crude calculations show that it is difficult to obtain good approximations from the formulations for the total scattering amplitude. The formulations for the phase shifts are much easier to use and, as noted by others, give fairly good answers. Scattering amplitudes calculated from these phase shifts automatically conserve probability. The only disadvantage in using the phase shifts is that, at intermediate and high energies, one must calculate large numbers of them to obtain the scattering amplitude.

In closing, it might be worth mentioning a method which removes some of this difficulty and gives good results in an essentially closed form. It can be obtained by combining the Born scattering amplitude with a few variationally determined phase shifts. This is done by re-writing the Born scattering amplitude, f_B, given in Eq. (17a), in the form

$$f_B(\theta)=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\tan\delta_{lB}P_l(\cos\theta), \qquad (56)$$

where $\tan\delta_{lB}$ is given by Eq. (26). Subtracting Eq. (56) from the usual phase shift expansion, Eq. (12), we have

$$f(\theta)-f_B(\theta)=\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\left[\frac{\tan\delta_l}{1-i\tan\delta_l}-\tan\delta_{lB}\right]P_l(\cos\theta).$$

Since the Born phase shifts are not far from the correct ones for the higher l values when the phase shifts become small for most potentials, the right side converges satisfactorily. Even if only one or two terms are retained in the sum, the results are surprisingly good.[9] Consequently, only one or two phase shifts need be determined and for that purpose, variational methods are quite satisfactory.

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Two-Component Fermion Theory*

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The relativistic generalization of Pauli's equation for the electron has recently motivated Feynman and Gell-Mann to propose a specific form of universal Fermi interaction. Particles satisfying this equation are studied from the viewpoint of their electromagnetic interaction. It is found that an anomalous magnetic moment similar to the Pauli moment violates the Hermiticity of the theory, and that only vector and axial vector interactions lead to simple two-component equations, even with parity conservation. Rules for calculation in quantum electrodynamics are developed which have definite advantages over the usual Dirac formulation.

BY considering a description of the known fermions in terms of two-component spinors satisfying the relativistic generalization of the Pauli equation and requiring that only direct (nonderivative) couplings act in the weak interactions, Feynman and Gell-Mann[1] have been led to a universal Fermi interaction which is essentially unique[2] and which has a considerable measure of agreement with experiment. While the solutions of the Dirac equation even in the presence of the electromagnetic field are exactly equivalent to those of the relativistic Pauli equation, the latter equation is more restrictive in several useful ways: it is CP-invariant, but not separately C- and P-invariant (though electromagnetic *effects* have the full invariance); it does not permit the simple addition of an anomalous intrinsic magnetic moment; it leads to a unique Fermi interaction when derivative couplings are excluded. It leads to *simple* interaction forms only of the vector and axial vector types, even for parity-conserving interactions (though this is not certain to be an advantage).

The present work is mainly concerned with a detailed formulation of the electromagnetic interaction of the two-component fermion and its relation to the Dirac formalism. New physical results do not, and should not, appear since the present predictions of quantum electrodynamics are in substantial agreement with experiment. However, one is led to calculational methods which are simpler; also, well-known results can be exhibited with a transparency which is sometimes clouded in the Dirac formalism. For example, the parity-conserving property of electromagnetic interactions will be seen to follow from a symmetry between the fermion spinor and a certain derivative spinor or,

* Work supported in part by the National Science Foundation.
[1] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).
[2] R. E. Marshak and G. Sudarshan, as well as J. J. Sakurai, and others, have also proposed this interaction.

more simply, from the velocity dependence of the interaction.

For brevity we shall restrict ourselves to the essential features of the theory, to the extent of treating only the c-number formalism. One of the advantages of the two-component theory is that, unlike the Dirac equation, it is susceptible to treatment by the path-integral method as well as by the standard quantum field-theoretical techniques. However, such a development is not considered in this paper.

I. NOTATION, ETC.

The theory is most conveniently formulated in terms of 2×2 matrices which have properties similar to the 4×4 $\not{p}=p_\mu\gamma_\mu$ introduced by Feynman. Accordingly, we define[3]

$$p^+=p_0+\boldsymbol{\sigma}\cdot\mathbf{p}\equiv\bar{\sigma}_\mu p_\mu, \tag{1a}$$

$$p^-=p_0-\boldsymbol{\sigma}\cdot\mathbf{p}\equiv\sigma_\mu p_\mu. \tag{1b}$$

Here p_μ is an ordinary four-vector having components $(p_0,\mathbf{p})$ and we adopt the metric $g_{00}=1$, $g_{11}=g_{22}=g_{33}=-1$. σ_1, σ_2, σ_3 are the usual Pauli matrices, always taken in the standard representation, and we may regard $\bar{\sigma}_\mu$ as the matrix four-vector $(1, -\boldsymbol{\sigma})$ while σ_μ is $(1,\boldsymbol{\sigma})$. We have, therefore,

$$\bar{\sigma}_\mu\sigma_\nu=g_{\mu\nu}+h_{\mu\nu}, \tag{2a}$$

$$\sigma_\mu\bar{\sigma}_\nu=g_{\mu\nu}+h_{\mu\nu}', \tag{2b}$$

where $g_{\mu\nu}$ is the metric tensor, and the antisymmetric tensors $h_{\mu\nu}$ and $h_{\mu\nu}'$ have

$$\begin{aligned} &h_{0i}=-h_{i0}=-h_{0i}'=h_{i0}'=\sigma_i;\\ &h_{ij}=h_{ij}'=-\tfrac{1}{2}(\sigma_i\sigma_j-\sigma_j\sigma_i);\quad i,j=1,2,3. \end{aligned} \tag{2c}$$

From the above it follows that for ordinary four-vectors k_μ, q_μ,

$$k^+q^-+q^+k^-=k^-q^++q^-k^+=2k_\mu q_\mu=2k\cdot q; \tag{3}$$

while for $p_\mu=i\partial/\partial x_\mu$ and A_μ, the electromagnetic potential (times the electric charge),

$$p^+A^-+A^+p^-=p\cdot A+A\cdot p+ih_{\mu\nu}\partial A_\nu/\partial x_\mu. \tag{4}$$

