Multilevel Formula for the Fission Process*

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The dispersion theory formalism of Wigner and Eisenbud has been applied to the problem of interference between resonances in the slow-neutron fission cross section. The development presented here assumes that the process involves one neutron channel and a large number of channels for the capture process. Although the treatment, for simplicity, assumes that only one fission channel is open, the relevant expressions for the fission cross section are presented in such a form that the generalization to any number of fission channels is readily apparent. Although no restriction as to the number of levels is necessary, the results are of practical use only when the number of assumed fission channels is small. Multilevel expressions for the radiative capture and scattering cross sections are also presented for one fission channel. The radiative capture cross section in the absence of fission is given and shown to differ formally from a sum of single-level Breit-Wigner terms.

I. INTRODUCTION

In recent years numerous measurements of slow-neutron resonances in the thermally fissile nuclides have been made.[1] As the experimental techniques, and consequently the data, have improved, it has become apparent that some of these resonances exhibit a definite asymmetry in the fission cross section[2]; their shape is not describable by the usual single-level Breit-Wigner formula. As an explanation of these observed asymmetries, two possibilities have been suggested: (1) they are due to the presence of small unresolved levels near the more prominent ones; and (2) they are due to interference between the resonance levels. While it is quite likely that in some cases small unresolved resonances are indeed responsible for this effect, there is an increasing amount of experimental evidence which points toward interference as being a more frequent cause. Some of the evidence for this latter hypothesis are the following:

(1) The analysis of the experiments of Shore and Sailor[3] on U^{235} indicates that, although some of the resonances in the U^{235} fission cross section are asymmetric, the corresponding resonances in the radiative capture cross section show the expected symmetry of a Breit-Wigner shape.

(2) The quantity η or $\nu/(1+\alpha)$ exhibits a net slope in the region of an asymmetric resonance. This behavior may be interpreted as evidence for interference.[4]

(3) For a given process, the size distribution of the reduced widths of a large number of levels gives some information about the number of channels open to the process.[5] For the fissile nuclei the distributions of fission widths are consistent with a small number of fission channels.

In the presence of interference between levels, one should use a multilevel formula, since a single-level Breit-Wigner formula is not expected to describe the individual resonances. One such multilevel formula which has been widely used is the approximate form given in a paper by Feshbach, Porter, and Weisskopf.[6] In this formula, the individual interfering resonance amplitudes are additive, *viz.*,

$$\sigma_{ij}\sim\left|\sum_{\lambda}\frac{y_{\lambda i}y_{\lambda j}}{E_\lambda-E-\frac{1}{2}i\Gamma_\lambda}\right|^2.$$

As Feshbach, Porter, and Weisskopf point out, however, because of the approximations made in its derivation, this formula may not yield an accurate description of the cross section in regions where the level widths are of the same order of magnitude as the spacing. In the case of the fission cross section of U^{233},[7] for example, it appears that the approximations in the FPW formula may not be good ones. It is thus of interest to develop a multilevel formula which contains no assumptions concerning the relation of the level widths to their separation.

The formalism developed in the following section is presently being applied to the slow-neutron cross sections of U^{233}.[8]

II. DEVELOPMENT OF THE FORMULA

The basis of the development is the relation between the derivative matrix (**R**) and collision matrix (**S**) as

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[1] *Neutron Cross Sections*, compiled by D. J. Hughes and J. A. Harvey, Brookhaven National Laboratory Report BNL-325 (Superintendent of Documents, U. S. Government Printing Office, Washington, D. C., 1955); D. J. Hughes and R. B. Schwartz, Supplement No. 1 to BNL-325 (1957).

[2] V. L. Sailor, *International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1955* (United Nations, New York, 1956), Vol. IV, p. 199.

[3] F. J. Shore and V. L. Sailor, Bull. Am. Phys. Soc. Ser. II, 2, 70 (1957).

[4] L. M. Bollinger, in Atomic Energy Research Establishment, Harwell Report NP/R 2076 (revised), edited by N. J. Pattenden, (1957), p. 21.

[5] C. E. Porter and R. G. Thomas, Phys. Rev. **104**, 483 (1956).

[6] Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954).

