Meissner Effect and Gauge Invariance*

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It is shown from a manifestly gauge-invariant Hamiltonian that the Meissner effect can follow from an energy-gap model of superconductivity. The superconductor is described by Fröhlich's Hamiltonian and the superconducting properties at the absolute zero are determined by a method due to Bogoliubov. In the weak-coupling limit ($T_c \ll \Theta_D$) there is an energy gap which leads to a Meissner effect. The method of Bogoliubov is extended to apply at general temperatures and the current is calculated in the weak-coupling limit. The results are in essential agreement with those of Bardeen, Cooper, and Schrieffer.

1. INTRODUCTION

QUESTIONS have been raised[1] concerning the gauge invariance of the theory of superconductivity of Bardeen, Cooper, and Schrieffer.[2] The idealized Hamiltonian which they used is not gauge invariant because of the momentum dependent cutoff on the effective interaction between electrons. While, as shown by Anderson,[3] errors arising from this cutoff are small in the weak coupling limit ($T_c \ll \Theta_D$), it is more convincing to start from a Hamiltonian which is manifestly gauge invariant.

The Hamiltonian used by BCS is based on one derived by Bardeen and Pines[4] with use of a canonical transformation which replaces the electron-phonon interaction by an effective interaction between electrons. Since the canonical transformation introduces extra terms in the expression for the magnetic interaction and current density (which again are small in the weak-coupling limit), it is desirable to start from the original Hamiltonian which includes the electron-phonon interactions and which is gauge invariant.

Since Coulomb interactions do not play an essential role in the explanation of the Meissner effect, we have simplified the problem by starting with Frölich's Hamiltonian[5] from which Coulomb interactions are omitted. For further simplicity we have calculated the current in the gauge for which $\operatorname{div}\mathbf{A}=0$. However, because the original Hamiltonian is gauge invariant it is clear that this simplification does not affect the existence of the Meissner effect. Indeed, Anderson[3] has shown by introducing the collective excitations of the system how the calculation is to be performed in another gauge.

The approach is that used by Bogoliubov[6] to obtain the ground-state energy and excitation energies of the superconducting state at the absolute zero. In Sec. 2 the method is outlined and in Sec. 3 the Meissner effect and an expression for the current at the absolute zero similar to that of BCS are obtained. In Sec. 4 the results are extended to higher temperatures.

Because the phonons have not been eliminated from the Hamiltonian, the calculations are complicated by the existence of the electron self-energies. BCS have assumed in comparing their results with experiment that the electron energies include the self-energy arising from the interaction of the normal electrons with the phonons. It is in principle possible to prove this assumption from the approach of this paper. This is carried out to some extent incidentally, but a complete proof requires that the calculation be taken to higher order, a program which is outside the scope of this paper.

2. OUTLINE OF BOGOLIUBOV'S METHOD

The aim of this section is to obtain the lowest eigenstates and eigenfunctions of Fröhlich's Hamiltonian,

$$H=\sum_{\mathbf{k},\sigma} \bar{\epsilon}_{\mathbf{k}} C_{\mathbf{k},\sigma}^* C_{\mathbf{k},\sigma}+\sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}} b_{\mathbf{q}}^* b_{\mathbf{q}} + \sum_{\substack{\mathbf{k},\mathbf{k}',\sigma\\ \mathbf{q}=\mathbf{k}'-\mathbf{k}}} g\left(\frac{\hbar\omega_{\mathbf{q}}}{2\Omega}\right)^{\frac{1}{2}} (C_{\mathbf{k},\sigma}^* C_{\mathbf{k}',\sigma} b_{\mathbf{q}}^* + \text{comp. conj.}), \quad (1)$$

where $\bar{\epsilon}_{\mathbf{k}}$ is the Bloch energy of an electron, $\hbar\omega_{\mathbf{q}}$ is the energy of a phonon of wave number $\mathbf{q}$, g is the interaction constant, and Ω is the volume. $b_{\mathbf{q}}^*$ is the operator which creates a phonon of wave vector $\mathbf{q}$ and $C_{\mathbf{k},\sigma}^*$ is the operator which creates an electron of wave vector $\mathbf{k}$ and spin σ. Throughout this paper it will be assumed that ω is a constant. To solve this problem Bogoliubov[6] has introduced the canonical transformation