We can also write

$$ih_{\mu\nu}\partial A_\nu/\partial x_\mu=\tfrac{1}{2}ih_{\mu\nu}F_{\mu\nu}=i\boldsymbol{\sigma}\cdot(\mathbf{E}+i\mathbf{B}) \tag{5a}$$

while

$$\tfrac{1}{2}ih_{\mu\nu}'F_{\mu\nu}=-i\boldsymbol{\sigma}\cdot(\mathbf{E}-i\mathbf{B}). \tag{5b}$$

Thus, with $\Pi_\mu\equiv p_\mu-A_\mu$, and $\boldsymbol{\Pi}^\pm\equiv p^\pm-A^\pm$, we have

$$\boldsymbol{\Pi}^+\boldsymbol{\Pi}^-=\Pi_\mu\Pi_\mu-\tfrac{1}{2}ih_{\mu\nu}F_{\mu\nu}, \tag{6a}$$

$$\boldsymbol{\Pi}^-\boldsymbol{\Pi}^+=\Pi_\mu\Pi_\mu-\tfrac{1}{2}ih_{\mu\nu}'F_{\mu\nu}. \tag{6b}$$

We take $\hbar=c=1$ throughout.

[3] Matrices of this type have been used for the representation of relativistically invariant quantities by many authors, especially by B. L. van d. Waerden, *Die gruppentheoretische Methode in der Quantenmechanik* (Verlag Julius Springer, Berlin, 1932); and by H. A. Kramers, *Quantum Mechanics* (Interscience Publishers, Inc., New York, 1957, translated by D. ter Haar).

II. CANONICAL FORMALISM

It is our purpose to formulate a description of a relativistic fermion of mass m in terms of a single two-component spinor ψ. For convenience, however, we introduce a second two-component spinor Ω, which at the outset will be independent of ψ but which we will shortly relate to ψ in such a way that the observables of the theory will be Hermitian.

Our Lagrangian density is († denotes Hermitian adjoint)

$$\mathfrak{L}=m^{-1}(\boldsymbol{\Pi}^+\Omega)^\dagger(\boldsymbol{\Pi}^-\psi)-m\Omega^\dagger\psi. \tag{7}$$

We note that Π^+ and Π^- are Hermitian operators and use

$$\delta\int\mathfrak{L}d\tau=0$$

to obtain

$$\boldsymbol{\Pi}^+\boldsymbol{\Pi}^-\psi=m^2\psi, \tag{8}$$

$$\boldsymbol{\Pi}^-\boldsymbol{\Pi}^+\Omega=m^2\Omega, \tag{9}$$

and their Hermitian adjoint equations.

We will discuss the invariance properties of the theory in more detail below. For the present we remark that

$$\sigma_2(\boldsymbol{\Pi}^-\boldsymbol{\Pi}^+)^*\sigma_2=(\boldsymbol{\Pi}^+\boldsymbol{\Pi}^-)_c, \tag{10}$$

where the subscript c means the sign of the charge is changed. On the other hand,

$$P\boldsymbol{\Pi}^-=\boldsymbol{\Pi}^+, \tag{11}$$

where P is the parity operation. Therefore, the equations of motion (8) and (9) transform into each other under C or P but are unchanged, except for the sign of the charge, under "combined inversion" CP.

The fields canonically conjugate to ψ and $\Omega^\dagger$ are

$$\chi^\dagger=\frac{\partial\mathfrak{L}}{\partial(\partial\psi/\partial t)}=(i/m)(\boldsymbol{\Pi}^+\Omega)^\dagger, \tag{12}$$

$$\varphi=\frac{\partial\mathfrak{L}}{\partial(\partial\Omega^\dagger/\partial t)}=-(i/m)\boldsymbol{\Pi}^-\psi. \tag{13}$$

From Eqs. (8) and (13),

$$im\boldsymbol{\Pi}^+\varphi=m^2\psi, \tag{14}$$

$$\boldsymbol{\Pi}^-\boldsymbol{\Pi}^+\varphi=m^2\varphi. \tag{15}$$

Similarly,

$$\boldsymbol{\Pi}^+\boldsymbol{\Pi}^-\chi=m^2\chi. \tag{16}$$

A comparison of (16) with (8) suggests the identification

$$i\chi=\psi. \tag{17}$$

If we now operate on both sides of Eq. (17) with $\boldsymbol{\Pi}^-$ and use the adjoint of (12), as well as (9) and (13), we deduce that

$$i\varphi=\Omega. \tag{18}$$

The last two equations can now be rewritten as [using (12) and (13)]

$$\Pi^{+}\Omega=m\psi, \tag{19}$$

$$\Pi^{-}\psi=m\Omega, \tag{20}$$

which expresses the relation between ψ and Ω that will make the observables Hermitian. It is important to note that we can consider ψ to be obtained first by solving Eq. (8); the auxiliary spinor Ω is then obtained by the operation indicated in Eq. (20).

Equations (19) and (20) can be written as a single Dirac equation[4]

$$(\Pi_0-\boldsymbol{\alpha}\cdot\boldsymbol{\Pi}-\beta m)\begin{pmatrix}\Omega\\ \psi\end{pmatrix}=0, \tag{21}$$

with

$$\boldsymbol{\alpha}=\begin{pmatrix}-\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix},\quad \beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{22}$$

For $\gamma_\mu=(\beta,\beta\boldsymbol{\alpha})$ and $\gamma_5=i\gamma_1\gamma_2\gamma_3\gamma_0$ we then have

$$\gamma_\mu=\begin{pmatrix}0 & \sigma_\mu\\ \bar{\sigma}_\mu & 0\end{pmatrix},\quad \gamma_5=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{23}$$

The standard form of the Dirac equation is obtained from Eq. (21) by the unitary transformation

$$\mathbf{S}=\exp(i\rho_2\pi/4)=\frac{1}{\sqrt{2}}(1+i\rho_2),\quad \rho_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}. \tag{24}$$

That is,

$$\mathbf{S}\begin{pmatrix}\Omega\\ \psi\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}\psi+\Omega\\ \psi-\Omega\end{pmatrix}$$

satisfies the usual Dirac equation.

