real vector. Assume that the S-matrix element for (5.4), denoted by $S_{np}(P,k)$, can be analytically continued to such a complex value of k and has a pole at the value of k given by (5.11), then the relation between (5.4) and (5.7) is given by

$$S_d=-ic\lim_{P\rightarrow P_d}(P^2+M^2)S_{np}(P,k), \quad (5.12)$$

where S_d is the S matrix element for (5.5). Of course P and k are not independent of one another but related through (5.11). With the help of (5.11) one also writes (5.12) as

$$S_d=4ic\lim(k^2+\gamma^2)S_{np}(P,k), \quad (5.13)$$

where $\gamma^2=(m-\frac{1}{2}M)(m+\frac{1}{2}M)\approx mB$, B being the binding energy of the deuteron. This formula shows the possibility of relating (5.5) to (5.4), and the problem is reduced to the determination of the normalization constant c in Eq. (5.8). This problem is answered in the following way: we start from the equation

$$\frac{-i}{2(2\pi)^3}\int(dx')e^{iP\cdot(x-x')}K_{x'}{}^M\langle\Omega,\ T[\varphi_p(x'-\tfrac{1}{2}\xi)$$
$$\times\varphi_n(x'+\tfrac{1}{2}\xi)\varphi_n^*(y+\tfrac{1}{2}\eta)\varphi_p^*(y-\tfrac{1}{2}\eta)]\Omega\rangle$$
$$=\sum_{\text{spin}}\langle\Omega,\ T[\varphi_p(x-\tfrac{1}{2}\xi)\varphi_n(x+\tfrac{1}{2}\xi)]\Phi_P{}^d\rangle$$
$$\times\langle\Phi_P{}^d,\ T[\varphi_n^*(y+\tfrac{1}{2}\eta)\varphi_p^*(y-\tfrac{1}{2}\eta)]\Omega\rangle, \quad (5.14)$$

where $\Phi_P{}^d$ represents a deuteron state with momentum P. Next we introduce f_d and $\tilde{f}_d$ satisfying the normalization condition

$$\int(d\xi)\tilde{f}_d(x,\xi)$$
$$\times\langle\Omega,\ T[\varphi_p(x-\tfrac{1}{2}\xi)\varphi_n(x+\tfrac{1}{2}\xi)]\Phi_P{}^d\rangle=1/2(2\pi)^3,$$
$$\int(d\eta)f_d(y,\eta)$$
$$\times\langle\Phi_d,\ T[\varphi_n^*(y+\tfrac{1}{2}\eta)\varphi_p^*(y-\tfrac{1}{2}\eta)]\Omega\rangle=1/2(2\pi)^3. \quad (5.15)$$

Multiplying (5.14) by $\tilde{f}_d(x,\xi)$ and $f_d(y,\eta)$ and integrating over ξ and η, we get from (5.15)

$$\int(dx')e^{iP\cdot(x-x')}K_{x'}{}^M\int(d\xi)(d\eta)\tilde{f}_d(x,\xi)$$
$$\times\langle\Omega,\ T[\varphi_p(x'-\tfrac{1}{2}\xi)\varphi_n(x'+\tfrac{1}{2}\xi)$$
$$\times\varphi_n^*(y+\tfrac{1}{2}\eta)\varphi_p^*(y-\tfrac{1}{2}\eta)]\Omega\rangle f_d(y,\eta)=i/2(2\pi)^3,$$

where we have interchanged the order of integration between x' and ξ, η.

If f_d and $\tilde{f}_d$ are of the form (5.6) and (5.8), one arrives at

$$-c^2(2\pi)^{-4}\lim_{P\rightarrow P_d}(P^2+M^2)S(P,k)=i/2(2\pi)^3, \quad (5.16)$$

where $S(n+p\rightarrow p+n)=\delta^4(P_i-P_f)S(P,k)$. In the center-of-mass system $S(P,k)$ is the function of $\mathbf{k}$ alone and we denote it by $S(\mathbf{k})$, then (5.16) is rewritten as

$$-4ic^2\pi^{-1}\lim_{|\mathbf{k}|\rightarrow i\gamma}(\mathbf{k}^2+\gamma^2)S(\mathbf{k})=1, \quad (5.17)$$

from which the normalization constant c is determined.

Equation (5.17) shows, in agreement with the S-matrix theory of bound states, that $S(\mathbf{k})$ has a pole at $|\mathbf{k}|=i\gamma$.

