Transient Analysis of the Townsend Discharge

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The transient growth of currents in a one-dimensional Townsend gas discharge system under uniform dc field conditions is examined. It is shown that a general solution for arbitrary external conditions can be given in terms of a "unit pulse" solution. Two cases are examined in detail; in one case the discharge is initiated by a single pulse from the cathode, in the second case the discharge develops under constant external illumination of the cathode. Expressions are derived which permit reasonably rapid calculation of quantities pertinent to the transient characterization of the discharge. For the two cases considered in detail, numerical illustrations are given and comparison is made with the results of other investigators.

I. INTRODUCTION

IN the past five years there has been a considerable revival of interest in the theoretical analysis of transient phenomena in an idealized Townsend discharge.[1–4] Reference to the earlier literature may be found in the references cited. The current interest in the Townsend model seems to be twofold in origin. The idealized Townsend model is certainly the simplest model of a dc gas discharge amenable to reasonably rigorous analysis. Furthermore, recent improvements in experimental techniques have shown that the predictions of the Townsend model at steady state, below breakdown voltages, is in very good agreement with observation. It is therefore of considerable interest to examine experimentally whether transient behavior in a carefully controlled discharge can be approximated by the predictions of the Townsend model.

Unfortunately, the transient analysis of the idealized Townsend model turns out to be fairly complicated in comparison with the steady-state analysis. Although the authors cited below all treat essentially the same model, they have used somewhat different approaches in obtaining the solution, and their results are not readily comparable. The principal objection which may be raised, however, to previously published results is that they have invariably been presented in a form that is not particularly transparent to the experimentalist desiring to compare his observed results with those calculated from theory.

In the present paper we shall attempt to present a systematic development of certain transient phenomena encountered in the Townsend model. Most of the results to be presented have already appeared in one form or another; however, it is hoped that by collecting these results in one place it will be possible to achieve sufficient mathematical transparency to aid the experimentalist.

After a description of the model we present a brief description of the formal solution. It will be shown that in principle, the general transient characterization of the idealized Townsend model can be constructed from the solutions of the "unit pulse" case; we term unit pulse the situation where the avalanche process starts off with a unit pulse of charge and no other charge enters the system from external sources after inception. In the following section we shall develop in considerable detail the solution of the unit pulse case. For purposes of mathematical tractability, we shall consider in detail only situations where a single secondary mechanism is operative. It will be shown that for short times the rigorous solutions may be obtained from a rapidly convergent infinite series. For long times the solutions may be obtained from a formula equivalent to the one originally proposed by Davidson; the time variation here is given by a single exponential term.

In addition to the unit-pulse case, we shall also investigate in some detail the case of constant illumination. The last section will discuss the results of numerical computations. The application of the present analysis to experimental results will be postponed for a subsequent publication.

II. IDEALIZED TOWNSEND MODEL

We shall consider a discharge gap bounded by infinite plane parallel electrodes, such that the discharge may be considered unidimensional. The origin of the coordinate x is placed at the cathode and the anode is at δ cm from the cathode. At some time t_0, a constant dc voltage is placed across the electrodes and it is assumed that a spatially uniform field develops throughout the gap. The field remains constant and uniform during the current buildup; space-charge distortions are neglected. At the same time that the voltage is applied, an electronic current is made to appear at the cathode. This current may be induced by any external agency, such as ultraviolet irradiation of the cathode, cosmic rays, etc. The externally induced current density is denoted by $i_0(t)$ and may be an arbitrary but specified function of time, t, after inception at t_0.

In the idealized Townsend model, the secondary mechanisms which regenerate electrons in the discharge gap are restricted to the cathode. Ordinarily two mech-

[1] Dutton, Haydon, Jones, and Davidson, British J. Appl. Phys. **4**, 170 (1953). P. M. Davidson, Phys. Rev. **99**, 1072 (1955); **103**, 1897 (1956).
[2] H. W. Bandel, Phys. Rev. **95**, 1117 (1954).
[3] P. L. Auer, Phys. Rev. **98**, 320 (1955); **101**, 1243 (1956).
[4] Y. Miyoshi, Phys. Rev. **103**, 1609 (1956).

anisms are considered important, one due to positive ions impinging on the cathode, another due to photoelectron emission due to photons generated in the gap by molecular excitation, ionization, or recombination. Still another mechanism important in certain gases is secondary production by energetic neutrals reaching the cathode. We shall consider the first two only, in general, and restrict the detailed analysis to one, in particular.