Returning now to the two-component formulation and Eqs. (17) and (18), we write the Hamiltonian density

$$\mathfrak{H}=\chi^{\dagger}\frac{\partial\psi}{\partial t}+\frac{\partial\Omega^{\dagger}}{\partial t}\varphi-\mathfrak{L}, \tag{25a}$$

$$\mathfrak{H}=i\left(\psi^{\dagger}\frac{\partial\psi}{\partial t}-\frac{\partial\Omega^{\dagger}}{\partial t}\Omega\right)-m(\psi^{\dagger}\Omega-\Omega^{\dagger}\psi). \tag{25b}$$

Using (19) and (20) to eliminate $\partial\psi/\partial t$ and $\partial\Omega/\partial t$, we obtain

$$\mathfrak{H}=m(\psi^{\dagger}\Omega+\Omega^{\dagger}\psi)+\psi^{\dagger}A_0\psi+\Omega^{\dagger}A_0\Omega+\psi^{\dagger}\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}\psi-(\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}\Omega)^{\dagger}\Omega, \tag{26}$$

which is evidently Hermitian. This is also true of the current density

$$s_\nu=-i(\partial\mathfrak{L}/\partial\psi_\nu)\psi+i\Omega^{\dagger}(\partial\mathfrak{L}/\partial\Omega_\nu{}^{\dagger})=\psi^{\dagger}\sigma_\nu\psi+\Omega^{\dagger}\bar{\sigma}_\nu\Omega. \tag{27}$$

[4] A different representation of the Dirac matrices is used in reference 1. See also Appendix IV.

The current, of course, obeys the continuity equation as

$$i\partial s_\nu/\partial x_\nu=-(\not{p}^{-}\psi)^{\dagger}\psi+\psi^{\dagger}\not{p}^{-}\psi-(\not{p}^{+}\Omega)^{\dagger}\Omega+\Omega^{\dagger}\not{p}^{+}\Omega, \tag{28}$$

and since

$$\not{p}^{-}\psi=m\Omega+\not{A}^{-}\psi,\quad \text{etc.},$$

we have

$$i\partial s_\nu/\partial x_\nu=0. \tag{29}$$

The component

$$s_0=\psi^{\dagger}\psi+\Omega^{\dagger}\Omega \tag{30}$$

is positive definite and must therefore be interpreted as the particle density, not the charge density, despite the equations being second order in the time.

III. FREE-PARTICLE STATES

Setting $A=0$ in Eq. (8), we obtain as elementary solutions:

$$\psi(x)=\psi e^{-ip\cdot x}. \tag{31}$$

ψ is an arbitrary constant two-component spinor. [In (31), and hereafter, in referring to free-particle states we shall mean a constant spinor whenever no explicit argument is written.] Particle and antiparticle states have p_0 respectively positive and negative; the four-momentum of an antiparticle is $-p$. The auxiliary spinor $\Omega(x)$ is then, from (20),

$$\Omega(x)=\Omega e^{-ip\cdot x}, \tag{32}$$

with

$$\not{p}^{-}\psi=m\Omega,\quad \not{p}^{+}\Omega=m\psi. \tag{33}$$

On substitution of (31) and (32) in Eq. (26) with $A=0$, we obtain the energy density:

$$E=p_0(\psi^{\dagger}\psi+\Omega^{\dagger}\Omega)=p_0 s_0. \tag{34}$$

States of negative energy are to be reinterpreted as antiparticle states of positive energy according to the method of Feynman.

We next consider the normalization of the particle density. We use (33) to write Eq. (30) as

$$s_0=m^{-2}\psi^{\dagger}(m^2+\not{p}^{-}\not{p}^{-})\psi. \tag{35}$$

Since

$$m^2+\not{p}^{-}\not{p}^{-}=p_0{}^2+\mathbf{p}^2+m^2-2p_0\boldsymbol{\sigma}\cdot\mathbf{p}=2p_0\not{p}^{-},$$

we have

$$s_0=(2p_0/m^2)\psi^{\dagger}\not{p}^{-}\psi=(2p_0/m)\psi^{\dagger}\Omega=(2p_0/m)\Omega^{\dagger}\psi. \tag{36}$$

Setting $s_0=1$ corresponds to the Dirac normalization $(u^{\dagger}u)=1$. A more convenient normalization, however, is to take

$$\psi^{\dagger}\Omega=\Omega^{\dagger}\psi=p_0/|p_0|\equiv\epsilon. \tag{37}$$

This results in

$$s_0=(2p_0/m)\epsilon=2|p_0|/m. \tag{38}$$

Transition amplitudes are then proportional to matrix elements of the form $(\Omega_F(x),\psi_I(x))$.

Now introduce normalized spin eigenfunctions[5]

[5] Note that in the free particle case $\psi(x)$ may be chosen as an eigenfunction of spin in an arbitrary direction, unlike the Dirac case. However, the direction of the momentum is still the most convenient choice for most purposes.

$\chi_s=\chi_\pm$ such that

$$\sigma\cdot\chi\hat{\mathbf{p}}_s=s\chi_s,\quad \chi_s^\dagger\chi_{s'}=\delta_{ss'}. \tag{39}$$

Then if $\psi=a_s\chi_s$, we find

$$\Omega=m^{-1}(p_0-s|\mathbf{p}|)\psi=m(p_0+s|\mathbf{p}|)^{-1}\psi. \tag{40}$$

From Eqs. (37) and (39), we obtain

$$\epsilon=m(p_0+s|\mathbf{p}|)^{-1}|a_s|^2 \tag{41}$$

and therefore, in accord with (37),

$$a_s=\epsilon^{\frac{1}{2}}|(p_0+s|\mathbf{p}|)/m|^{\frac{1}{2}}. \tag{42}$$

In summary of this paragraph, we have

$$\psi=a_s\chi_s,\quad \Omega=a_{-s}\chi_s,\quad a_sa_{-s}=\epsilon. \tag{43}$$

To calculate the probability of transitions between plane wave states caused by an interaction matrix M, we frequently have to evaluate the sum over spins of

$$|(\Omega_2,M\psi_1)|^2=(\Omega_2^\dagger M\psi_1)(\psi_1^\dagger M^\dagger\Omega_2). \tag{44}$$