[7] Moore, Miller, and Reich, Bull. Am. Phys. Soc. Ser. II, **1**, 327 (1956).

[8] J. E. Evans and R. G. Fluharty, *Proceedings of the International Conference on Neutron Interaction with Nuclei*, Columbia University, 1957 (to be published). Also, M. S. Moore and C. W. Reich (to be published).

given by Wigner and Eisenbud[9]:

$$\mathbf{S}=\boldsymbol{\omega}\frac{1+i(\mathbf{BRB}+\mathbf{C})}{1-i(\mathbf{BRB}+\mathbf{C})}\boldsymbol{\omega}. \quad (1)$$

The cross sections are then related to the elements, S_{ij}, of the collision matrix by relations of the form

$$\sigma_{ij}=\pi\lambda\!\!\!^{-}{}_i^2 g|\delta_{ij}-S_{ij}|^2. \quad (2)$$

These cross sections are those due only to resonance levels of the same total angular momentum and parity. In order to arrive at the experimentally measured cross sections, one must add to this the contributions from the other spin state. The elements of the diagonal matrices $\boldsymbol{\omega}$, $\mathbf{B}$, and $\mathbf{C}$ contain the properties of the external regions in the various channels. The $\mathbf{R}$, or derivative, matrix contains the specifically nuclear terms, and has the following form:

$$\mathbf{R}=\sum_\lambda \frac{\boldsymbol{\gamma}_\lambda\times\boldsymbol{\gamma}_\lambda}{E_\lambda-E}, \quad (3)$$

where $\boldsymbol{\gamma}_\lambda\times\boldsymbol{\gamma}_\lambda$ is the direct product of the vectors $\boldsymbol{\gamma}_\lambda$.[10] The elements of $\boldsymbol{\gamma}_\lambda$ are reduced width parameters, and are usually treated as adjustable parameters in the application of the theory to a given set of data. The sum is over levels in the compound nucleus with the same total angular momentum and parity. Since the contributions of interfering levels are additive in the derivative matrix, it is a convenient quantity with which to work.

For one level, the expression for the elements of the collision matrix has the form $S_{ij}\sim y_{\lambda i}y_{\lambda j}/[E_\lambda-E-\frac{1}{2}i\Gamma_\lambda]$, the usual Breit-Wigner amplitude. One sees that the FPW approximation is equivalent to assuming that the contributions from interfering levels in the compound nucleus are additive in the collision matrix.

The relationship (1) is quite general, containing as it does an infinite number of parameters (the $\boldsymbol{\gamma}_\lambda$ and E_λ). It is of practical use only when the number of parameters is reasonably small. For the present situation, it must describe a nuclear reaction in which three processes—scattering, fission, and radiative capture—occur.

It is assumed that there is only one neutron channel, since, in the range of neutron energy under consideration, only neutrons with $l=0$ are expected to contribute. For the reasons previously mentioned, it is quite likely that there is a small number of fission channels open; and, for simplicity, this number is taken to be one. The result for one fission channel is quite easily generalized to include any number of such channels. For the emission of capture radiation, however, it is assumed that there is a large number of exit channels available. The relative constancy of the radiation widths for levels of the same spin and parity within a given nucleus would tend to support this.[5] The effects of the closed channels[11] are neglected in the following development. Implicit in this treatment, of course, is the assumption that the dispersion theory is a valid representation of the fission resonances.

Subject to these assumptions, the derivative matrix may be written in the form

$$\mathbf{R}=\begin{pmatrix} R_{11} & R_{12} & R_{13} & \cdots & R_{1\,n+2} \\ R_{21} & R_{22} & R_{23} & \cdots & R_{2\,n+2} \\ R_{31} & R_{32} & R_{33} & \cdots & R_{3\,n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ R_{n+2\,1} & R_{n+2\,2} & R_{n+2\,3} & \cdots & R_{n+2\,n+2} \end{pmatrix}.$$

The subscripts 1 and 2 refer to the neutron and fission channels, respectively. The subscripts 3, $\cdots n+2$ refer to the n radiative capture channels.