$$\begin{aligned}\gamma_{\mathbf{k}0}&=U_{\mathbf{k}}C_{\mathbf{k}\uparrow}-V_{\mathbf{k}}C_{-\mathbf{k}\downarrow}^*,\\ \gamma_{\mathbf{k}1}&=U_{\mathbf{k}}C_{-\mathbf{k}\downarrow}+V_{\mathbf{k}}C_{\mathbf{k}\uparrow}^*,\end{aligned} \quad (2)$$

where $U_{\mathbf{k}}$, $V_{\mathbf{k}}$ are real constants that satisfy

$$U_{\mathbf{k}}^2+V_{\mathbf{k}}^2=1.$$

It is easily verified that $\gamma_{\mathbf{k}0}^*$, $\gamma_{\mathbf{k}1}^*$ satisfy the commutation relations for Fermi operators. They are operators which create quasi-particle excitations from the ground state, $|0\rangle$, defined by

$$\gamma_{\mathbf{k}0}|0\rangle=0, \quad \gamma_{\mathbf{k}1}|0\rangle=0 \quad \text{for all } \mathbf{k}.$$

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[5] H. Fröhlich, Proc. Roy. Soc. (London) **A215**, 291 (1952).
[6] N. N. Bogoliubov, Nuovo cimento **7**, 794 (1958).

With the choice of $U_{\mathbf{k}}$, $V_{\mathbf{k}}$ given below, Bogoliubov has shown that this state, which corresponds with the superconducting ground-state wave function of BCS, has a lower energy than the normal state. The operators, $\gamma_{\mathbf{k}}$, have been introduced independently by Valatin[7] to simplify the formalism of BCS. It is convenient, as BCS found, to allow the number of particles to vary in the wave function so that the expectation value of

$$N=\sum_{\mathbf{k},\sigma}C_{\mathbf{k},\sigma}^*C_{\mathbf{k},\sigma},$$

in the eigenstates is N_0, the number of electrons present. This is achieved by adding a term $-\lambda N$ to the Hamiltonian and choosing the Fermi level, λ, so that

$$\bar{N}=N_0. \tag{3}$$

In terms of the new operators the total Hamiltonian is given by the system of equations

$$H=H_0+H_1{}^\alpha+H_1{}^\beta+H_2{}^\alpha+U,$$

$$H_0=\sum_k E_k(\gamma_{k0}{}^*\gamma_{k0}+\gamma_{k1}{}^*\gamma_{k1})+\sum_q \hbar\omega_q b_q{}^*b_q,$$

$$\begin{aligned}H_1{}^\alpha=&\sum_{\substack{\mathbf{k},\mathbf{k}'\\ \mathbf{q}=\mathbf{k}'-\mathbf{k}}} g\left(\frac{\hbar\omega_q}{2\Omega}\right)^{\frac{1}{2}}\\ &\times[(U_{\mathbf{k}}V_{\mathbf{k}'}+U_{\mathbf{k}'}V_{\mathbf{k}})(\gamma_{\mathbf{k}0}{}^*\gamma_{\mathbf{k}'1}{}^*+\gamma_{\mathbf{k}1}\gamma_{\mathbf{k}'0})b_{\mathbf{q}}{}^*\\ &+(U_{\mathbf{k}}U_{\mathbf{k}'}-V_{\mathbf{k}}V_{\mathbf{k}'})(\gamma_{\mathbf{k}0}{}^*\gamma_{\mathbf{k}'0}\\ &\qquad+\gamma_{\mathbf{k}'1}{}^*\gamma_{\mathbf{k}1})b_{\mathbf{q}}{}^*+\text{comp. conj.}],\end{aligned} \tag{4}$$

$$H_1{}^\beta=\sum_k 2(\bar{\epsilon}_{\mathbf{k}}-\lambda)U_{\mathbf{k}}V_{\mathbf{k}}(\gamma_{\mathbf{k}0}{}^*\gamma_{\mathbf{k}1}{}^*+\gamma_{\mathbf{k}1}\gamma_{\mathbf{k}0}),$$

$$H_2{}^\alpha=\sum_{\mathbf{k}}[(\bar{\epsilon}_{\mathbf{k}}-\lambda)(U_{\mathbf{k}}{}^2-V_{\mathbf{k}}{}^2)-E_{\mathbf{k}}](\gamma_{\mathbf{k}0}{}^*\gamma_{\mathbf{k}0}+\gamma_{\mathbf{k}1}{}^*\gamma_{\mathbf{k}1}),$$

$$U=2\sum_{\mathbf{k}}(\bar{\epsilon}_{\mathbf{k}}-\lambda)V_{\mathbf{k}}{}^2.$$

The term $H_2{}^\alpha$ has been introduced to take account of the renormalization of the energies of the excited particles. For the discussion of the Meissner effect it is useful to introduce this renormalization explicitly into the Hamiltonian.