While $\sum_{s_1}\psi_1\psi_1^\dagger$ is not very simple, we can make use of

$$\sum_{s_1}\psi_1\Omega_1^\dagger=\sum_{s_1}\Omega_1\psi_1^\dagger=\epsilon_1(\chi_+\chi_+^\dagger+\chi_-\chi_-^\dagger)=\epsilon_1 I, \tag{45}$$

where I is the unit matrix, to write

$$\sum_{s_1}|(\Omega_2,M\psi_1)|^2=\sum_{s_1}(\Omega_2^\dagger M\psi_1)(\Omega_1^\dagger\bar{M}\psi_2)$$
$$=\epsilon_1(\Omega_2^\dagger M\bar{M}\psi_2), \tag{46}$$

with

$$\bar{M}=m^{-2}p_1^+M^\dagger p_2^-. \tag{47}$$

Similarly, on summing over the final spins, we obtain

$$\sum_{s_1,s_2}|(\Omega_2,M\psi_1)|^2=\epsilon_1\epsilon_2\,\mathrm{Tr}[M\bar{M}]. \tag{48}$$

To obtain results for particular spin situations, one inserts the conventional spin projection operators,[5] e.g., $\frac{1}{2}(1\pm\sigma\cdot\hat{p})$. The usual "energy" projection operators are *not* required. It may appear that they are already contained in the definition of $\bar{M}$, but we shall see in the next section that this is not the case.

IV. QUANTUM ELECTRODYNAMICS

We write Eq. (8) in the form

$$(p^2-m^2)\psi(x)=\mathcal{J}\psi(x), \tag{49}$$

with

$$\mathcal{J}=p^+A^-+A^+p^--A^+A^-, \tag{50}$$

where $p_\mu=i\partial/\partial x_\mu$ and for A^+A^- we can equivalently write A^2. The corresponding integral equation,

$$\psi(x)=\psi_0(x)+(p^2-m^2)^{-1}\mathcal{J}\psi(x), \tag{51}$$

can be interated and transformed to momentum representation, yielding the usual S-matrix formulation of the Klein-Gordon equation except that the "vertex interactions" are given by (50) instead of the usual

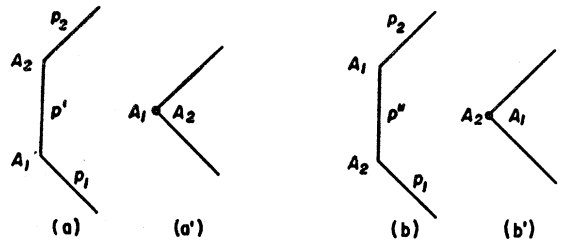

FIG. 1. Diagrams for the second-order interaction. The interaction matrix is the sum (superscripts have been omitted) of (a) $(p'^2-m^2)^{-1}(p_2A_2+A_2p')(p'A_1+A_1p_1)$; (a') $-A_2A_1$; (b) $(p''^2-m^2)^{-1}(p_2A_1+A_1p'')(p''A_2+A_2p_1)$; (b') $-A_1A_2$.

(in the Klein-Gordon theory) scalar form. For comparison with the diagrams usually drawn for the Dirac equation (see Fig. 1) it is convenient to associate with each fermion line containing successive single vertices (say, in the order A_1, A_2) a double vertex at which the interaction is $-A_2^+A_1^-$. As there will always occur the corresponding diagram with the "single" interactions in the reverse order, the two "double" interactions will yield

$$-A_1^+A_2^--A_2^+A_1^-=-2A_1\cdot A_2. \tag{52}$$

Although the notation is rather cumbersome, it will be seen shortly that the superscripts $\pm$ can safely be omitted and that transition probabilities will assume a simple form.

Consider the "single" vertex interaction

$$M^{(1)}=p_2^+A^-+A^+p_1^-. \tag{53}$$

Since p_2^+, A^-, etc., are Hermitian matrices, we have

$$M^{(1)\dagger}=A^-p_2^++p_1^-A^+. \tag{54}$$

From Eq. (47) and $p_1^2=p_2^2=m^2$, we get

$$\bar{M}^{(1)}=p_1^+A^-+A^+p_2^-. \tag{55}$$

That is, $\bar{M}^{(1)}$ is $M^{(1)}$ with the factors written in reverse order and with superscripts plus and minus interchanged. This is, indeed, the general rule for writing $\bar{M}$ for an M of any order. (The proof of this "backwards rule" is contained in Appendix I.) Thus in writing M, $\bar{M}$ or $\mathrm{Tr}[M\bar{M}]$ there is a uniform alternation of "plus" and "minus." We can therefore omit the superscripts. Furthermore, the traces are evaluated by successive applications of

$$p^+q^-+q^+p^-=2p\cdot q, \tag{56}$$

which does not spoil the alternation, and since these are precisely the algebraic relations for the Dirac p, q, the procedures for evaluating traces are unchanged.

A further simplification, as well as additional physical insight, can be gained by introducing into (53) the momentum transfer $q=p_2-p_1$:

$$M^{(1)}=\tfrac{1}{2}(q^+A^--A^+q^-)+(p_1+p_2)\cdot A. \tag{57}$$

It can be seen that $\bar{M}^{(1)}$ in the form (55) exchanges

subscripts 1 and 2 in $M^{(1)}$. But this is equivalent to writing (57) in reverse order and exchanging superscripts. The "backwards rule" is thus valid also for the form (57) and one need not apply the definition (47). The same remark applies to the term $-A^+A^-$ (see proof in Appendix I).

In momentum representation the interaction $\mathcal{J}$ consists of $\mathcal{J}_C+\mathcal{J}_S$ where the "charge" or Klein-Gordon interaction is

$$\mathcal{J}_C=(p_1+p_2)\cdot A-A^2, \tag{58}$$

and the "spin," or additional Dirac, interaction is

$$\mathcal{J}_S=\tfrac{1}{2}(q^+A^--A^+q^-). \tag{59}$$

This separation of the interaction can be useful for approximate calculations. Evidently only $\mathcal{J}_C$ contributes for small momentum transfers. $\mathcal{J}_S$ can also be written as

$$\mathcal{J}_S=q^+A^--q\cdot A=-A^+q^-+q\cdot A, \tag{60}$$

which is useful as $q\cdot A$ is often zero (e.g., for free photon interactions or potential interactions). $\mathcal{J}_C$ is merely a number and does not enter in the calculation of traces. Since there are no "energy" projection operators and only momentum transfer and polarization vectors occur in $\mathcal{J}_S$, traces are *always* simpler than those written in a straightforward way from the Dirac theory. Examples are given in Appendix II.