Thus the problem is formally answered. In the S-matrix theory[12,13] the levels of bound states could be obtained as the poles of the S matrix, but in the present theory not only the levels of bound states but also the S-matrix elements of reactions involving composite particles can be derived. In this sense the present theory exhibits a possible generalization of the S-matrix theory of bound states.

In what follows we shall illustrate this method by assuming the possible form of $S_{np}(P,k)$ phenomenologically. From now on, we go over to spinor nucleons; then we have to replace the Klein-Gordon operators for nucleons by Dirac operators.[6] So far as only nonrelativistic nucleons are concerned, however, the formulas given above are still useful and we shall determine the normalization constant c in the nonrelativistic approximation. In the normalization employed in this paper the S matrix in the ^3S state is given by

$$S(\mathbf{k})=S(k)=\frac{m}{k}\frac{e^{2i\delta}}{2\pi}, \qquad (5.18)$$

where δ is the phase shift in the ^3S state. For simplicity we dropped the ^3D state and retained only the ^3S state to describe the deuteron state. In the effective-range theory,[14] the phase shift δ is given by

$$k\cot\delta=-\frac{1}{a}+\frac{1}{2}r_0k^2. \qquad (5.19)$$

Combining (5.17) and (5.19), we get

$$c^2=\frac{\pi^2}{8}\frac{1-r_0\gamma}{m\gamma}. \qquad (5.20)$$

The cross sections for (5.1) and (5.2) are given by

$$\sigma_{np}=\frac{4\pi^2}{B}\int\frac{d\mathbf{q}}{q_0}\frac{d\mathbf{p}_n}{p_{n0}}\frac{d\mathbf{p}_p}{p_{p0}}\delta^4(P_f-P_i)|S_{np}|^2, \qquad (5.21)$$

$$\sigma_d=\frac{4\pi^2}{B}\int\frac{d\mathbf{q}}{q_0}\frac{d\mathbf{P}}{P_0}\delta^4(P_f-P_i)|S_d|^2, \qquad (5.22)$$

where q is the four-momentum of the produced pion and B is given in terms of the four-momenta $p^{(1)}$ and $p^{(2)}$ of the incident protons by

$$B=[-(p_\mu^{(1)}p_\nu^{(2)}-p_\mu^{(2)}p_\nu^{(1)})^2/4]^{\frac{1}{2}}$$
$$=[(\mathbf{p}^{(1)}E^{(2)}-\mathbf{p}^{(2)}E^{(1)})^2-(\mathbf{p}^{(1)}\times\mathbf{p}^{(2)})^2]^{\frac{1}{2}}.$$

From (5.22) the differential cross section for pion production is given by

$$\frac{d\sigma_d}{d\Omega}=\frac{4\pi^2}{B}\frac{q}{E}|S_d|^2, \qquad (5.23)$$

where E is the total energy of the system in the center-of-mass reference, and q denotes the absolute value of the three-vector $\mathbf{q}$. Then in order to simplify (5.21) let us study the kinematics of the system. Let M be the rest mass of the $n+p$ system; then the total four-momentum of this system is given by

$$P=[-\mathbf{q},\ (M^2+\mathbf{q}^2)^{\frac{1}{2}}].$$

Since k is orthogonal to P, we get

$$k_0=-\mathbf{k}\cdot\mathbf{q}/(M^2+\mathbf{q}^2)^{\frac{1}{2}}.$$

Upon inserting this result into the first equation of (5.11), it follows that

$$-(M^2/4)+m^2+\mathbf{k}^2-(\mathbf{k}\cdot\mathbf{q})^2/(M^2+\mathbf{q}^2)=0.$$

Since the final nucleons are assumed to be nonrelativistic, we make an approximation $k_0\approx 0$. Then we have

$$M^2=4(m^2+\mathbf{k}^2),\quad P_0=[4(m^2+\mathbf{k}^2)+\mathbf{q}^2]^{\frac{1}{2}}.$$

In this approximation the differential cross section is given by

$$\frac{d^2\sigma_{np}}{dq_0d\Omega}=\frac{4\pi^2}{B}\frac{4\pi kq}{P_0}|S_{np}|^2, \qquad (5.24)$$

where $k=|\mathbf{k}|$. k and q_0 are not independent, but they are related to one another by

$$q_0=(E^2+\mu^2-M^2)/2E,$$

which follows from the conservation of energy. M is a function of k, and we have

$$(q_0)_{\max}-q_0=(M^2-M_{\min}{}^2)/2E=2k^2/E,$$

or

$$T_0-T=2k^2/E, \qquad (5.25)$$

where T is the kinetic energy of the produced pion and T_0 is the maximum value of T.