The problem is complicated by the fact that even in the normal state an electron is surrounded by a cloud of phonons which gives rise to a self-energy. The energy, $\epsilon_{\mathbf{k}}$, of the electron in the normal state is therefore different from $\bar{\epsilon}_{\mathbf{k}}$, the energy of a "bare" electron. Since we wish to compare the superconducting state with the normal state, we replace $\bar{\epsilon}_{\mathbf{k}}$ by $\epsilon_{\mathbf{k}}$ in Eqs. (4) and compensate for this by adding the terms $H_2{}^\beta$ and H_2 given below to the Hamiltonian.

$$\begin{aligned}H_2{}^\beta&=\sum_{\mathbf{k}}(\bar{\epsilon}_{\mathbf{k}}-\epsilon_{\mathbf{k}})(U^2-V^2)(\gamma_{\mathbf{k}0}{}^*\gamma_{\mathbf{k}0}+\gamma_{\mathbf{k}1}{}^*\gamma_{\mathbf{k}1}),\\ H_2&=\sum_{\mathbf{k}}2(\bar{\epsilon}_{\mathbf{k}}-\epsilon_{\mathbf{k}})UV(\gamma_{\mathbf{k}0}{}^*\gamma_{\mathbf{k}1}{}^*+\gamma_{\mathbf{k}1}\gamma_{\mathbf{k}0}).\end{aligned} \tag{5}$$

The difference $(\bar{\epsilon}_{\mathbf{k}}-\epsilon_{\mathbf{k}})$ is found below [Eq. (9)]. We could take into account the renormalization of ω but this will not enter the following discussion.

The constants $U_{\mathbf{k}}$ and $V_{\mathbf{k}}$ are determined from the following considerations. In the weak-coupling limit of g small we can try to solve the problem by perturbation theory with H_0 as the zero-order Hamiltonian. We shall refer to the various terms of the expansion by referring to the corresponding Feynman graphs. In these graphs (see Fig. 1) the dashed line indicates a phonon, a line with an arrow to the left a particle "0" and a line with an arrow to the right a particle "1." Now there will be some terms of the perturbation expansion, for instance, those illustrated by the graphs of Fig. 1, in which two excited particles are created from the vacuum. The corresponding integrals will contain denominators $(2E_{\mathbf{k}})$. If the unrenormalized energies appeared, the integrals would diverge and for this reason they are called "dangerous" by Bogoliubov. Even when the renormalized energies are used the integrals will be large. Therefore, we choose $U_{\mathbf{k}}$, $V_{\mathbf{k}}$ to make these contributions vanish. To second order in g, the contributions of Fig. 1 must cancel. (H_2 will give a contribution of higher order.) Therefore

$$2(\epsilon_{\mathbf{k}}-\lambda)U_{\mathbf{k}}V_{\mathbf{k}}-\sum_{\mathbf{k}'}\frac{g^2\hbar\omega}{\Omega}\frac{(U'U-V'V)}{\hbar\omega+E+E'}(U'V+V'U)=0, \tag{6}$$

$$E'=E(\mathbf{k}'),\quad\text{etc.}$$

This is an integral equation for U, V which has been solved by Bogoliubov for small g. The solution is given by the following equations when g and ω are constants:

$$U_{\mathbf{k}}{}^2=\frac{1}{2}\left(1+\frac{\xi_{\mathbf{k}}}{(\xi_{\mathbf{k}}{}^2+c_{\mathbf{k}}{}^2)^{\frac{1}{2}}}\right),\quad V_{\mathbf{k}}{}^2=\frac{1}{2}\left(1-\frac{\xi_{\mathbf{k}}}{(\xi_{\mathbf{k}}{}^2+c_{\mathbf{k}}{}^2)^{\frac{1}{2}}}\right),$$

$$\xi_{\mathbf{k}}=\epsilon_{\mathbf{k}}-\lambda-\frac{g^2}{2\Omega}\sum_{\mathbf{k}'}\frac{\hbar\omega}{\hbar\omega+E+E'}(U'^2-V'^2),$$

$$c_{\mathbf{k}}=\frac{g^2}{2\Omega}\sum_{\mathbf{k}'}\frac{\hbar\omega}{\hbar\omega+E+E'}\frac{c'}{(\xi'^2+c'^2)^{\frac{1}{2}}},$$

$$c_{\mathbf{k}0}=c_0=2\hbar\omega\exp[-1/\rho],\quad \rho=N(0)g^2/\Omega. \tag{7}$$