The following notation will be used:

$$\begin{aligned} i(x,t) &= \text{electron current density in gap,}\\ j(x,t) &= \text{positive ion current density in gap,}\\ i_0(t) &= \text{externally induced electron current density at cathode,}\\ v &= \text{electron velocity,}\\ -w &= \text{positive-ion velocity,}\\ u &= wv/(w+v) = \text{mean velocity.} \end{aligned} \tag{1}$$

According to the Townsend model the equations of continuity become

$$\begin{aligned} \frac{1}{v}\frac{\partial i}{\partial t}+\frac{\partial i}{\partial x} &= \alpha i,\\ -\frac{1}{w}\frac{\partial j}{\partial t}+\frac{\partial j}{\partial x} &= -\alpha i, \end{aligned} \tag{2}$$

where α is the primary ionization coefficient and has units of reciprocal length.

If initially there is no charge present in the gap, the solutions of (2) are subject to the boundary conditions

$$\begin{aligned} i(0,t) &= f(t),\\ j(\delta,t) &= 0, \end{aligned} \tag{3}$$

where $f(t)$ is zero for $t<t_0$ and is specified through the secondary mechanisms. Physically, of course, $f(t)$ is simply the cathode electron current density. The most general solutions of (2) are

$$\begin{aligned} i(x,t) &= 0, \qquad t<x/v;\\ i(x,t) &= f(t-x/v)e^{\alpha x}, \quad t\geqslant x/v;\\ j(x,t) &= 0, \qquad t<x/v;\\ j(x,t) &= \alpha\int_x^{u(t+x/w)} i(s;\,t+x/w-s/w)ds, \quad t+x/w<\delta/u;\\ j(x,t) &= \alpha\int_x^{\delta} i(s;\,t+x/w-s/w)ds, \quad t+x/w\geqslant\delta/u; \end{aligned} \tag{4}$$

where we have set the starting time t_0 equal to zero.

If initially there is a distribution of negative charge density $n(x)$ and positive charge density $p(x)$ present in the gap, we shall have to add expressions

$$vn(x-vt)e^{\alpha x} \tag{5a}$$

to the electron current in (4), and

$$wp(x+wt) \tag{5b}$$

to the positive-ion current in (4), with the requirement

$$n(x)=0=p(x), \quad x<0 \quad \text{or} \quad x>\delta. \tag{5c}$$

The inclusion of initial charge in the gap complicates the analysis of the secondary process to some extent and will not be included in the present discussion. It could be included, however, by an obvious generalization of what will follow.

In order to discuss the secondary process, it is useful to introduce the definitions of the electron and ion transit times:

$$\begin{aligned} \tau_1 &= \delta/v = \text{electron transit time,}\\ \tau_2 &= \delta/w = \text{positive-ion transit time,}\\ \tau_3 &= \delta/u = \tau_1+\tau_2. \end{aligned} \tag{6}$$

The expression in (4) for the positive-ion current density may be rewritten

$$\begin{aligned} j(x,t) &= \alpha u\int_{t-\tau_3+x/w}^{t-x/v} ds\, f(s)e^{\alpha u(t+x/w-s/u)},\\ f(t) &= 0, \quad t<0. \end{aligned} \tag{7}$$

Considering the photon and positive-ion secondary mechanisms only, the Townsend model postulates in general that

$$\begin{aligned} f(t) &= i_0(t)+\frac{\omega}{\alpha}j(0,t)+\beta\int_0^{\delta} i(x,t)dx;\\ i_0(t) &= f(t)=0, \quad t<0. \end{aligned} \tag{8}$$

Some authors denote $\gamma_i=\omega/\alpha$, $\gamma_p=\beta/\alpha$; the first quantity represents the ion secondary coefficient, the second represents the photon secondary coefficient. With the use of (4) and (7), (8) becomes in the absence of initial charges in the gap

$$f(t)=i_0(t)+\omega u\int_{t-\tau_3}^{t} e^{\alpha u(t-s)}f(s)ds +\beta v\int_{t-\tau_1}^{t} e^{\alpha v(t-s)}f(s)ds. \tag{9}$$

The above equation represents an integro-difference equation. It is complicated by the fact that two different transit times, τ_1 and τ_3, are present. For purely mathematical convenience we shall consider only the case where one secondary is operative; in particular we assume it to be a photon mechanism. To convert from a pure photon case to a pure positive ion case, it is only necessary to interchange the roles of β and ω, v and u, τ_1 and τ_3, in the integral equation. It is obvious from (4) and (7) that once a solution to (9) is obtained, the current characteristics of the gap are fully specified. Before considering detailed solution of (9), we wish to discuss the current in the external circuit.