V. ANOMALOUS MOMENT AND OTHER INTERACTIONS

In the terms of the preceding discussion one is led to inquire whether it is possible to consider an arbitrary mixture of "charge" and "spin" interaction. That is, can we write

$$\mathcal{J}=\mathcal{J}_C+(1+\lambda)\mathcal{J}_S? \tag{61}$$

If so, this would correspond to an intrinsic anomalous magnetic moment $\lambda e/2m$ having different dynamical properties from the familiar Pauli addition to the Dirac Hamiltonian. [It can be seen by reference to Eq. (6a) that (61) is equivalent to multiplying the term $h_{\mu\nu}F_{\mu\nu}$ by $(1+\lambda)$, and that such a term cannot be formed by interation of the Pauli-Dirac Hamiltonian.] Like the Pauli addition, it violates what Gell-Mann has called the principle of minimal electromagnetic interaction. In this sense a theory in which it would be impossible to add such a term would be distinctly advantageous. This appears to be true of the two-component theory.[6]

[6] This situation has been discussed from a somewhat different point of view by R. P. Feynman [Phys. Rev. 84, 108 (1951), Appendix D]. There the iterated Dirac equation is considered from the standpoint of the path-integral formulation and the connection with the classical limit. It is pointed out that transition probabilities are positive definite only if $\lambda=0$ (fermions) or $\lambda=-1$ (bosons). Thus $\lambda\neq 0$ for fermions would violate not only the Hermiticity of the theory but the connection between spin and statistics as well. The conclusion made in the above work that projection operators are necessary does not hold in our case, but arises there from the fact that the iterated Dirac equation has four components.

To obtain this interaction let us add to the Lagrangian density a term $\Omega^\dagger\mathcal{J}_S'\psi$, where by $\mathcal{J}_S'$ we mean $(\lambda/m)\times ih_{\mu\nu}F_{\mu\nu}$. Equations (8) and (9) become

$$\boldsymbol{\Pi}^+\boldsymbol{\Pi}^-\psi(x)=(m^2-\mathcal{J}_S')\psi(x), \tag{62}$$

$$\boldsymbol{\Pi}^-\boldsymbol{\Pi}^+\Omega(x)=(m^2-\mathcal{J}_S')\Omega(x). \tag{63}$$

Since $\mathcal{J}_S'$ contains no differential operators, the canonically conjugate fields (12) and (13) are unchanged. Consequently, Eqs. (12) and (13) hold and

$$im\boldsymbol{\Pi}^+\varphi(x)=(m^2-\mathcal{J}_S')\psi(x), \tag{64}$$

$$\boldsymbol{\Pi}^-\boldsymbol{\Pi}^+\varphi(x)=(m^2-\mathcal{J}_S')\varphi(x)+(i/m)[\boldsymbol{\Pi}^-,\mathcal{J}_S']\psi(x). \tag{65}$$

Thus $\varphi(x)$ and $\Omega(x)$ satisfy the same equation only if the commutator of $\boldsymbol{\Pi}^-$ and $\mathcal{J}_S'$ vanishes (e.g., for constant fields), and we cannot in general identify them. Similarly, we cannot identify $\chi(x)$ with $\psi(x)$. However, if we examine the Hamiltonian density, Eq. (25a), using only Eqs. (12) and (13) to eliminate the time derivatives, it reads, for $A_0=0$,

$$\mathcal{H}=m(\chi^\dagger\varphi+\Omega^\dagger\psi)-i\chi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}\psi-i(\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}\Omega)^\dagger\varphi. \tag{66}$$

There is essentially only the choice previously made:

$$i\chi=\psi,\quad i\varphi=\Omega, \tag{67}$$

to make (66) Hermitian. That is, we must take $\lambda=0$,[7] since other values are excluded by (65).

It is tempting to try to generalize the above conclusion and to see what other interactions, for example with meson fields, have a natural place in this theory. Clearly the exclusion of the interaction $\mathcal{J}_S'$ is connected with our inability to factor the second-order equation into a pair of equations of first order in the space and time derivatives, i.e., to write it as the Dirac equation.

Conversely, we may ask which of the relativistically invariant interactions possible in the four-component theory lead to simple linear second-order two-component equations. By consulting Eq. (23) one sees that the vector and axial vector interactions are represented by "odd" Dirac operators and scalar, tensor, and pseudoscalar by "even" Dirac operators. Thus only the V and A interactions can be written simply in the two-component form.[8] This conclusion holds even for the strong parity-conserving interactions. (In the next section we show that the class of parity-conserving interactions contains only those which are symmetrical to exchange of ψ with Ω.)

For example, the axial vector interaction with a pseudoscalar meson field φ gives (with $\varphi_\mu=i\partial\varphi/\partial x_\mu$)

$$(p_\mu-i\gamma_5\varphi_\mu(x))\gamma_\mu\begin{pmatrix}\Omega(x)\\ \psi(x)\end{pmatrix}=m\begin{pmatrix}\Omega(x)\\ \psi(x)\end{pmatrix}. \tag{68}$$

[7] There is, of course, the possibility $\lambda=-1$ with $\psi=\Omega$. But this is just the Klein-Gordon theory.

[8] This has been remarked previously by R. P. Feynman.

From Eq. (23) we get

$$(p^- - i\varphi^-(x))\psi(x) = m\Omega(x), \tag{69}$$

$$(p^+ + i\varphi^+(x))\Omega(x) = m\psi(x), \tag{70}$$

or

$$(p^+ + i\varphi^+)(p^- - i\varphi^-)\psi(x) = m^2\psi(x). \tag{71}$$

On the other hand, the pseudoscalar interaction gives

$$p_\mu\gamma_\mu\begin{pmatrix}\Omega(x)\\ \psi(x)\end{pmatrix} = (m + i\gamma_5\varphi(x))\begin{pmatrix}\Omega(x)\\ \psi(x)\end{pmatrix}, \tag{72}$$

$$p^-\psi(x) = (m + i\varphi(x))\Omega(x), \tag{73}$$

and

$$p^+\Omega(x) = (m - i\varphi(x))\psi(x), \tag{74}$$

$$p^+(m + i\varphi)^{-1}p^-\psi(x) = (m - i\varphi)\psi(x). \tag{75}$$

Similarly the true Pauli anomalous moment (or other tensor interaction) will introduce the fields unpleasantly in the denominator.