Here k_0 is the momentum at the Fermi surface, $N(0)$ is the density of states at the Fermi surface, c_0 is the same as ϵ_0 of BCS, and ρ is the same as $N(0)V$ of BCS. λ is determined according to (3) so that ξ is zero at the Fermi surface. ξ is then the single particle energy in the normal state as measured from the normal Fermi surface, λ_0, i.e.,

$$\xi=\epsilon_{\mathbf{k}}-\lambda_0.$$

The renormalized energy in the superconducting state, $E_{\mathbf{k}}$, is determined from the fact that since it is

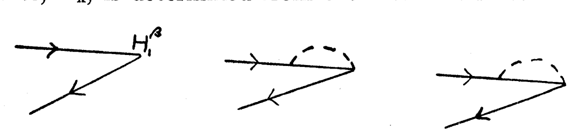

FIG. 1. Graphs which give rise to small energy denominators at the absolute zero.

[7] J. G. Valatin, Nuovo cimento 7, 843 (1958).

the exact energy of the state $\gamma_k^*|0\rangle$ (relative to the energy of $|0\rangle$) the corrections to the energy of this state must vanish. To order g^2, we find from second-order perturbation theory

$$(\bar{\epsilon}-\lambda)(U^2-V^2)-E_k - \frac{g^2\hbar\omega}{2\Omega}\sum_{\mathbf{k}'}\frac{(U'U-V'V)^2}{(\hbar\omega+E'-E)}-\frac{(U'V+UV')^2}{(\hbar\omega+E'+E)}=0.$$

Neglecting terms of order $(c_0/\hbar\omega)^2$, we have

$$E_k=(\xi^2+c_0^2)^{\frac{1}{2}}-(g^2\hbar\omega/2\Omega)E\sum_{\mathbf{k}''}[(\hbar\omega+E'')^2-E^2]^{-1} +(\bar{\epsilon}_k-\epsilon_k)\xi(\xi^2+c_k^2)^{-\frac{1}{2}}. \quad (8)$$

In the sum the energy, E_k, can be replaced by the zero-order approximation, $\xi^2(\xi^2+c_k^2)^{-\frac{1}{2}}$. Using the fact that when $c_0\rightarrow 0$, $E\rightarrow|\xi|$, one obtains

$$\bar{\epsilon}_k-\epsilon_k=\xi\frac{g^2\hbar\omega}{2\Omega}\sum_{\mathbf{k}''}[(\hbar\omega+\xi'')^2-\xi^2]^{-1},$$

$$E_k=(\xi^2+c_0^2)^{\frac{1}{2}}. \quad (9)$$

Terms of order $\rho c_0(\xi^2+c_0^2)^{\frac{1}{2}}$ have been neglected in obtaining this result. For most superconductors ρ has values $\sim$0.2 to 0.4 so that these terms are not negligible. For a complete justification of the formula of BCS it would be necessary to carry the perturbation expansion to terms of the fourth order in g.

3. MEISSNER EFFECT AT THE ABSOLUTE ZERO

In terms of the creation and annihilation operators for "bare" electrons the interaction between the electrons and the electromagnetic field is, to first order in the vector potential $\mathbf{A}(\mathbf{r})$

$$H_A=\left(\frac{e\hbar}{2mc}\right)\frac{(2\pi)^{\frac{3}{2}}}{\Omega}\sum_{\mathbf{k},\mathbf{k}',\sigma}C_{\mathbf{k}'\sigma}^*C_{\mathbf{k}\sigma}\mathbf{a}(\mathbf{k}'-\mathbf{k})\cdot(\mathbf{k}'+\mathbf{k}) +\frac{e^2}{2mc^2\Omega}\int d^3r\,\mathbf{A}^2(\mathbf{r})\sum_{\mathbf{k},\mathbf{k}',\sigma}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}C_{\mathbf{k}'\sigma}^*C_{\mathbf{k}\sigma},$$

where

$$\mathbf{a}(\mathbf{q})=(2\pi)^{-\frac{3}{2}}\int d^3r\,\mathbf{A}(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}.$$