The diagonal matrices $\boldsymbol{\omega}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{1}$ are also $(n+2)\times(n+2)$. One may make the formal manipulation of the matrix products somewhat more simple by partitioning the matrices in the following manner:

$$\mathbf{R}=\begin{pmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22}\end{bmatrix} & \begin{bmatrix} R_{13} & \cdots & R_{1\,n+2} \\ R_{23} & \cdots & R_{2\,n+2}\end{bmatrix} \\ \begin{bmatrix} R_{31} & R_{32} \\ \vdots & \vdots \\ R_{n+2\,1} & R_{n+2\,2}\end{bmatrix} & \begin{bmatrix} R_{33} & \cdots & R_{3\,n+2} \\ \vdots & & \vdots \\ R_{n+2\,3} & \cdots & R_{n+2\,n+2}\end{bmatrix} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{R}_{2\times2} & \mathbf{R}_{2\times n} \\ \mathbf{R}_{n\times2} & \mathbf{R}_{n\times n}\end{pmatrix}.$$

One may then treat the matrices as 2×2 matrices whose elements are themselves matrices and may form the product $(\mathbf{BRB}+\mathbf{C})$ in the usual manner. Let submatrices of the matrix $\mathbf{a}=(\mathbf{BRB}+\mathbf{C})$ be defined as follows:

$$\mathbf{a}\equiv\begin{pmatrix} \mathbf{a}_{2\times2} & \mathbf{a}_{2\times n} \\ \mathbf{a}_{n\times2} & \mathbf{a}_{n\times n}\end{pmatrix},$$

where

$$\begin{aligned} \mathbf{a}_{2\times2}&=\mathbf{B}_{2\times2}\mathbf{R}_{2\times2}\mathbf{B}_{2\times2}+\mathbf{C}_{2\times2}, \\ \mathbf{a}_{2\times n}&=\mathbf{B}_{2\times2}\mathbf{R}_{2\times n}\mathbf{B}_{n\times n}, \\ \mathbf{a}_{n\times2}&=\mathbf{B}_{n\times n}\mathbf{R}_{n\times2}\mathbf{B}_{2\times2}, \\ \mathbf{a}_{n\times n}&=\mathbf{B}_{n\times n}\mathbf{R}_{n\times n}\mathbf{B}_{n\times n}+\mathbf{C}_{n\times n}. \end{aligned}$$

Equation (1) may be written

$$\begin{aligned} \boldsymbol{\omega}^{-1}\mathbf{S}\boldsymbol{\omega}^{-1} &=\mathbf{1}+2i[\mathbf{1}-i(\mathbf{BRB}+\mathbf{C})]^{-1}[\mathbf{BRB}+\mathbf{C}] \\ &=\mathbf{1}+2i\begin{pmatrix} \mathbf{1}-i\mathbf{a}_{2\times2} & -i\mathbf{a}_{2\times n} \\ -i\mathbf{a}_{n\times2} & \mathbf{1}-i\mathbf{a}_{n\times n}\end{pmatrix}^{-1}\begin{pmatrix} \mathbf{a}_{2\times2} & \mathbf{a}_{2\times n} \\ \mathbf{a}_{n\times2} & \mathbf{a}_{n\times n}\end{pmatrix}. \quad (4) \end{aligned}$$

In this instance, the problem of inverting $\mathbf{1}-i(\mathbf{BRB}+\mathbf{C})$ is formally quite simple. The inverse of a 2×2 matrix,

[9] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).
[10] E. P. Wigner, Phys. Rev. **70**, 606 (1946).
[11] T. Teichmann and E. P. Wigner, Phys. Rev. **87**, 123 (1952).

each element of which is itself a matrix, is[12]

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}^{-1} = \begin{pmatrix} [\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C}]^{-1} & [\mathbf{B}\mathbf{D}^{-1}\mathbf{C}-\mathbf{A}]^{-1}\mathbf{B}\mathbf{D}^{-1} \\ [\mathbf{C}\mathbf{A}^{-1}\mathbf{B}-\mathbf{D}]^{-1}\mathbf{C}\mathbf{A}^{-1} & [\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}]^{-1} \end{pmatrix}.$$

With this identification, the expression for $\boldsymbol{\omega}^{-1}\mathbf{S}\boldsymbol{\omega}^{-1}$ follows from (4).