In order to compare the present theory with Watson's theory,[15] we make a nonrelativistic approximation for the produced pion to get

$$q=(2\mu T)^{\frac{1}{2}}.$$

To simplify the formula, we may roughly put

$$P_0=E-q_0\approx E\approx 2m.$$

We shall further assume that this reaction takes place mainly in the following states:

$$\underset{^1S}{p+p}\rightarrow \underset{^3S}{n+p}+\underset{p}{\pi^+}, \qquad (5.26)$$

[12] W. Heisenberg, Z. Physik **120**, 513, 673 (1943); Z. Naturforsch. **1**, 608 (1946).
[13] C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **23**, No. 1 (1945); **24**, No. 19 (1946).
[14] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).
[15] K. M. Watson, Phys. Rev. **88**, 1163 (1952). For the comparison of Watson's theory with experiments, we refer to A. H. Rosenfeld, Phys. Rev. **96**, 139 (1954).

at least at low energies. Then we may take the q-dependence of S_{np} as linear:

$$S_{np}=qR. \tag{5.27}$$

Thus we finally get

$$\frac{d^2\sigma_{np}}{d\Omega dT}=\frac{4\pi^2}{B}\frac{4\pi\sqrt{2}\mu^{\frac{3}{2}}T^{\frac{3}{2}}(T_0-T)^{\frac{1}{2}}}{m^{\frac{1}{2}}}|R|^2. \tag{5.28}$$

We do not know the precise form of R, but Watson's theory on the final-state interactions yields the basis for assuming

$$R=\frac{A}{k(\cot\delta-i)}, \tag{5.29}$$

where A is a constant for a fixed angle. Thus for a fixed angle one obtains

$$\frac{d^2\sigma_{np}}{d\Omega dT}=\frac{\sqrt{2}}{\pi}\left(\frac{m}{\mu}\right)^3 T^{\frac{3}{2}}(T_0-T)^{\frac{1}{2}}\left(\frac{\sin^2\delta}{k^2}\right)G, \tag{5.30}$$

where

$$G=\frac{4\pi^2}{B}\left(\frac{4\pi^2A^2\mu^3}{m^2}\right), \tag{5.31}$$

and G is generally a function of the angle between the momenta of the incident proton and the produced pion. S_d is now readily obtained as

$$\begin{aligned} S_d&=-8c\gamma\lim_{k\to i\gamma}(k-i\gamma)S_{np}\\ &=-8c\gamma qA\frac{i}{1-r_0\gamma}. \end{aligned} \tag{5.32}$$

Inserting the expression for c^2, (5.20), into this relation we have

$$|S_d|^2=\frac{8\pi^2\gamma}{m}\left(\frac{q^2A^2}{1-r_0\gamma}\right), \tag{5.33}$$

$$\frac{d\sigma_d}{d\Omega}=\left(\frac{\gamma(q/\mu)^3}{1-r_0\gamma}\right)G. \tag{5.34}$$

Watson's formula for $d\sigma_d/d\Omega$ agrees completely with (5.34), and although in both theories (5.28) and (5.29) were assumed for the reaction (5.1), this agreement still seems to be very interesting. In his derivation Watson utilized Heisenberg's method of normalizing a bound-state wave function. Namely, let the asymptotic form of a bound state wave function be

$$\psi\sim C\frac{e^{-\gamma r}}{2\pi r\sqrt{2}}; \tag{5.35}$$

then the normalization constant C is given by

$$|C|^2=\oint_{i\gamma}dk\,S(k). \tag{5.36}$$

One readily notices that (5.36) corresponds to (5.17) in the present formulation. In (5.17), c^2 was inversely proportional to the residue of the S matrix, but this does not contradict (5.36) since what really corresponds to the wave function ψ is not f_d but the Feynman amplitude $\langle\Omega, T(\psi_n\psi_p)\Phi_d\rangle$ which is inversely proportional to f_d.

Rigorously speaking, the results obtained in this section are just conjectures, since it still remains to be justified that the analytic continuation of the S-matrix elements to the complex values of k is really possible.