In terms of the operators for the creation of the new quasi-particle excitations the interaction becomes

$$H_A=H_A'+H_A'',$$

$$H_A'=\left(\frac{e\hbar}{2mc}\right)\frac{(2\pi)^{\frac{3}{2}}}{\Omega}\sum_{\mathbf{k},\mathbf{k}'}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{a}(\mathbf{k}'-\mathbf{k}) \times\{(\gamma_{\mathbf{k}'0}^*\gamma_{\mathbf{k}0}-\gamma_{\mathbf{k}1}^*\gamma_{\mathbf{k}'1})(U'U+V'V) +(\gamma_{\mathbf{k}'0}^*\gamma_{\mathbf{k}1}^*-\gamma_{\mathbf{k}'1}\gamma_{\mathbf{k}0})(U'V-UV')\},$$

$$H_A''=\frac{e^2}{2mc^2\Omega}\int d^3r\,\mathbf{A}^2(\mathbf{r})\sum_{\mathbf{k},\mathbf{k}'}e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} \times\{(\gamma_{\mathbf{k}'0}^*\gamma_{\mathbf{k}0}+\gamma_{\mathbf{k}1}^*\gamma_{\mathbf{k}'1})(UU'-VV') +(\gamma_{\mathbf{k}'0}^*\gamma_{\mathbf{k}1}^*+\gamma_{\mathbf{k}'1}\gamma_{\mathbf{k}0})(U'V+UV')+2V_{\mathbf{k}}^2\delta_{\mathbf{k}\mathbf{k}'}\}. \quad (10)$$

In order to calculate the current, $\mathbf{j}(\mathbf{r})$, we shall first calculate the energy, $E[\mathbf{A}]$, to second order in $\mathbf{A}(\mathbf{r})$ and then use the result

$$\mathbf{j}(\mathbf{r})=-c\{\delta E[\mathbf{A}]/\delta\mathbf{A}(\mathbf{r})\}.$$

We choose the gauge so that div$\mathbf{A}$ is zero. This means that the perturbation does not introduce the collective excitations of the system.[3] To zero order in g, one easily finds that

$$\mathbf{j}(\mathbf{r})=\left(\frac{e\hbar}{m}\right)^2\frac{(2\pi)^{\frac{3}{2}}}{2c\Omega^2}\sum_{\mathbf{k},\mathbf{k}'}(\mathbf{k}+\mathbf{k}')[(\mathbf{k}+\mathbf{k}')\cdot\mathbf{a}(\mathbf{k}-\mathbf{k}')] \times e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}\frac{(U'V-UV')^2}{E+E'}-\frac{Ne^2}{mc}\mathbf{A}(\mathbf{r}). \quad (11)$$

This expression is of the same form as that of BCS [see Eqs. (5.14) and (5.15) of reference 2] and implies a Meissner effect. However, since important terms of order g^2 are implicitly included in Eq. (11) we must calculate the current to this higher order to ensure that they are not canceled.

Many of the graphs of order g^2 give contributions to the energy that are zero or of order $(\hbar\omega/E_F)$ which can be neglected. Those graphs which contain an intermediate state in which only two particles of equal momenta are present cancel because of the condition imposed on U and V. The remaining graphs which contain H_A'' involve only the diagonal part of this operator. The corresponding contributions to the energy are convergent sums of the form

$$\sum_{\mathbf{k}}(U^2-V^2)f(E),$$

where $f(E)$ is a function of E but not of ξ. This sum is of order $\hbar\omega/E_F$. The graphs, in which the two vertices at which H_A' acts are joined only by lines containing other vertices, give zero contributions because of the choice of gauge. The proof of this result is the same for all these terms so we shall prove it only for the particular term corresponding to the following sequence of processes.

Particles of momenta, $\mathbf{k}$, $\mathbf{k}+\mathbf{q}$, are created from the vacuum by H_A'—the particle of momentum, $\mathbf{k}+\mathbf{q}$, is scattered to $\mathbf{l}+\mathbf{q}$ with the emission of a phonon—the particle of momentum $\mathbf{k}$ is scattered to $\mathbf{l}$ with absorption of a phonon—the particles are destroyed by H_A'.

