PHYSICAL REVIEW VOLUME 111, NUMBER 3 AUGUST 1, 1958

Equation for Vertex Part Corresponding to Fermion-Fermion Scattering

K. TER-MARTIROSYAN
Department of Physical Mathematic Sciences, U.S.S.R. Academy of Sciences, Moscow, U.S.S.R.
(Received December 26, 1957)

A system of equations is deduced whereby the vertex operator for fermion-fermion scattering and the Green's function are expressible in terms of the sum of contributions from a subset of all irreducible diagrams for the process.

IN the following we obtain the equations for the vertex part $\Gamma(p_4,p_3;p_2,p_1)$ (represented by the diagram to the left of the equal sign in Fig. 3) corresponding to fermion-fermion scattering. Similar equations were obtained earlier[1] for the vertex part corresponding to boson-boson scattering; they somewhat resemble the Bethe-Salpeter equation,[2] but here we present the method of the summation of the irreducible diagrams in two dimensions[3] instead of one as in the Bethe-Salpeter equation.

We consider a case where the operator for the fermion interaction has the form[4]

$$V=2\pi^2 g_0\int(\bar{\psi}O_j\psi)(\bar{\psi}O_j\psi)dv, \quad (1)$$

the simplest diagram [Fig. 1(a)] corresponding [within a factor $(2\pi)^2 ig_0$ which is omitted everywhere in the quantities corresponding to the vertex-part diagrams] to the value

$$\alpha_0=(O_j)_{\xi_4\xi_2}(O_j)_{\xi_3\xi_1}-(O_j)_{\xi_4\xi_1}(O_j)_{\xi_3\xi_2}\equiv\alpha_0(\xi_4,\xi_3;\xi_2,\xi_1),$$

where ξ_i are the indices of the spin, the states of the fermion energy, the charge, the strangeness, etc. Analogously, to the second-order diagrams, [see the upper parts of Figs. 2(a) and 2(b)] will correspond

$$F_1(p_4\xi_4,p_3\xi_3;p_2\xi_2,p_1\xi_1)=\frac{ig_0}{2}\int\alpha_0(\xi_4,\xi_3;\mu,\lambda)\times G_{\mu\nu}(l)G_{\lambda\sigma}(l')\alpha_0(\nu,\sigma;\xi_2,\xi_1)d^4l, \quad (2)$$

$$\Phi_1(p_4\xi_4,p_3\xi_3;p_2\xi_2,p_1\xi_1)=\frac{1}{i}g_0\int\alpha_0(\xi_4,\mu;\xi_2,\lambda)\times G_{\lambda\sigma}(l'')\alpha_0(\sigma,\xi_3;\nu,\xi_1)G_{\nu\mu}(l)d^4l, \quad (3)$$

where $l'=p_1+p_2-l$, $l''=l-p_2+p_4$, $d^4l=dl_0dl_1dl_2dl_3/(2\pi)^2$, and $G_{\mu\nu}(l)$ is the Green function of the fermion; the repeated indices μ, ν, λ, σ are to be summed over always.

Among the infinite number of diagrams determining Γ there are some which consist of two parts connected only by two lines: of the type shown in Fig. 2(a), the two external lines coming up to one part of the diagram and coming out of the other, and of the type in Figs. 2(b) and (c), where the fermion lines enter each part and come out of the *same* one they enter. We shall call these diagrams reducible and shall denote the sum of the contributions of all the diagrams of the Fig. 2(a) type by $F(p_4,p_3;p_2,p_1)$ ("horizontal brick"),[5] and of the diagrams of the type of Fig. 2(b) by $\Phi(p_4,p_3;p_2,p_1)$ ("vertical brick)[5]; we denote the sum of all

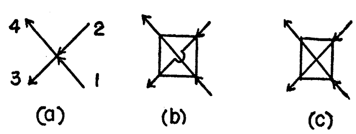

FIG. 1. Simplest irreducible diagrams.