This suggests the postulate that only V and A interactions occur in quantum theory. At each vertex either V (electromagnetic) or A (meson) interactions occur. When V and A interactions can occur simultaneously with a given field, parity nonconservation results.

VI. INVARIANCE PROPERTIES

We have noted before [Eq. (10)] that charge conjugation C consists of complex conjugation followed by the unitary transformation $\mathbf{S} = \sigma_2$. Thus, under C, Eqs. (8) and (9) become

$$(\mathbf{\Pi}^-\mathbf{\Pi}^+)_C\sigma_2\psi^*(x) = m^2\sigma_2\psi^*(x). \tag{76}$$

$$(\mathbf{\Pi}^+\mathbf{\Pi}^-)_C\sigma_2\Omega^*(x) = m^2\sigma_2\Omega^*(x), \tag{77}$$

where c means that the sign of the charge is changed (or $A_\mu \rightarrow -A_\mu$). We can express this result by defining

$$\psi^C(x) = \sigma_2\Omega^*(x), \quad \Omega^C(x) = \sigma_2\psi^*(x). \tag{78}$$

A theory will be invariant under C if its physical consequences are independent of the exchange of $\psi(x)$ and $\Omega(x)$.

Similarly, under the parity operation P,

$$P\mathbf{\Pi}^+ = \mathbf{\Pi}^-, \quad P\mathbf{\Pi}^- = \mathbf{\Pi}^+. \tag{79}$$

That is, $P\psi(x)$ satisfies the Ω equation and $P\Omega(x)$ satisfies the ψ equation. The theory is P-invariant if it is invariant under $\psi \leftrightarrow \Omega$. Electrodynamics is C- and P-invariant since

$$(\Omega_2, M\psi_1)^* = (\Omega_1, \bar{M}\psi_2). \tag{80}$$

Under $\psi_1 \leftrightarrow \Omega_1$, $\psi_2 \leftrightarrow \Omega_2$, the results will be unchanged except that "plus" and "minus" superscripts are exchanged, which has no physical consequences. Under combined inversion CP, Eq. (8) becomes

$$(\mathbf{\Pi}^+\mathbf{\Pi}^-)_C\psi^{CP}(x) = m^2\psi^{CP}(x), \tag{81}$$

where $\psi^{CP}(x) = \sigma_2\psi^*(x)$, and we need not refer to Ω; $\psi^{CP}(x)$ describes the antiparticle. Of course we did not really *need* to introduce $\Omega(x)$ for discussing parity conservation; it is merely a convenience. Instead we could have referred to $\psi(x)$ and its "derivative function."

The Feynman–Gell-Mann proposal can now be briefly stated. It is that the parity-nonconserving weak interactions are to involve only one of the pair $\psi(x)$, $\Omega(x)$, and no derivatives. In the Dirac formulation, since

$$\gamma_5\begin{pmatrix}\Omega\\ 0\end{pmatrix} = \begin{pmatrix}\Omega\\ 0\end{pmatrix},$$

we have

$$\gamma_5\begin{pmatrix}0\\ \psi\end{pmatrix} = -\begin{pmatrix}0\\ \psi\end{pmatrix}, \tag{82}$$

using Ω only, say, is equivalent to writing everywhere for u the expression $\frac{1}{2}(1+\gamma_5)u$.

VII. ACKNOWLEDGMENTS

I would like to thank R. P. Feynman for introducing me to this subject some years ago, as well as for recent conversations and a reading of the manuscript. Thanks are also due Y. Nambu who has been most generous in discussion and suggestions.

APPENDIX I. PROOF OF "BACKWARDS RULE"

Consider two of the diagrams for an nth order process, as in Fig. 2. We can write the corresponding part of the interaction matrix (omitting superscripts) as

$$M = \cdots(p_3e_2 + e_2p_2)\kappa^{-1}(p_2e_1 + e_1p_1) - \cdots e_2e_1 \tag{A1}$$

with $\kappa = p_2^2 - m^2$. Then, by Eq. (47), we have

$$\bar{M} = m^{-2}[p_1(e_1p_2 + p_1e_1)\kappa^{-1}(e_2p_3 + p_2e_2)\cdots p_f - p_1e_1e_2\cdots p_f]. \tag{A2}$$

At the position of κ^{-1} in (A2) insert p_2p_2/p_2^2 and multiply one factor p_2 to the left, one to the right (and use $p_1^2 = m^2$):

$$\bar{M} = m^{-2}[(p_1e_1p_2^2 + m^2e_1p_2)(\kappa p_2^2)^{-1} \times (p_2e_2p_3 + p_2^2e_2)\cdots p_f - p_1e_1e_2\cdots p_f]. \tag{A3}$$

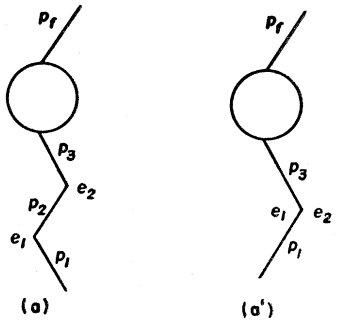

FIG. 2. Two diagrams for higher order process. The circle represents a vertex interaction of arbitrary complexity.