The corresponding contribution to the energy is

$$-(e\hbar/2mc)^2(2\pi/\Omega)^3 g^2\hbar\omega \sum_{\mathbf{l},\mathbf{k},\mathbf{q}} [2\mathbf{l}\cdot\mathbf{a}(\mathbf{q})]$$
$$\times(U_lV_{l+q}-U_{l+q}V_l)(U_lU_k-V_lV_k)$$
$$\times(U_{k+q}U_{l+q}-V_{k+q}V_{l+q})(U_kV_{k+q}-U_{k+q}V_k)$$
$$\times[2\mathbf{k}\cdot\mathbf{a}(\mathbf{q})][E(\mathbf{l})+E(\mathbf{l}+\mathbf{q})]^{-1}[E(\mathbf{k})+E(\mathbf{k}+\mathbf{q})]^{-1}$$
$$\times[\hbar\omega+E(\mathbf{k})+E(\mathbf{l}+\mathbf{q})]^{-1}.$$

When we proceed to the limit $\Omega\to\infty$ and replace the sums by integrals we have an expression of the form

$$\int d^3q\int d^3k\ \mathbf{a}(\mathbf{q})\cdot\mathbf{k}\left[\mathbf{a}(\mathbf{q})\cdot\int \mathbf{l}d^3l\ f(k,l,|\mathbf{k}+\mathbf{q}|,|\mathbf{l}+\mathbf{q}|)\right],$$

where f is a scalar function. In the integration over $\mathbf{l}$ the only direction defined is that of $\mathbf{q}$. Hence the integral is proportional to $(\mathbf{a}\cdot\mathbf{q})$ which is zero. This proof depends on the simplifying assumption that ω is a constant.

There are four graphs in which H_A' scatters one of the particles just once and these contribute to the current an amount

$$\left(\frac{e\hbar}{m}\right)^2\frac{(2\pi)^{\frac{3}{2}}}{c\Omega^2}g^2\hbar\omega\sum_{\mathbf{k},\mathbf{k}',\mathbf{k}''}\frac{(UV'-U'V)(UU'+VV')(U''U-V''V)(UV''+VU'')}{(\hbar\omega+E''+E)(\hbar\omega+E''+E')(E+E')}\times(\mathbf{k}+\mathbf{k}')[(\mathbf{k}+\mathbf{k}')\cdot a(\mathbf{k}-\mathbf{k}')]e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}.$$

This vanishes in the normal metal and also in the London limit where the important values of $\mathbf{k}$ and $\mathbf{k}'$ are such that $|\mathbf{k}-\mathbf{k}'|\hbar v_0\ll\epsilon_0$. In the Pippard limit, $|\mathbf{k}-\mathbf{k}'|\hbar v_0\gg\epsilon_0$, which is applicable to penetration phenomena this term is of order $(\epsilon_0{}^2/\hbar v_0|(\mathbf{k}-\mathbf{k}'|\hbar\omega)$ and can still be neglected.

The remaining graphs of which there are sixteen are connected with the self-energies of the particles. They are such that H_A' creates and destroys two particles; $H_2{}^\beta$, $H_2{}^\gamma$ or a phonon interacts with one of the particles. The contributions of these graphs to the current are

$$\mathbf{j}_2(\mathbf{r})=\left(\frac{e\hbar}{m}\right)^2\frac{(2\pi)^{\frac{3}{2}}}{2c\Omega^2}\times\sum_{\mathbf{k},\mathbf{k}'}(\mathbf{k}+\mathbf{k}')[(\mathbf{k}+\mathbf{k}')\cdot a(\mathbf{k}-\mathbf{k}')]e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}\times[f(E,E',\epsilon_0)+f(E',E,\epsilon_0)],\quad(12)$$

where

$$f(E,E',\epsilon_0)=(U'V-UV')^2\left\{-\frac{(\bar{\epsilon}-\lambda)(U^2-V^2)-E}{(E+E')^2}+\frac{g^2\hbar\omega}{2\Omega}\sum_{\mathbf{k}''}\frac{(UU''-VV'')^2}{(E+E')^2(\hbar\omega+E''+E')}-\frac{(UV''+VU'')^2(\hbar\omega+2E+E''+E')}{(E+E')^2(\hbar\omega+E''+E)^2}+(U'U+V'V)^2\times\frac{g^2\hbar\omega}{2\Omega}\sum_{\mathbf{k}''}\frac{(UV''+VU'')^2}{(\hbar\omega+E+E'')^2(\hbar\omega+E''+E')}\right\}.\quad(13)$$