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APPENDIX A. DERIVATION OF THE RECURSION FORMULAS

In this appendix we shall derive the recursion formulas (2.25) by generalizing the relation between the Feynman amplitudes and the S matrix.

Let D be a finite space-time domain, and we introduce an external source $Q(x)$ in this domain. Since D is finite, we may assume that the asymptotic conditions at $t\to\pm\infty$ are not invalidated. The S matrix is then a functional of the external source and is given by

$$S_{\beta\alpha}[Q]=\langle\Phi_\beta^{(-)}[Q],\Phi_\alpha^{(+)}[Q]\rangle. \tag{A.1}$$

The state vectors $\Phi^{(\pm)}$ cannot be time-independent because of the interaction with the external source; so, choosing a definite space-like surface σ, we shall fix them by

$$\Phi_\alpha^{(+)}[Q]=U(\sigma,-\infty)\Phi_\alpha^{(+)},$$

$$\Phi_\beta^{(-)}[Q]=U(\infty,\sigma)^+\Phi_\beta^{(-)}, \tag{A.2}$$

where

$$U(\sigma_2,\sigma_1)=T\exp\left(-i\int_{\sigma_1}^{\sigma_2}(dx)H_{\text{ext}}(x)\right), \tag{A.3}$$

and $\Phi_\alpha^{(+)}$ and $\Phi_\beta^{(-)}$ are the state vectors in the absence of the external source.

Hence we obtain

$$S_{\beta\alpha}[Q]=\langle\Phi_\beta^{(-)},U(\infty,-\infty)\Phi_\alpha^{(+)}\rangle. \tag{A.4}$$

The occurrence of $U(\infty,-\infty)$ does not damage the asymptotic condition, since

$$U(\infty,-\infty)=T\exp\left(-i\int_D H_{\text{ext}}(x)(dx)\right),$$

where D is a finite domain.

In what follows we assume $H_{\text{ext}}(x)$ to be of the form

$$H_{\text{ext}}(x)=\varphi(x)Q(x); \tag{A.5}$$

then, expanding (A.4) in powers of $Q(x)$, one obtains

$$S_{\beta\alpha}[Q]=\langle\Phi_\beta^{(-)},\Phi_\alpha^{(+)}\rangle -i\int(dx_1)Q(x_1)\langle\Phi_\beta^{(-)},\varphi(x_1)\Phi_\alpha^{(+)}\rangle +\frac{(-i)^2}{2!}\int(dx_1)(dx_2)Q(x_1)Q(x_2) \times\langle\Phi_\beta^{(-)},T[\varphi(x_1)\varphi(x_2)]\Phi_\alpha^{(+)}\rangle+\cdots. \quad (A.6)$$

Corresponding to (2.22), we get in this case the equation

$$\int(dx_1')(dx_2')\cdots\Delta^{(+)}(x_1-x_1')\Delta^{(+)}(x_2-x_2')\cdots \times K_{x_1'}K_{x_2'}\cdots\langle\Omega^{(-)}[Q],T[\varphi_Q(x_1')\varphi_Q(x_2')\cdots]\Phi_\alpha^{(+)}[Q]\rangle =\sum_{(\beta)}S_{\beta\alpha}'[Q]F_\beta(x_1,x_2,\cdots), \quad (A.7)$$

where $\varphi_Q(x)$ is defined by

$$\begin{aligned}\varphi_Q(x)&=U(\sigma_x,\sigma)^{-1}\varphi(x)U(\sigma_x,\sigma) \quad \text{if} \quad \sigma_x>\sigma\\ &=U(\sigma,\sigma_x)\varphi(x)U(\sigma,\sigma_x)^{-1} \quad \text{if} \quad \sigma_x<\sigma,\end{aligned} \quad (A.8)$$

and as for $\Omega^{(-)}[Q]$, $\Phi_\alpha^{(+)}[Q]$, $S[Q]$ we refer to (A.2) and (A.4). Inserting (A.8) into (A.7), one finds

$$\int(dx_1')(dx_2')\cdots\Delta^{(+)}(x_1-x_1')\Delta^{(+)}(x_2-x_2')\cdots \times K_{x_1'}K_{x_2'}\cdots\langle\Omega,T[U(\infty,-\infty)\varphi(x_1')\varphi(x_2')\cdots]\Phi_\alpha^{(+)}\rangle =\sum_{(\beta)}S_{\beta\alpha}'[Q]F_\beta(x_1,x_2,\cdots), \quad (A.9)$$

where $S'[Q]$ is related to $S[Q]$ by (2.23).