For m^2 in the first bracket insert $p_2^2-\kappa$:

$$\bar{M}=m^{-2}[(\boldsymbol{p}_1\boldsymbol{e}_1+\boldsymbol{e}_1\boldsymbol{p}_2)\kappa^{-1}(\boldsymbol{p}_2\boldsymbol{e}_2\boldsymbol{p}_3+p_2^2\boldsymbol{e}_2)\cdots\boldsymbol{p}_f -\boldsymbol{e}_1(\boldsymbol{e}_2\boldsymbol{p}_3+\boldsymbol{p}_2\boldsymbol{e}_2)\cdots\boldsymbol{p}_f-\boldsymbol{p}_1\boldsymbol{e}_1\boldsymbol{e}_2\cdots\boldsymbol{p}_f]. \quad \text{(A4)}$$

Now in the first term of (A4) write $\kappa+m^2$ for p_2^2. The term from κ cancels the last term and part of the second term of (A4) and we obtain

$$\bar{M}=m^{-2}[(\boldsymbol{p}_1\boldsymbol{e}_1+\boldsymbol{e}_1\boldsymbol{p}_2)\kappa^{-1}(\boldsymbol{p}_2\boldsymbol{e}_2\boldsymbol{p}_3+m^2\boldsymbol{e}_2)\cdots\boldsymbol{p}_f -\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{p}_3\cdots\boldsymbol{p}_f]. \quad \text{(A5)}$$

The second bracket of the first term of (A5) has now a form similar to the first bracket of the first term of (A3) and the same procedure can be continued (including now the term containing the double vertex $\boldsymbol{e}_2\boldsymbol{e}_3$, etc.).

To see what happens when the last fermion line $\boldsymbol{p}_f$ is reached, let $\boldsymbol{p}_f=\boldsymbol{p}_3$. Then as $\boldsymbol{p}_f^2=m^2$, we find

$$\bar{M}=(\boldsymbol{p}_1\boldsymbol{e}_1+\boldsymbol{e}_1\boldsymbol{p}_2)\kappa^{-1}(\boldsymbol{p}_2\boldsymbol{e}_2+\boldsymbol{e}_2\boldsymbol{p}_3)-\boldsymbol{e}_1\boldsymbol{e}_2. \quad \text{(A6)}$$

This completes the proof that $\bar{M}$ is M written in reverse order.

APPENDIX II. CALCULATIONS IN QUANTUM ELECTRODYNAMICS

We give here two short examples of calculations with the methods described in the text. Naturally, the advantages of the method are better exhibited in longer examples, such as the bremsstrahlung problem.

a. Compton Scattering with Photon Polarization Vectors Parallel (Fig. 3)

Assume $e_1=e_2=e=(0,\boldsymbol{e})$; initial electron at rest. Then, as $e\cdot k_1=e\cdot k_2=e\cdot p_1=0$, by momentum conservation $e\cdot p_2=0$. Since $(p_1+k_1)^2-m^2=2m\omega_1$, $(p_1-k_2)^2-m^2=-2m\omega_2$, the terms of Fig. 3 contribute

(I) $(2m\omega_1)^{-1}(-\boldsymbol{k}_2\boldsymbol{e})(\boldsymbol{k}_1\boldsymbol{e})=(2m\omega_1)^{-1}(-\boldsymbol{k}_2\boldsymbol{k}_1)$,

(II) $(2m\omega_2)^{-1}(\boldsymbol{k}_1\boldsymbol{e})(\boldsymbol{k}_2\boldsymbol{e})=(2m\omega_2)^{-1}\boldsymbol{k}_1\boldsymbol{k}_2$,

(III) $-2e\cdot e=2$.

The Klein-Gordon result is entirely contained in (III).

Adding, squaring, and taking the spin average [the only nontrivial trace is $\mathrm{Tr}(\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_1\boldsymbol{k}_2)=4(k_1\cdot k_2)^2$], we obtain

$$(2m\omega_1\omega_2)^{-2}[8m\omega_1\omega_2(\omega_1-\omega_2)(k_1\cdot k_2) -4\omega_1\omega_2(k_1\cdot k_2)^2]+4. \quad \text{(A7)}$$

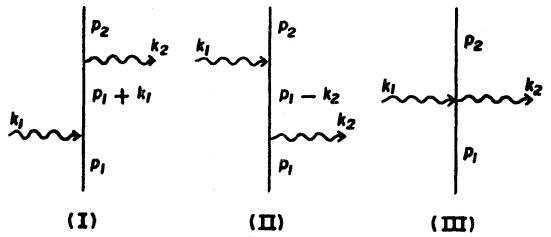

FIG. 3. Diagrams for the Compton effect.

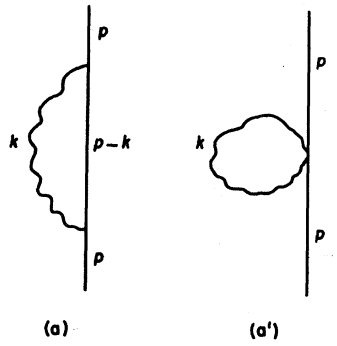

FIG. 4. Diagrams for the self-mass.

As $k_1\cdot k_2=m(\omega_1-\omega_2)$, the result is

$$4+(\omega_1\omega_2)^{-1}(\omega_1-\omega_2)^2=\omega_1/\omega_2+\omega_2/\omega_1+2. \quad \text{(A8)}$$

b. Self-Mass

To illustrate the interaction with a virtual photon, consider Fig. 4. The contribution of diagram (a') is $-\bar{\sigma}_\mu\sigma_\mu=-4$, where the sum is taken over four directions of polarization of the virtual photon of momentum k. The single vertex operator at the upper vertex of (a) is $\boldsymbol{p}\boldsymbol{e}+\boldsymbol{e}(\boldsymbol{p}-\boldsymbol{k})$, which is replaced by $2p_\mu-\bar{\sigma}_\mu\boldsymbol{k}$ for summing over the polarizations; similarly at the lower vertex. The mass correction Δm is then $(e^2/2m\pi i)$ times

$$\int\frac{d^4k}{k^2}\left[\frac{(2p_\mu-\bar{\sigma}_\mu\boldsymbol{k})(2p_\mu-\boldsymbol{k}\sigma_\mu)}{(p-k)^2-m^2}-4\right] =4\int\frac{d^4k}{k^2}\left[\frac{p^2+k^2-p\cdot k}{(p-k)^2-m^2}-1\right] =4\int(p\cdot k+m^2)(k^2-2p\cdot k)^{-1}d_4k/k^2, \quad \text{(A9)}$$

where in the last step we have put $p^2=m^2$. This result is equivalent to the usual result and can be treated by the usual convergence techniques. Note, however, that no matrices are left in the result (A9).