III. EXTERNAL CIRCUIT CURRENTS

In an experimental investigation of transient discharge characteristics, one ordinarily observes the current in the external circuit. In a dc discharge not limited by external resistance, the electronic contribution and ionic contribution to the external current are, respectively,

$$I_e(t)=\delta^{-1}\int_0^\delta i(x,t)dx,$$
$$I_p(t)=\delta^{-1}\int_0^\delta j(x,t)dx. \qquad (10)$$

With neglect of initial charge in the gap, (10) may be recast with the aid of (4) into the form

$$I_e(t)=\tau_1^{-1}\int_{t-\tau_1}^{t} e^{\alpha v(t-s)}f(s)ds,$$
$$I_p(t)=\tau_2^{-1}\int_{t-\tau_1}^{t}[e^{\alpha v(t-s)}-e^{\alpha u(t-s)}]f(s)ds \qquad (11)$$
$$+\tau_2^{-1}\int_{t-\tau_3}^{t-\tau_1}[e^{\alpha\delta}-e^{\alpha u(t-s)}]f(s)ds.$$

The second of these expressions is obtained after an integration by parts and some rearrangement. It will be noted that the lower limit of all integrals is to be taken as zero when the indicated limit is less than zero, since $f(t)=0$ for $t<0$. For times less than an electron transit time, the second integral in I_p is identically zero. The total observed current in the external circuit, $I(t)$, is the sum of I_e and I_p. The expressions given by (10) and (11) are valid for any combination of secondary mechanisms. Experimentally, of course, one usually observes the sum of the electronic and ionic components. On the basis of (11), however, it is possible to have them contribute on different time scales. For purposes of rough estimation we can assume that

$$\tau_2/\tau_1\sim v/u\sim 100.$$

The first integral in I_p as given by (11) is then always negligible in comparison with I_e. The variation of the second integral in I_p relative to I_e depends in detail on the nature of the secondary and on the extent to which the gap has an overvoltage or undervoltage. We postpone a fuller discussion of this point until we obtain solutions of the integral equation for $f(t)$.

IV. UNIT PULSE SOLUTION

We shall now treat the case where only a photon secondary is active. The relation of this situation to the one where a pure ion secondary exists has already been discussed. The following notation will be useful:

$$\xi=t/\tau_1=\text{time in units of } \tau_1,$$
$$\sigma=\alpha\delta=\text{avalanche size}, \qquad (12)$$
$$\lambda=\beta\delta=\gamma_p\sigma=\text{size of secondary}.$$

For an arbitrary externally induced current, $i_0(t)$, the pertinent integral equation becomes

$$f(\xi)=i_0(\xi)+\lambda\int_{\xi-1}^{\xi} e^{\sigma(\xi-\eta)}f(\eta)d\eta;$$
$$f(\xi)=i_0(\xi)=0,\quad \xi<0. \qquad (13)$$

For initial times ($t<\tau_1$), the lower limit of integration in (13) is zero and the equation is readily inverted to give

$$f(\xi)=i_0(\xi)+\lambda\int_0^{\xi} e^{(\lambda+\sigma)(\xi-\eta)}i_0(\eta)d\eta;\quad 0\leqslant\xi\leqslant 1. \qquad (14)$$

Thus, in the interval 0 to τ_1 for pure photon secondary, or in the interval 0 to τ_3 for pure ion secondary, (14) presents a complete solution to the transient characterization of the discharge.

For arbitrary times one can readily verify that the solution of (13) may be written in the form

$$f(\xi)=\int_0^{\xi} i_0(\xi-\eta)g(\eta)d\eta; \qquad (15)$$

where the quantity $g(\xi)$ is the unit pulse solution and satisfies the equation

$$g(\xi)=\delta(\xi)+\lambda\int_{\xi-1}^{\xi} e^{\sigma(\xi-\eta)}g(\eta)d\eta, \qquad (16)$$

with $\delta(\xi)$ denoting the usual delta function.