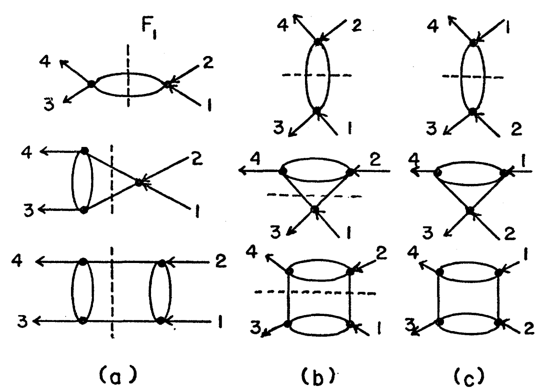

FIG. 2. Simplest irreducible diagrams.

[1] For analysis of the role of diagrams corresponding to meson-meson scattering in the question of vanishing of the renormalized charge, see I. T. Diatlov and K. A. Ter-Martirosyan, Zhur. Exptl. i Teort. Fiz. **30**, 416 (1956) [translation: Soviet Phys. JETP **3**, 454 (1956)]. Pomeranchuk, Sudakov, and Ter-Martirosyan, Phys. Rev. **103**, 784 (1956); Diatlov, Sudakov, and Ter-Martirosyan, Zhur. Eksptl. i Teoret. Fiz. **32**, 767 (1957) [translation: Soviet Phys. JETP **5**, 631 (1951)].

[2] E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951).

[3] As a result each irreducible diagram in the equation gives rise to an infinite set of diagrams reducible in two directions.

[4] If the interaction between the nucleons goes through mesons, the operators O_j will depend on the Green's functions of the mesons; the operator $O_j\times O_j$ should possess the symmetry properties indicated by Critchfield and Wigner [C. L. Critchfield and E. P. Wigner, Phys. Rev. **60**, 412 (1941); C. L. Critchfield, Phys. Rev. **63**, 417 (1943)] if the fermions are neutral.

[5] The terms "horizontal" or "vertical" correspond to Fig. 2 in which (as elsewhere in the following) the incoming fermion ends are arranged at the right-hand side and the outgoing at the left-hand side.

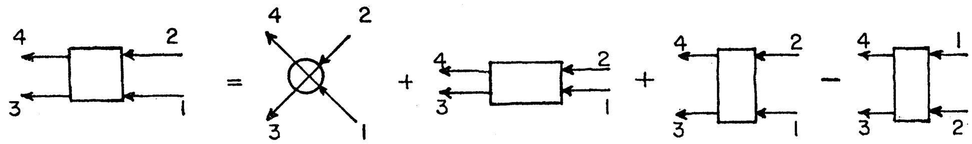

FIG. 3. Graphical representation of Eq. (4).

other irreducible diagrams (examples of such are given in Fig. 1) by $\Gamma_0(p_4,p_3;\,p_2,p_1)$. These three quantities are represented by the second, third, and first diagrams, respectively, to the right of the equal sign in Fig. 3. Since any diagram is either reducible or irreducible, then

$$\Gamma(p_4,p_3;\,p_2,p_1)=\Gamma_0(p_4,p_3;\,p_2,p_1)+F(p_4,p_3;\,p_2,p_1)+\Phi(p_4,p_3;\,p_2,p_1)-\Phi(p_4,p_3;\,p_1,p_2), \quad (4)$$

an equation whose diagrammatic equivalent is shown in Fig. 3.

In analogy with the case of bosons, equations can be obtained by which F and Φ (i.e., the whole sum Γ) can be expressed in terms of Γ_0. For this purpose consider an arbitrary reducible diagram and draw the line dividing it into two parts as far as possible to the left for horizontal diagrams [Figs. 2(a) and 4(a)] and as far up as possible for vertical ones [Figs. 2(b) and 4(b)].[6]

If all the diagrams of the type shown in Fig. 2(a) and 4(a) are summed, then to the left of the dotted line only diagrams of the type in Fig. 1 and Fig. 2(b) or 4(b) will appear [and analogous ones of the type in Fig. 2(c), in which the two incoming lines have changed places], to which corresponds the value

$$R(p_4,p_3;\,p_2,p_1)=\Gamma_0(p_4,p_3;\,p_2,p_1)+\Phi(p_4,p_3;\,p_2,p_1)-\Phi(p_4,p_3;\,p_1,p_2)\equiv\Gamma-F, \quad (5)$$

while the aggregate Γ of all the diagrams will again be at the right. Therefore, in analogy to (2),

$$F(p_4\xi_4,p_3\xi_3;\,p_2\xi_2,p_1\xi_1)=\frac{ig_0}{2}\int R(p_4\xi_4,p_3\xi_3;\,l\mu,l'\lambda)\times G_{\mu\nu}(l)G_{\lambda\sigma}(l')\Gamma(l\nu,l'\sigma;\,p_2\xi_2,p_1\xi_1)d^4l. \quad (6)$$