$$\boldsymbol{\omega}^{-1}\mathbf{S}\boldsymbol{\omega}^{-1} = \begin{bmatrix} \dfrac{1+i\mathbf{a}_{2\times2}+i\mathbf{a}_{2\times n}(1-i\mathbf{a}_{n\times n})^{-1}i\mathbf{a}_{n\times2}}{1-i\mathbf{a}_{2\times2}-i\mathbf{a}_{2\times n}(1-i\mathbf{a}_{n\times n})^{-1}i\mathbf{a}_{n\times2}} & \dfrac{2i\mathbf{a}_{2\times n}(1-i\mathbf{a}_{n\times n})^{-1}}{1-i\mathbf{a}_{2\times2}-i\mathbf{a}_{2\times n}(1-i\mathbf{a}_{n\times n})^{-1}i\mathbf{a}_{n\times2}} \\ \dfrac{2i\mathbf{a}_{n\times2}(1-i\mathbf{a}_{2\times2})^{-1}}{1-i\mathbf{a}_{n\times n}-i\mathbf{a}_{n\times2}(1-i\mathbf{a}_{2\times2})^{-1}i\mathbf{a}_{2\times n}} & \dfrac{1+i\mathbf{a}_{n\times n}+i\mathbf{a}_{n\times2}(1-i\mathbf{a}_{2\times2})^{-1}i\mathbf{a}_{2\times n}}{1-i\mathbf{a}_{n\times n}-i\mathbf{a}_{n\times2}(1-i\mathbf{a}_{2\times2})^{-1}i\mathbf{a}_{2\times n}} \end{bmatrix}. \qquad (5)$$

In order to obtain the various partial cross sections, it is necessary to calculate the elements of the first row of $\mathbf{S}$. Before this can be done, however, one must evaluate the matrix $(1-i\mathbf{a}_{n\times n})^{-1}$. Neglecting the effects of the elements of the matrix $\mathbf{C}_{n\times n}$, one has

$$\begin{aligned} 1-i\mathbf{a}_{n\times n} &= 1-i\mathbf{B}_{n\times n}\mathbf{R}_{n\times n}\mathbf{B}_{n\times n} \\ &= 1-i\sum_\lambda \frac{\boldsymbol{\beta}_\lambda\times\boldsymbol{\beta}_\lambda}{E_\lambda-E}, \end{aligned} \qquad (6)$$

where

$$\boldsymbol{\beta}_\lambda \equiv \mathbf{B}\boldsymbol{\gamma}_\lambda.$$

It is important to note that the components of both $\mathbf{B}$ and $\boldsymbol{\gamma}_\lambda$ in (6) refer to the capture channels only.

Following Wigner,[10] one assumes an expansion of the form

$$\begin{aligned} (1-i\mathbf{a}_{n\times n})^{-1} &= \left(1-i\sum_\lambda \frac{\boldsymbol{\beta}_\lambda\times\boldsymbol{\beta}_\lambda}{E_\lambda-E}\right)^{-1} \\ &= 1+i\sum_{\mu,\nu} A_{\mu\nu}(\boldsymbol{\beta}_\mu\times\boldsymbol{\beta}_\nu), \end{aligned}$$

where the subscripts μ and ν refer to the interfering levels in the compound nucleus. In order that the inverse have the form indicated, the $A_{\mu\nu}$ must satisfy the following equation[10]:

$$\delta_{\mu\lambda} = A_{\mu\lambda}(E_\lambda-E) - i\sum\nolimits_\nu A_{\mu\nu}(\boldsymbol{\beta}_\nu\cdot\boldsymbol{\beta}_\lambda),$$

where $\delta_{\mu\lambda}$ is the usual Kronecker delta, and

$$(\boldsymbol{\beta}_\nu\cdot\boldsymbol{\beta}_\lambda) \equiv \sum_j \beta_{\nu j}\beta_{\lambda j} = \sum_{j=3}^{n+2} B_j^2\gamma_{\lambda j}\gamma_{\nu j}.$$