Expanding both sides of (A.9) in powers of $Q(x)$, we arrive at

$$\int(dx_1')(dx_2')\cdots \times\Delta^{(+)}(x_1-x_1')\Delta^{(+)}(x_2-x_2')\cdots K_{x_1'}K_{x_2'}\cdots \times\langle\Omega,T[\varphi(z_1)\cdots\varphi(z_m)\varphi(x_1')\varphi(x_2')\cdots]\Phi_\alpha^{(+)}\rangle =\sum_\beta\sum_\lambda(-1)^\lambda\langle\Phi_{\beta/\lambda}^{(-)},T[\varphi(z_1)\cdots\varphi(z_m)]\Phi_{\alpha/\lambda}^{(+)}\rangle \times F_\beta(x_1,x_2,\cdots), \quad (A.10)$$

from which we immediately get (2.25a). In a similar way we can derive (2.25b) from the rules for evaluating the asymptotic forms for $t\to-\infty$.

APPENDIX B. PROOF OF THEOREM II

Let the slice of Minkowsky space $[t_1,t_2]$ be called D, and let $\mathfrak{R}$ be the ring generated from field operators whose arguments belong to the domain D. For an operator O of $\mathfrak{R}$ we define the R_D product by

$$R_D[O:\varphi(x_1)\cdots\varphi(x_n)] =\sum_{\text{perm.}(x_i)}(-i)^n\theta(t_1-x_1)\theta(x_1-x_2)\cdots\theta(x_{n-1}-x_n) \times[\cdots[[O,\varphi(x_1)]\varphi(x_2)]\cdots\varphi(x_n)]. \quad (B.1)$$

For this new kind of product, we have the following lemma:

Lemma.—If the recursion formulas for R-products hold, then the following relation holds for an arbitrary element O of $\mathfrak{R}$:

$$\int(dz)\langle\Phi_b,\varphi(z)\Omega\rangle K_z \times\langle\Phi_\beta^{(+)},R_D[O:\varphi(x_1)\cdots\varphi(x_n)\varphi(z)]\Phi_\alpha^{(+)}\rangle =\langle\Phi_{\beta b}^{(+)},R_D[O:\varphi(x_1)\cdots\varphi(x_n)]\Phi_\alpha^{(+)}\rangle -\langle\Phi_\beta^{(+)},R_D[O:\varphi(x_1)\cdots\varphi(x_n)]\Phi_{\alpha/b}^{(+)}\rangle. \quad (B.2)$$

(Proof) If Eq. (B.2) is true for O_1 and O_2, then it is also true for $c_1O_1+c_2O_2$. Then making use of the relation

$$R_D[O_1O_2:x_1\cdots x_n] =\sum_{\text{comb.}(x_i)}R_D[O_1:x_1\cdots x_k]R_D[O_2:x_{k+1}\cdots x_n], \quad (B.3)$$

one can prove that it is also true for the product O_1O_2. Therefore if (B.2) is true for $\varphi(x)$, where x belongs to D, then (B.2) is true for an arbitrary operator O of the form

$$O=c_0+\int_D c_1(x)\varphi(x)(dx)+\int_D c_2(x_1,x_2) \times\varphi(x_1)\varphi(x_2)(dx_1)(dx_2)+\cdots, \quad (B.4)$$

and hence for an arbitrary element of $\mathfrak{R}$. So we shall prove (B.2) for $O=\varphi(x)$, x being an element of D. Let us assume $t_1>x_1,\cdots,x_n$ in (B.2), since otherwise both sides of (B.2) vanish and the statement becomes trivial. Then, for $O=\varphi(x)$, $x\epsilon D$, the right-hand side is rewritten by the ordinary R-product,

$$[\text{right-hand side of (B.2)}] =\langle\Phi_{\beta b}^{(+)},R(x:x_1\cdots x_n)\Phi_\alpha^{(+)}\rangle -\langle\Phi_\beta^{(+)},R(x:x_1\cdots x_n)\Phi_{\alpha/b}^{(+)}\rangle, \quad (B.5)$$

while the left-hand side is also equal to the usual R-product except for the case $x_0\geq z_0\geq t_1$, and can generally be written

$$(\text{left-hand side})=\int(dz)\langle\Phi_b,\varphi(z)\Omega\rangle \times K_z\langle\Phi_\beta^{(+)},R(x:x_1\cdots x_nz)\Phi_\alpha^{(+)}\rangle -\int(dz)\langle\Phi_b,\varphi(z)\Omega\rangle K_z\{[\theta(x-z)-\theta(t_1-z)] \times\langle\Phi_\beta^{(+)},R(x:x_1\cdots x_nz)\Phi_\alpha^{(+)}\rangle\}.$$