Likewise, summing all the diagrams of Figs. 2(b)–4(b) type, we get, in analogy with (3),

$$\Phi(p_4\xi_4,p_3\xi_3;\,p_2\xi_2,p_1\xi_1)=\frac{1}{i}g_0\int R_1(p_4\xi_4,l\mu;\,p_2\xi_2,l''\lambda)\times G_{\lambda\sigma}(l'')\Gamma(l''\sigma,p_3\xi_3;\,l\nu,p_1\xi_1)G_{\nu\mu}(l)d^4l, \quad (7)$$

where

$$R_1(p_4,p_3;\,p_2,p_1)=\Gamma_0(p_4,p_3,;\,p_2,p_1)+F(p_4,p_3;\,p_2,p_1)-\Phi(p_4,p_3;\,p_1,p_2)\equiv\Gamma-\Phi(p_4,p_3;\,p_2,p_1). \quad (8)$$

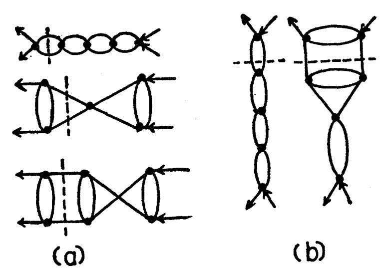

FIG. 4. Arbitrary diagrams showing method of division.

These equations are represented graphically in the upper and lower parts, respectively, of Fig. 5. Together with the Dyson-Schwinger equation for the Green's function,

$$\left\{(-i\gamma\cdot p-m)_{\xi'\zeta}-\frac{g_0^2}{2}\int\alpha_0(\mu,\xi';\,\lambda,\epsilon)G_{\lambda\sigma}(l-l')\times G_{\epsilon\eta}(p+l')\Gamma(l-l',\,\sigma,\,p+l',\,\eta;\,l\nu,p\zeta)\times G_{\nu\mu}(l)d^4ld^4l'\right\}G_{\zeta\xi}(p)=\delta_{\xi'\xi} \quad (9)$$

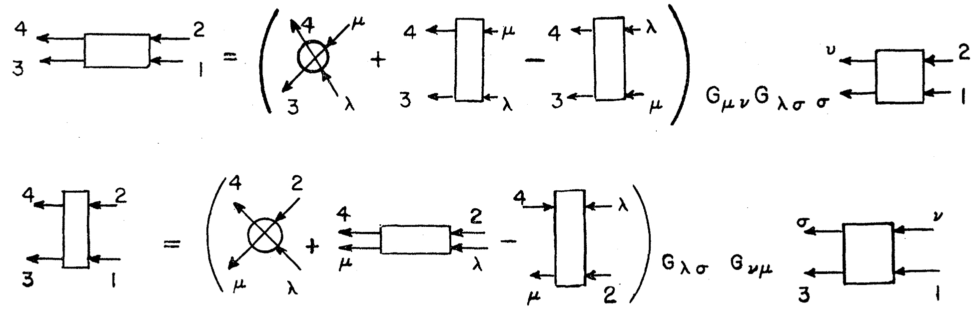

FIG. 5. Graphical representation of Eqs. (6) (upper part) and (7) (lower part).

[6] Or, on the contrary, as far to the right and as close to the bottom as possible; the result will be the same.

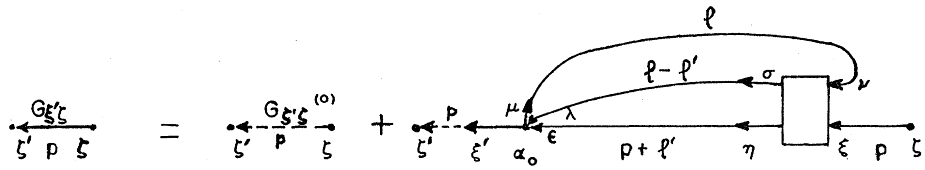

FIG. 6. Graphical representation of Eq. (9).