In the London limit this becomes

$$\mathbf{j}_2(\mathbf{r})=\left(\frac{e\hbar}{m}\right)^2\frac{\mathbf{A}(\mathbf{r})}{c}\frac{2}{3}\frac{g^2\hbar\omega}{\Omega^2}\sum_{\mathbf{k},\mathbf{k}''}\frac{k^2(UV''+VU'')^2}{(\hbar\omega+E+E'')^3}=N(0)\frac{g^2}{\Omega}\frac{2e^2}{3c}\bar{v}_0{}^2N(0)\mathbf{A}(\mathbf{r}),$$

where

$$\bar{v}_0=\hbar k_0/m=(1/\hbar)|\nabla_{\mathbf{k}}\bar{\epsilon}_{\mathbf{k}}|.$$

Therefore the total current in this limit is

$$\mathbf{j}(\mathbf{r})=-\tfrac{2}{3}[\bar{N}(0)\bar{v}_0{}^2e^2/c](1-\rho)\mathbf{A}(\mathbf{r}).$$

This limiting form implies a Meissner effect.[8] The factor $(1-\rho)$ takes into account the renormalization of the velocity at the Fermi surface to order g^2. For this renormalized velocity is

$$v_0=(1/\hbar)|\nabla_{\mathbf{k}}\epsilon|.$$

With the use of Eq. (9) this can be rewritten

$$v_0=\bar{v}_0/(1+\rho)\approx\bar{v}_0(1-\rho).$$

Hence the limiting form of the current is

$$\mathbf{j}(\mathbf{r})=-\tfrac{2}{3}[N(0)v_0{}^2e^2/c]\mathbf{A}(\mathbf{r}),$$

in terms of the parameters of the normal metal. This is in agreement with the result of BCS.

To evaluate the current in the general case it is useful to split off the corresponding current in the normal state, that is,

$$\mathbf{j}_2(\mathbf{r})=\mathbf{j}_2'(\mathbf{r})+\mathbf{j}_2''(\mathbf{r}),$$

where

$$\mathbf{j}_2''(\mathbf{r})=\left(\frac{e\hbar}{m}\right)^2\frac{(2\pi)^{\frac{3}{2}}}{2c\Omega^2}\sum_{\mathbf{k},\mathbf{k}'}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{a}(\mathbf{k}-\mathbf{k}')e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}\times[f(|\xi|,|\xi'|,0)+f(|\xi'|,|\xi|,0)],$$

and f is defined in Eq. (13). Combined with the corresponding contribution of zero order in g, $\mathbf{j}_2''(\mathbf{r})$ gives the Landau diamagnetism of a normal metal and the zero-order correction to it. The interesting part of the current in the superconductor is $\mathbf{j}_2'(\mathbf{r})$. From Eq. (12) we can follow the analysis of BCS which leads from their

[8] M. R. Schafroth, Helv. Phys. Acta **24**, 645 (1951).

Eq. (5.15) to (5.37). In this way we obtain

$$\mathbf{j}_2'(\mathbf{r})=\frac{e^2\hbar^2}{2mc^2\pi^4}\int d^3r'\,\frac{[G_{2s}(R)-G_{2n}(R)]\mathbf{R}[\mathbf{A}(\mathbf{r}')\cdot\mathbf{R}]}{R^6},$$

where

$$G_{2s}(R)-G_{2n}(R)=+\tfrac{1}{2}k_0^4R^2\left[\left(\frac{dk}{d\bar{\epsilon}}\right)^2\right]_{k=k_0}\pi^2\epsilon_0 2\rho J_2(R),$$

$$J_2(R)=I(R,0)-I(R,\epsilon_0),$$

and

$$I(R,\epsilon_0)=-\frac{1}{2\rho\pi^2\epsilon_0}\int_{-\infty}^{+\infty}\int d\bar{\epsilon}\,d\bar{\epsilon}'[f(E,E',\epsilon_0)+f(E',E,\epsilon_0)]\times\cos\left[(\bar{\epsilon}-\bar{\epsilon}')\frac{R}{\hbar\bar{v}_0}\right].$$

The important contributions to the integrals are obtained when $|\xi|$ and $|\xi'|$ are of order ϵ_0. For these values of $|\xi|$ and $|\xi'|$ we have, neglecting terms of order $(\epsilon_0/\hbar\omega)^2$

$$f(E,E',\epsilon_0)+f(E',E,\epsilon_0)=\frac{g^2\hbar\omega}{2\Omega}\sum_{\mathbf{k}''}\left\{\frac{(U'V-V'U)^2}{E+E'}\times\left[-\frac{2}{(\hbar\omega+E'')^2}+\frac{2(E+E')}{(\hbar\omega+E'')^3}\right]+\frac{2(UU'+VV')^2}{(\hbar\omega+E'')^3}\right\}$$

$$=-\frac{g^2\hbar\omega}{2\Omega}N(0)\left\{\frac{(U'V-V'U)^2}{E+E'}\frac{4}{\hbar\omega}+\frac{1}{(\hbar\omega)^2}\right\}.$$