Since the first term is equal to (B.5), one has to prove

$$\int (dz)\langle\Phi_b, \varphi(z)\Omega\rangle K_z\{[\theta(x-z)-\theta(t_1-z)] \times\langle\Phi_\beta^{(+)}, R(x:x_1\cdots x_nz)\Phi_\alpha^{(+)}\rangle\}=0. \quad (B.6)$$

This expression remains only for $x_0\geq z_0\geq t_1$, and in this case $z\geq x_1, \cdots, x_n$ holds. Hence the R-product in (B.6) is given by

$$\sum_{\text{perm.}(x_i)} \theta(z-x_1)\cdots\theta(x_{n-1}-x_n) \times[\cdots[[\varphi(x), \varphi(z)]\varphi(x_1)]\cdots\varphi(x_n)].$$

Therefore it suffices to prove the lemma if one can show that the Fourier transform of $[\theta(x-z)-\theta(t_1-z)] \times[\varphi(x), \varphi(z)]$ with regard to z has no pole at $p^2+m^2=0$. However, this is easily proved since z can move only in the finite time interval $(x_0\geq z_0\geq t_1)$. This argument is also true for composite particles. It is clear that a similar lemma holds for A-products.

Next we shall introduce the generating functionals of R- and A-products by

$$\varphi_R(x:Q)=U^{-1}T[U\varphi(x)], \quad \varphi_A(x:Q)=T[U\varphi(x)]U^{-1}, \quad (B.7)$$

from which we have

$$\varphi_A(x:Q)U=U\varphi_R(x:Q). \quad (B.8)$$

In a similar way one can introduce $O_R(Q)$ and $O_A(Q)$, for an operator O of $\mathfrak{R}$, satisfying

$$O_A(Q)U=UO_R(Q), \quad (B.9)$$

with the understanding that under the time-ordering operation O is regarded to have the time coordinate t_1. By differentiating (B.9) with respect to Q, we get

$$\sum_{\text{comb.}} (-1)^m(-i)^{n-m} \times\langle\Omega, A_D[O:x_1\cdots x_m]T(x_{m+1}, \cdots, x_n)\Omega\rangle = \sum_{\text{comb.}} (-i)^l\langle\Omega, T(x_1\cdots x_l)R_D[O:x_{l+1}\cdots x_n]\Omega\rangle. \quad (B.10)$$

We apply $\tilde{D}^\beta=\prod\tilde{D}^{b_i}$ on this equation, then due to the positive energy condition only the term $n=m$ survives on the left-hand side, which is given by

$$\langle\Phi_\beta^{(-)}, O\Omega\rangle$$

as a result of the lemma. The right-hand side is given again with the help of the lemma by

$$\sum_{\beta_1\times\beta_2=\beta}\sum_\gamma (-i)^{\beta_1}\tilde{D}^{\beta_1}\langle\Omega, T(\cdots)\Phi_\gamma^{(+)}\rangle \times\tilde{D}^{\beta_2}\langle\Phi_\gamma^{(+)}, R_D[O:\cdots]\Omega\rangle$$
$$= \sum_{\beta_1\times\beta_2=\beta}\sum_\gamma (-i)^{\beta_1}\tilde{D}^{\beta_1}\langle\Omega, T(\cdots)\Phi_\gamma^{(+)}\rangle\langle\Phi_{\gamma\beta_2}^{(+)}, O\Omega\rangle$$
$$= \sum_{\beta_1\times\beta_2=\beta}\sum_\gamma (-i)^{\beta_1}\tilde{D}^{\beta_1}\langle\Omega, T(\cdots)\Phi_{\gamma/\beta_2}^{(+)}\rangle\langle\Phi_\gamma^{(+)}, O\Omega\rangle$$
$$=\sum_\gamma T_{\beta\gamma}\langle\Phi_\gamma^{(+)}, O\Omega\rangle, \quad (B.11)$$

where $T_{\beta\gamma}$ is defined by

$$T_{\beta\gamma}=\sum_{\beta_1\times\beta_2=\beta} (-i)^{\beta_1}\tilde{D}^{\beta_1}\langle\Omega, T(\cdots)\Phi_{\gamma/\beta_2}^{(+)}\rangle. \quad (B.12)$$