(the graphical equivalent of which is shown in Fig. 6), they are a closed system, making it possible to express Γ and G in terms of Γ_0.

In conclusion the author wishes to acknowledge his indebtedness to I. Y. Pomeranchuk for valuable discussions.

PHYSICAL REVIEW VOLUME 111, NUMBER 3 AUGUST 1, 1958

Variational Methods in Scattering Problems

MILDRED MOE* AND DAVID S. SAXON†
University of California, Los Angeles, California
(Received April 9, 1958)

Modifications of the Hulthén-Kohn variational principle are introduced with the hope of increasing the usefulness of variational methods. Although no simple formulation of general utility is found, it is shown that there exists a great variety of stationary expressions which make possible a greater freedom in the choice of variational principles than has hitherto been demonstrated. Some criteria for the selection of special forms are discussed.

I. INTRODUCTION

CONSIDERABLE attention has been given in the past to the development of variational principles for scattering problems.[1–4] However, the utility of these principles is limited by the difficulty of finding good trial functions and evaluating the necessary integrals, particularly so for the total scattering amplitude. In Schwinger's variational principle, the impediment is the evaluation of the double integral containing the Green's function. Even for plane waves, which are the simplest trial functions, this integral is not easy. On the other hand, the Hulthén-Kohn variational principle, which involves simpler integrations, is limited by the difficulty of finding adequate trial functions. The structure of this variational principle is such that it requires better trial functions than Schwinger's principle. For example, plane wave trial functions, which give a result similar to the second Born approximation in the Schwinger case, yield merely the first Born approximation in the Hulthén-Kohn case.

Our main purpose in the present paper is to discuss what might be done to overcome these difficulties. As we shall show, the Hulthén-Kohn and Schwinger principles are not the only stationary expressions for the scattering amplitude. There is a limitless number of other forms which may be obtained in a simple way. However, we have been unable to exploit this freedom sufficiently to construct variational principles entirely free of the troubles mentioned above. In any case, the existence of this great variety of forms is of independent interest.

II. FORMS OF THE VARIATIONAL PRINCIPLE

Schrödinger's equation for a two-body interaction can be written in the dimensionless form

$$[\nabla^2+k^2-V(\mathbf{x})]\psi(\mathbf{x})=0, \qquad (1)$$

where

$$\mathbf{x}=\mathbf{r}/a, \quad k^2=2ma^2E/\hbar^2,$$

and

$$V(\mathbf{x})=(2ma^2/\hbar^2)V_0(a\mathbf{x}).$$

Here m is the reduced mass, E is the energy in the center-of-mass system, V_0 is the potential energy of interaction, and a is a characteristic length associated with the range of the potential.

The well-known integral equation corresponding to Eq. (1) is

$$\psi_{\mathbf{k}}(\mathbf{x})=e^{i\mathbf{k}\cdot\mathbf{x}}-\int G(\mathbf{x},\mathbf{x}')V(\mathbf{x}')\psi_{\mathbf{k}}(\mathbf{x}')d\mathbf{x}', \qquad (2)$$

* Present address, The Ramo-Wooldridge Corporation, Los Angeles, California.
† Supported in part by the National Science Foundation.

[1] L. Hulthén, Kgl. Fysiograf. Sällskap. hund Förh. **14**, 257 (1944); Den 10 Skandinaviske Matematiker Kongres (1946), p. 201; Arkiv Mat. Astron. Fysik **35A**, No. 25 (1948).
[2] J. Schwinger, lectures on nuclear physics, Harvard University, 1947 (unpublished); H. Levine and J. Schwinger, Phys. Rev. **74**, 958 (1948).
[3] W. Kohn, Phys. Rev. **74**, 1763 (1948); **84**, 495 (1951).
[4] H. Feshbach and S. I. Rubinow, Phys. Rev. **88**, 484 (1952); S. I. Rubinow, Phys. Rev. **98**, 183 (1955).