APPENDIX III. STRONG V AND A INTERACTIONS

In view of the possibility that only V and A interactions occur in nature, it is interesting to see how this suggestion can be motivated starting with the free-particle Dirac equation $i\nabla\psi(x)=m\psi(x)$ which is usually considered to have elementary solutions

$$\psi(x)=ue^{-ip\cdot x}, \quad \text{with} \quad \boldsymbol{p}u=mu. \quad \text{(A10)}$$

However, this is not the most general solution which is characterized by the four-vector p_μ, but rather

$$\psi(x)=ue^{-ip\cdot x}+a\gamma_5ue^{ip\cdot x} \quad \text{(A11)}$$

with a arbitrary, since $\boldsymbol{p}\gamma_5u=-mu$. Symmetry con-

siderations[9] suggest $a=\pm 1$, and to be definite let us choose $a=1$.

For a local interaction with any boson field φ (of any tensor rank, with indices suppressed) we have, say,

$$I=\int (\bar{\psi}_2(x)O\psi_1(x))\cdot\varphi(x)d\tau \tag{A12}$$

and using $(\overline{\gamma_5 u})=-\bar{u}\gamma_5$, we get

$$I=\{\langle O\rangle-\langle\gamma_5 O\gamma_5\rangle\}\cdot\varphi'\delta(p_2-p_1-q) +\{\langle O\gamma_5\rangle-\langle\gamma_5 O\rangle\}\cdot\varphi'\delta(p_2+p_1-q). \tag{A13}$$

We have assumed $\varphi(x)=\varphi' e^{-iq\cdot x}$ and let $(\bar{u}_2 O u_1)=\langle O\rangle$, etc. Evidently $I=0$ unless O and γ_5 anticommute. Thus V and A are selected. Momentum conservation insures that in any given interaction only one of the brackets of (A13) will contribute. Therefore parity is conserved. If, on the other hand, we could write

$$\psi(x)=(1+\gamma_5)ue^{-ip\cdot x}, \tag{A14}$$

then we could have both V and A and maximum parity nonconservation would follow. But (A14) does not solve the Dirac equation with $m\neq 0$, for

$$i\nabla\psi(x)=m\chi(x)\neq m\psi(x), \tag{A15}$$

and

$$i\nabla\chi(x)=m\psi(x). \tag{A16}$$

For $m=0$, (A14) *is* a solution and this is the basis of the two-component neutrino theory, since $(1+\gamma_5)u$ has a two-component representation. However, (A14) solves the relativistic Pauli equation even for finite mass, which is the basis of the Feynman–Gell-Mann theory.

APPENDIX IV. UNIQUENESS OF TWO-COMPONENT REPRESENTATION

As it may appear that some of the foregoing conclusions have been based upon a rather special representation, we shall study here the possible equivent two-component representations.

It is hardly necessary to remark that as the matrices $\boldsymbol{\sigma}$ appear only in rotational invariant forms, they are subject to their usual transformation group without disturbing the formalism. As this is necessary, we are restricted to Dirac matrices β and $\boldsymbol{\alpha}$ of the form

$$\beta=\boldsymbol{\varrho}\cdot\mathbf{u}+u_0, \quad \boldsymbol{\alpha}=(\boldsymbol{\varrho}\cdot\mathbf{v}+v_0)\boldsymbol{\sigma}, \tag{A17}$$

where ρ_1, ρ_2, ρ_3 are the 4×4 anticommuting matrices introduced by Dirac, having properties similar to σ_1, σ_2, σ_3 and commuting with the latter; also u_0 and v_0 are real numbers and $\mathbf{u}$ and $\mathbf{v}$ are real numerical vectors, their reality being demanded to make β and $\boldsymbol{\alpha}$ Hermitian. As $\beta^2=I$, where I is the unit matrix, we must have either $u_0=0$ or $u_1=u_2=0$, using the standard representation of the ρ_i. Similarly we show that either $v_0=0$ or $v_1=v_2=0$. However, the required anticommutation of β with a component of $\boldsymbol{\alpha}$ eliminates three of these four possibilities, leaving only $u_0=v_0=0$. We have, therefore, that

$$\beta=\boldsymbol{\varrho}\cdot\mathbf{u}=\rho_u, \quad \alpha=\boldsymbol{\varrho}\cdot\mathbf{v}\boldsymbol{\sigma}=\rho_v\boldsymbol{\sigma} \tag{A18}$$

with $\mathbf{u}$ and $\mathbf{v}$ orthogonal unit vectors. Define also $\mathbf{w}=\mathbf{u}\times\mathbf{v}$. Then

$$\boldsymbol{\gamma}=\beta\boldsymbol{\alpha}=i\boldsymbol{\varrho}\cdot\mathbf{w}\boldsymbol{\sigma}=i\rho_w\boldsymbol{\sigma}. \tag{A19}$$

The possibility of describing a Dirac particle by a spinor having two components zero in all coordinate frames connected by proper Lorentz transformations rests on the fact that γ_5 can be written in diagonal form. We consider, therefore, the possible choices of $\boldsymbol{\gamma}$ and $\gamma_0=\beta$ under the assumption that $\gamma_5=\rho_3$. As

$$\gamma_5=i\alpha_1\alpha_2\alpha_3=i\rho_v\sigma_1\sigma_2\sigma_3=-\rho_v, \tag{A20}$$

we have that $\mathbf{v}=(0,0,-1)$. Since $\mathbf{u}\cdot\mathbf{v}=0$, $\mathbf{u}=(l,m,0)$ with $l^2+m^2=1$; also $\mathbf{w}=(-m,l,0)$. Thus, with $\kappa=l-im$,

$$\gamma_0=\begin{pmatrix}0 & \kappa\\ \kappa^* & 0\end{pmatrix}, \quad \boldsymbol{\gamma}=\begin{pmatrix}0 & \kappa\boldsymbol{\sigma}\\ -\kappa^*\boldsymbol{\sigma} & 0\end{pmatrix}. \tag{A21}$$

The Dirac operator $\Pi_\mu\gamma_\mu$ applied to (Ω,ψ) then yields

$$\kappa^*\boldsymbol{\Pi}^+\Omega=m\psi, \tag{A22}$$

$$\kappa\boldsymbol{\Pi}^-\psi=m\Omega, \tag{A23}$$

which give, on substitution, precisely Eqs. (8) and (9) of the text.

[9] For example, we might require symmetry under mass reversal, or that $\gamma_5\psi$ be equivalent to $\dot{\psi}$.