By comparing (B.11) with

$$\langle\Phi_\beta^{(-)}, O\Omega\rangle=\sum_\gamma\langle\Phi_\beta^{(-)}, \Phi_\gamma^{(+)}\rangle\langle\Phi_\gamma^{(+)}, O\Omega\rangle =\sum_\gamma S_{\beta\gamma}\langle\Phi_\gamma^{(+)}, O\Omega\rangle, \quad (B.13)$$

we get

$$(S-T)O\Omega=0. \quad (B.14)$$

In the limit $t_1\to-\infty$, $t_2\to+\infty$, state vectors of the form $O\Omega$ span the whole Hilbert space, i.e.,

$$\mathfrak{R}\Omega=\mathfrak{H}. \quad (B.15)$$

This follows from assumption IV. To prove (B.15) let us first assume its inverse. Namely we assume that $\mathfrak{R}\Omega$ is a true subspace of $\mathfrak{H}$; then the projection operator to this subspace commutes with $\varphi(x)$ at every space-time point x. Hence it must be a c-number, and this contradicts the fact that $\mathfrak{R}\Omega$ is a true subspace of $\mathfrak{H}$. From (B.14) and (B.15) there follows the relation

$$S=T. \quad (B.16)$$

The explicit form of the S matrix is given by (B.12), i.e.,

$$S_{\beta\alpha}=\sum_\lambda\tilde{D}^{\beta/\lambda}\langle\Omega, \mathfrak{T}(\cdots)\Phi_{\alpha/\lambda}^{(+)}\rangle, \quad (B.17)$$

where $\mathfrak{T}(x_1,\cdots,x_n)=(-i)^nT(x_1,\cdots,x_n)$. If we start from

$$\varphi_A(x:Q)W=W\varphi_R(x:Q),$$

instead of (B.8), one can prove that the T-product in (B.17) can be replaced by the normal product.

From (B.7), or more explicitly from $U\varphi_R(x:Q) =T[U\varphi(x)]$, one can derive

$$\mathfrak{T}(xx_1\cdots x_n)=\sum_{\text{comb.}} \mathfrak{T}(x_1\cdots x_m)R(x:x_{m+1}\cdots x_n). \quad (B.18)$$

Therefore, one has

$$\tilde{D}^\beta\langle\Omega, \mathfrak{T}(x\cdots)\Phi_\alpha^{(+)}\rangle$$
$$=\sum_{\lambda,\gamma}\tilde{D}^{\beta/\lambda}\langle\Omega,\mathfrak{T}(\cdots)\Phi_\gamma^{(+)}\rangle\cdot\tilde{D}^\lambda\langle\Phi_\gamma^{(+)},R(x:\cdots)\Phi_\alpha^{(+)}\rangle$$
$$=\sum_{\lambda=\mu\nu,\gamma}\tilde{D}^{\beta/\lambda}\langle\Omega, \mathfrak{T}(\cdots)\Phi_\gamma^{(+)}\rangle(-1)^\mu \times\langle\Phi_{\gamma\nu}^{(+)}, \varphi(x)\Phi_{\alpha/\mu}^{(+)}\rangle$$
$$=\sum_{\lambda=\mu\nu,\gamma}\tilde{D}^{\beta/\mu\nu}\langle\Omega, \mathfrak{T}(\cdots)\Phi_{\gamma/\nu}^{(+)}\rangle(-1)^\mu \times\langle\Phi_\gamma^{(+)}, \varphi(x)\Phi_{\alpha/\mu}^{(+)}\rangle$$
$$=\sum_{\lambda=\mu\nu,\gamma} S_{\beta/\mu,\gamma}\langle\Phi_\gamma^{(+)}, \varphi(x)\Phi_{\alpha/\mu}^{(+)}\rangle(-1)^\mu$$
$$=\sum_\mu(-1)^\mu\langle\Phi_{\beta/\mu}^{(-)}, \varphi(x)\Phi_{\alpha/\mu}^{(+)}\rangle,$$

where for the sake of simplicity the arguments upon which $\tilde{D}^\beta$ operates are omitted. By mathematical induction this result can be generalized to yield