The sum goes over all the exit channels for gamma radiation. For $\lambda=\nu$, this sum can be thought of as a sum of partial radiation widths since, on the single level definition, $\frac{1}{2}\Gamma_{\lambda j}=B_j^2\gamma_{\lambda j}^2$.[9] For $\lambda\neq\nu$, if it is assumed that the $\gamma_{\lambda j}$ have random signs and exhibit random size variations for the various channels j,[5] this sum, when taken over a large number of channels, would be expected to vanish, or at least to be small compared to $\Gamma_{\lambda\gamma}$. One then defines

$$(\boldsymbol{\beta}_\nu\cdot\boldsymbol{\beta}_\lambda) = \tfrac{1}{2}\Gamma_{\lambda\gamma}\delta_{\lambda\nu}.$$

With this identification, one obtains

$$A_{\mu\lambda} = \delta_{\mu\lambda}/(E_\lambda - E - \tfrac{1}{2}i\Gamma_{\lambda\gamma}),$$

and the expression for the inverse becomes

$$(1-i\mathbf{a}_{n\times n})^{-1} = 1+i\sum_\lambda \frac{\boldsymbol{\beta}_\lambda\times\boldsymbol{\beta}_\lambda}{E_\lambda-E-\frac{1}{2}i\Gamma_{\lambda\gamma}}.$$

The matrix multiplications indicated in the expressions for the 2×2 and $2\times n$ components of $\boldsymbol{\omega}^{-1}\mathbf{S}\boldsymbol{\omega}^{-1}$ in Eq. (5) may now be carried out.

The matrix $\mathbf{a}_{2\times2}+i\mathbf{a}_{2\times n}(1-i\mathbf{a}_{n\times n})^{-1}\mathbf{a}_{n\times2}$ appearing in the 2×2 portion of $\boldsymbol{\omega}^{-1}\mathbf{S}\boldsymbol{\omega}^{-1}$ has the simple form $(\mathbf{B}\mathbf{R}'\mathbf{B}+\mathbf{C})_{2\times2}$, where the subscript 2×2 indicates that all matrices are the 2×2 portions of their respective unpartitioned matrices, and where $\mathbf{R}'$ is defined by the equation

$$\mathbf{R}' = \sum_\lambda (\boldsymbol{\gamma}_\lambda\times\boldsymbol{\gamma}_\lambda)/(E_\lambda-E-\tfrac{1}{2}i\Gamma_{\lambda\gamma}).$$

The 2×2 portion of $\mathbf{S}$ may be written in the form

$$\mathbf{S} = \boldsymbol{\omega}\frac{1+i(\mathbf{B}\mathbf{R}'\mathbf{B}+\mathbf{C})}{1-i(\mathbf{B}\mathbf{R}'\mathbf{B}+\mathbf{C})}\boldsymbol{\omega}, \qquad (7)$$

all matrices being understood to be 2×2. The formal effect of a large number of capture channels on the 2×2 portion of $\mathbf{S}$ is to modify the definition of the derivative matrix, leaving the form of the relationship between it and the collision matrix unchanged. One may easily generalize this result to include any number of fission channels. In general, if one has $m-1$ fission channels, then the relation (7) holds, if all matrices are considered to be the $m\times m$ portions of their respective unpartitioned matrices. It might be pointed out, in addition, that the above development is not limited to the fission process, but is valid for any nuclear reaction which proceeds primarily through a few channels.

[12] A. C. Aitken, *Determinants and Matrices* (Oliver and Boyd, London, 1951), p. 139.

Fission Cross Section

For a single fission channel,

$$\sigma_{12}=4\pi\bar{\lambda}_1^2 g\left|\sum_\lambda \frac{\beta_{\lambda1}\beta_{\lambda2}}{E_\lambda-E-\frac{1}{2}i\Gamma_{\lambda\gamma}}\Big/ \left[\left(1-iC_1-i\sum_\lambda\frac{\beta_{\lambda1}^2}{E_\lambda-E-\frac{1}{2}i\Gamma_{\lambda\gamma}}\right)\left(1-iC_2-i\sum_\lambda\frac{\beta_{\lambda2}^2}{E_\lambda-E-\frac{1}{2}i\Gamma_{\lambda\gamma}}\right)+\left(\sum_\lambda\frac{\beta_{\lambda1}\beta_{\lambda2}}{E_\lambda-E-\frac{1}{2}i\Gamma_{\lambda\gamma}}\right)^2\right]\right|^2, \tag{8}$$

where the subscript 1 refers to the neutron channel, and the subscript 2 to the fission channel. The quantity g is the statistical weight factor, and $\bar{\lambda}_1$ is the neutron wavelength divided by 2π.