The second term is canceled by the corresponding term in $I(R,0)$ so that, neglecting terms of order $(\epsilon_0/\hbar\omega)^2$, we can take

$$I(R,\epsilon_0)=\frac{1}{\pi^2\epsilon_0}\int_{-\infty}^{+\infty}\int d\bar{\epsilon}\,d\bar{\epsilon}'\,\frac{(U'V-V'U)^2}{E+E'}\cos\left[\frac{(\bar{\epsilon}-\bar{\epsilon}')R}{\hbar\bar{v}_0}\right].$$

If the zero- and second-order terms are now combined, then

$$\mathbf{j}'(r)=-\frac{e^2}{2c}\frac{\bar{N}(0)\bar{v}_0\epsilon_0}{\hbar}\int d^3r'\,\frac{J(R)\mathbf{R}[\mathbf{A}(r')\cdot\mathbf{R}]}{R^4},$$

where

$$J(R)=(1-2\rho)[I(R,0)-I(R,\epsilon_0)].$$

Since to this order

$$d\bar{\epsilon}=d\epsilon(1+\rho),$$

we can replace unrenormalized quantities by renormalized ones in $J(R)$ and obtain just Eq. (5.42) of BCS with all parameters referring to electrons in the normal state. It is consistent with our approximations to replace the c_k which appears in the definition of U_k [see Eq. (7)] by $c_0=\epsilon_0$. The result is then exactly the same as that of BCS.

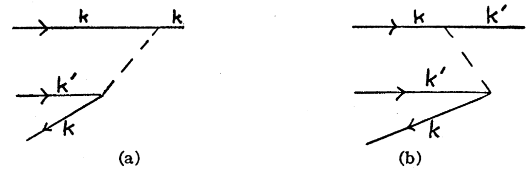

FIG. 2. Graphs which give rise to small energy denominators above the absolute zero.

4. MEISSNER EFFECT AT ANY TEMPERATURE

In this section we outline the extension of the method to a general temperature. We proceed formally as in Sec. 2 introducing new operators $\gamma_{\mathbf{k}0}$, $\gamma_{\mathbf{k}1}$ through Eqs. (2) and rewriting the Hamiltonian as in Eq. (4). However, $U_{\mathbf{k}}$ and $V_{\mathbf{k}}$ satisfy a different equation for there are more possibilities of obtaining small denominators. The zero-order state contains the quasi-particle excitations with probabilities given by the Fermi-Dirac function, $f(E)$. This means that besides the graphs of Fig. 1 there are "dangerous" graphs like those of Fig. 2. For all these graphs to cancel we must have

$$(\epsilon_{\mathbf{k}}-\lambda)U_{\mathbf{k}}V_{\mathbf{k}}=\sum_{\mathbf{k}'}\frac{g^2\hbar\omega}{2\Omega}(UV'+VU')(UU'-VV')\times\left\{\frac{1-f'}{\hbar\omega+E+E'}-\frac{f'}{\hbar\omega+E-E'}\right\}.$$

This equation leads to the same temperature-dependent energy gap as obtained by BCS, with $E_{\mathbf{k}}$ still given formally by Eq. (9) and c_0 temperature-dependent.

To the first order in A and zeroth order in g, one obtains for the paramagnetic current,

$$2\left(\frac{e\hbar}{2m}\right)^2\frac{(2\pi)^3}{c\Omega^2}\sum_{\mathbf{k}'\mathbf{k},}(\mathbf{k}+\mathbf{k}')[(\mathbf{k}+\mathbf{k}')\cdot a(\mathbf{k}'-\mathbf{k})]e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}\times\left\{\frac{(U'V-UV')^2}{(E+E')}(1-f-f')-\frac{(UU'+VV')^2}{(E-E')}(f-f')\right\}.$$

The diamagnetic current is the same as at the absolute zero. If the terms of order g^2 and $(\epsilon_0/\hbar\omega)$ are neglected the total current is formally the same as given by BCS. The effect of higher order terms has not so far been computed.

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