$$\tilde{D}^\beta\langle\Omega, \mathfrak{T}(x_1\cdots x_n\cdots)\Phi_\alpha{}^{(+)}\rangle$$
$$=\sum_\mu(-1)^\mu\langle\Phi_{\beta/\mu}{}^{(-)}, \mathfrak{T}(x_1\cdots x_n)\Phi_{\alpha/\mu}{}^{(+)}\rangle, \quad \text{(B.19)}$$

and similarly

$$D^\alpha\langle\Phi_\beta{}^{(-)}, \mathfrak{T}(x_1\cdots x_n\cdots)\Omega\rangle$$
$$=\sum_\mu(-1)^\mu\langle\Phi_{\beta/\mu}{}^{(-)}, \mathfrak{T}(x_1\cdots x_n)\Phi_{\alpha/\mu}{}^{(+)}\rangle. \quad \text{(B.20)}$$

The recursion formulas for T-products are the direct consequences of (B.19) and (B.20).

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Foldy-Wouthuysen Transformation. Exact Solution with Generalization to the Two-Particle Problem

ERIK ERIKSEN
Institute for Theoretical Physics, University of Oslo, Oslo, Norway
(Received April 14, 1958)

The Dirac Hamiltonian for a particle in a nonexplicitly time-dependent field is converted to an even Dirac matrix by means of a single canonical transformation. When the interaction term is an odd Dirac matrix, the transformed Hamiltonian is expressed in a very simple form. An exact transformation is also found for two-particle wave equations of Breit's type. The transformed Hamiltonian is then a uU-separating matrix, in Chraplyvy's sense.

In the nonrelativistic limit expansions in powers of $1/m$ or $1/c$ are made. The approximate wave equations are in agreement with previous transformation results.

INTRODUCTION

A SPIN $\frac{1}{2}$ particle in interaction with various types of external fields is described by a spinor ψ satisfying an equation of the Dirac type.[1–3] For different purposes it is of interest to have this equation converted to a two-component equation of the Pauli type. This was earlier achieved by an elimination method that gives an equation for the large components of ψ. In this equation, however, there are terms nonlinear in $\partial/\partial t$ and non-Hermitian interaction terms like the imaginary electric moment term. Furthermore, the exact interpretation remains in terms of the four-component wave function.

A different treatment is due to Foldy and Wouthuysen.[4] By means of a canonical transformation of the wave equation, a representation is found where the Hamiltonian is an even Dirac matrix. Then the Dirac equation splits into two uncoupled equations of the Pauli type, describing particles in positive- and negative-energy states, respectively. When the particle is free, the transformation is exhibited in a simple, closed form. In the presence of interactions, however, a transformation in closed form has not been found, but an infinite sequence of transformations can be made, each of which makes the Hamiltonian even to one higher order in the expansion parameter $1/m$.

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[3] W. A. Barker and Z. V. Chraplyvy, Phys. Rev. **89**, 446 (1953).
[4] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

Progress on this point has been made by Case[5] who found the transformation in closed form for spin $\frac{1}{2}$ particles and spin 0 particles in time-independent magnetic fields.

The Foldy-Wouthuysen transformation method has been extended to two-particle wave equations by Chraplyvy,[6,7] who found that in the case of equal masses, the postulate of an *even-even* transformed Hamiltonian is too far-reaching. When the less stringent requirement of a *uU-separating* (or an *lL-separating*) Hamiltonian was introduced, a whole class of usable transformations could be found, but none of them is given explicitly.

In the present paper it is found that the exact transformation of the Dirac equation for one particle, can easily be generalized to two-particle wave equations when Chraplyvy's less stringent requirement is used.

SUMMARY OF THE FOLDY-WOUTHUYSEN TRANSFORMATION

The wave function in the Dirac theory is a column matrix with four components ψ_ν, where ψ_1 and ψ_2 are called upper components and ψ_3 and ψ_4 lower components. ψ satisfies the wave equation,

$$i\hbar(\partial/\partial t)\psi = H\psi, \quad (1)$$

the Hamiltonian being a Hermitian four-by-four matrix,

$$H = \beta mc^2 + c\boldsymbol{\alpha}\cdot\mathbf{p} + \text{interaction terms.}$$

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[6] Z. V. Chraplyvy, Phys. Rev. **91**, 388 (1953).
[7] Z. V. Chraplyvy, Phys. Rev. **92**, 1310 (1953).