Other Partial Cross Sections

One may obtain a multilevel, multichannel expression for the radiative capture process. The elements of the $2\times n$ portion of $\mathbf{S}$ may be calculated, depending as they do essentially upon a knowledge of $(1-i\mathbf{a}_{n\times n})^{-1}$. For all j then, one knows the elements, S_{1j}, of the collision matrix. The capture cross section, $\sigma_{1\gamma}$, may be written as

$$\sigma_{1\gamma}=\pi\bar{\lambda}_1^2 g\sum_{j=3}^{n+2}|S_{1j}|^2.$$

This sum may be carried out with the following result, for one fission channel:

$$\sigma_{1\gamma}=4\pi\bar{\lambda}_1^2 g\frac{b_{11}[(1+b_{22})^2+a_{22}^2]+b_{22}a_{12}^2-b_{12}[2a_{12}a_{22}+b_{12}(2+b_{22})]}{|[1-i(a_{11}+ib_{11})][1-i(a_{22}+ib_{22})]+(a_{12}+ib_{12})^2|^2}, \tag{9}$$

where a_{ij} and b_{ij} are the real and imaginary parts of $C_i\delta_{ij}+\sum_\lambda[\beta_{\lambda i}\beta_{\lambda j}/(E_\lambda-E-\frac{1}{2}i\Gamma_{\lambda\gamma})]$, respectively. If one assumes no fission, (9) becomes

$$\sigma_{1\gamma}=4\pi\bar{\lambda}_1^2 g\frac{b_{11}}{(1+b_{11})^2+a_{11}^2}. \tag{9a}$$

This expression is not formally the sum of single-level Breit-Wigner terms. In the analysis of slow neutron resonance data in the medium and heavy elements, the assumption is commonly made[13] that the capture cross section is the sum of such terms. According to (9a), this assumption is quite good, providing that the neutron widths are much smaller than the radiation widths, since under these conditions a_{11} and b_{11} are much less than 1; and

$$\sigma_{1\gamma}\simeq\pi\bar{\lambda}_1^2 g\sum_\lambda\frac{\Gamma_{\lambda1}\Gamma_{\lambda\gamma}}{(E_\lambda-E)^2+\frac{1}{4}\Gamma_\lambda^2},$$

where $\Gamma_{\lambda1}=2\beta_{\lambda1}^2$ is the neutron width of the level λ, and $\Gamma_\lambda=\Gamma_{\lambda\gamma}+\Gamma_{\lambda1}\simeq\Gamma_{\lambda\gamma}$ is the total width. In such cases typical calculations show that the numerical differences between (9a) and a sum of Breit-Wigner terms are quite small.

In terms of the element, S_{11}, of the collision matrix, the expression for the S-wave neutron scattering cross section is

$$\sigma_{11}=\pi\bar{\lambda}_1^2 g|1-S_{11}|^2.$$

Written in terms of the quantities a_{ij} and b_{ij} defined above, the expression becomes

$$\sigma_{11}=\pi\bar{\lambda}_1^2 g\left|1-e^{-2ik_1r_1}\left[1+\frac{2i\{(a_{11}+ib_{11})-i(a_{11}+ib_{11})(a_{22}+ib_{22})+i(a_{12}+ib_{12})^2\}}{[1-i(a_{11}+ib_{11})][1-i(a_{22}+ib_{22})]+(a_{12}+ib_{12})^2}\right]\right|^2, \tag{10}$$

where k_1 is the neutron wave number, and r_1 is the neutron channel radius.

The elements of the matrix $\mathbf{C}$ appear in the formalism. Assuming the boundary condition of Wigner and Eisenbud,[9] one may show that

$$C_{s,l}=-\left[F_lF_l'+G_lG_l'+\frac{l}{\rho_s}(F_l^2+G_l^2)\right]\Bigg|_{\rho_s=k_sa_s},$$

where a_s is the radius in the channel (s,l), F_l and G_l are related to the regular and irregular wave functions, respectively, in the external region corresponding to the channel (s,l), and primes denote differentiation with respect to $\rho_s=k_sr_s$. For $l=0$ neutrons, this expression vanishes; and we thus take $C_1=0$. In the analysis of slow neutron resonances, where the neutron energies are varied by less than some tens of electron volts, $C_{s,l}$ in the fission channels is essentially constant. As such, it may, with the proper choice of boundary condition parameters,[11] be taken to be zero. However, it may be of interest to note the results of the calculation of $C_{s,l}$. In the fission channels, the F_l and G_l

[13] See, for example, Fluharty, Simpson, and Simpson, Phys. Rev. **103**, 1778 (1956).

become the usual Coulomb wave functions[14] and are completely specified by the parameters ρ_s, η_s, and l, where l specifies the relative orbital angular momentum, $\rho_s=k_s a_s$, and $\eta_s=Z_s Z_s' e^2/(\hbar v_s)$. In order to obtain values for ρ_s and η_s and thus $C_{s,l}$ in the fission channel, it is necessary to make some assumptions about the fission process. If one assumes that the kinetic energy of the fission fragments at large separations is equal to the potential energy at contact, then the boundary of the external region is defined by the condition $\rho_s=2\eta_s$.[15] On the basis of this crude model and for reasonable kinetic energies and charge splits between the two fragments, $\eta\sim 190$, and $C_{s,l}\sim\frac{1}{2}$ for $l=0$.

For simplicity, the elements of the matrix $\mathbf{C}_{n\times n}$ have been omitted. Krotkov[16] has shown that, to a first approximation, their inclusion leads to a redefinition of the $\Gamma_{\lambda\gamma}$ in terms of the $\gamma_{\lambda j}$ in the scalar product $(\boldsymbol{\beta}_\lambda\cdot\boldsymbol{\beta}_\nu)$. Since both the $\Gamma_{\lambda\gamma}$ and the $\gamma_{\lambda j}$ are adjustable parameters, in practice this distinction is unimportant.

The elements of the matrix $\mathbf{B}$ may also be calculated. For neutrons, $B_1=k_1^{\frac{1}{2}}$, where k_1 is the wave number of the relative motion in the neutron channel. This gives the usual $E^{\frac{1}{2}}$ energy dependence of the neutron width. For the fission channels, the elements of $\mathbf{B}$ are essentially constant over the range of neutron energy of interest. It is then convenient to consider the product $B_2\gamma_{\lambda 2}$ for the various fission channels as the adjustable parameter. The elements of matrix $\boldsymbol{\omega}$ are pure phases and thus do not enter the reaction cross sections. However, ω_1 does appear in the expression for the scattering cross section. For $l=0$ neutrons, $\omega_1=e^{-ik_1r_1}$, where r_1 is the radius in the neutron channel.

[14] Carl-Erik Fröberg, Revs. Modern Phys. **27**, 399 (1955).

[15] See, for example, K. A. Petrzhak, *Physics of Fission* (Supplement to Soviet Journal of Atomic Energy, translated by Consultant's Bureau, Inc., New York, 1957).

[16] R. Krotkov, Can. J. Phys. **33**, 622 (1955).

III. DISCUSSION

The parameters which occur naturally in this formalism are the characteristic energies, E_λ, and the various partial reduced width parameters, $\gamma_{\lambda s}$. For one fission channel, for instance, there would be four parameters (E_λ, $\gamma_{\lambda 1}$, $\beta_{\lambda 2}$, and $\Gamma_{\lambda\gamma}$) per level. In addition to these, the reduced width parameters are signed quantities. This choice of sign, relative to that of other levels, of course, is not completely arbitrary, since it must yield the proper type of interference.

The development presented here may be extended to include many fission channels. However, because of the rapidly increasing complexity as the assumed number of such channels increases, it is of practical use only when this number is small. In such cases, the inversion of the matrix $[\mathbf{1}-i(\mathbf{BR'B}+\mathbf{C})]$ in (7) is not difficult. The inclusion of a large number of levels, however, may be treated very simply; it merely involves working with more terms in the various sums.