As examples, consider first the case where the source of $i_0(t)$ is a finite charge density of size q_0 appearing at the cathode at only zero time. Then, according to (15),

$$i_0(t)=q_0\delta(t);$$
$$f(\xi)=i_0 g(\xi),\quad t\geqslant 0; \qquad (17a)$$
$$i_0=q_0/\tau_1.$$

On the other hand, consider the case where $i_0(t)$ arises from a constant source of background illumination. Then

$$i_0(t)=i_0=\text{const},\qquad t\geqslant 0;$$
$$f(\xi)=i_0\int_0^{\xi} g(\eta)d\eta,\quad t\geqslant 0. \qquad (17b)$$

The above two cases are the ones which have been treated by Miyoshi.[4] The other authors[1–3] have treated primarily the case described by (17b). On the basis of (15), however, the unit-pulse solution, $g(\xi)$, provides a method for the construction of solutions for arbitrary variation of $i_0(t)$.

The initial time span solution reduces in the case of the unit pulse to

$$g(\xi)=\delta(\xi)+\lambda e^{(\lambda+\sigma)\xi}, \quad 0\leqslant\xi\leqslant 1. \tag{18}$$

For times greater than zero it is useful to define a new function according to the scheme

$$g(\xi)=\lambda u(\xi)e^{\sigma\xi}, \quad \xi>0. \tag{19}$$

Substituting (19) into (16) for $\xi>0$ and differentiating with respect to ξ yields a difference equation:

$$\begin{aligned} &u'(\xi)=\lambda[u(\xi)-u(\xi-1)], \quad \xi>0;\\ &u(\xi)=e^{\lambda\xi}, \quad 0<\xi<1, \end{aligned} \tag{20}$$

where the prime denotes differentiation and the boundary condition for $u(\xi)$ is obtained from (18).

If we integrate (20) beyond the initial time span, we notice there is a discontinuity in $u(\xi)$ and $g(\xi)$ at $\xi=1$:

$$\begin{aligned} &\lim_{\epsilon\to 0} u(1-\epsilon)=e^{\lambda},\\ &\lim_{\epsilon\to 0} u(1+\epsilon)=e^{\lambda}-1, \\ &\lim_{\epsilon\to 0}[g(1-\epsilon)-g(1+\epsilon)]=\lambda e^{\sigma}. \end{aligned} \tag{21}$$

Physically, the reason the discontinuity appears is simply because the finite starting pulse leaves the discharge system after a unit transit time and its contribution to the secondary avalanche process disappears.

There are two convenient ways for solving (20). One is by the method of generating functions as discussed previously,[3] the other is by the method of Laplace transforms. We shall use the second method in this paper, since it can yield directly the results of all four authors cited previously.

We define the Laplace transform of $u(\xi)$ by

$$\begin{aligned} &\tilde{u}(p)=\mathrm{P}\int_0^\infty e^{-p\xi}u(\xi)d\xi,\\ &\mathrm{P}\int_0^\infty=\lim_{\epsilon\to 0}\int_0^{1-\epsilon}+\int_{1+\epsilon}^\infty. \end{aligned} \tag{22}$$

The integral in (22) is the principal-value integral; this must be used because of the discontinuity given by (21). One may verify independently using the method of generating functions that (22) is the proper way to describe the Laplace transform.

Applying (22) to (20) we obtain, with the aid of (21),

$$\tilde{u}(p)=\frac{1-e^{-p}}{p-\lambda(1-e^{-p})}. \tag{23}$$

We may recover the results of the method of generating functions by expanding (23) as

$$\tilde{u}(p)=\frac{1-e^{-p}}{p-\lambda}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\lambda}{p-\lambda}\right)^n e^{-np}; \tag{24}$$

and (24) may be inverted by standard techniques to yield the final solution

$$u(\xi)=e^{\lambda\xi}-\sum_{j=1}^{n}\left[\frac{\lambda^{j-1}(j-\xi)^{j-1}}{(j-1)!}-\frac{\lambda^{j}(j-\xi)^{j}}{j!}\right]e^{-(j-\xi)\lambda}, \quad n\leqslant\xi\leqslant n+1. \tag{25}$$

The solution given by (25) is exact but not very useful for computation purposes when ξ becomes greater than a few transit times. The trouble is that (25) is an oscillatory series and numerical calculations rapidly encounter significant-figure difficulties as ξ increases.

The problem in ease of computation can be rectified as follows. Using a Burmann-Lagrange series expansion[5] for $e^{\lambda\xi}$ in powers of $(\lambda e^{-\lambda})$, we obtain the identity

$$e^{\lambda\xi}=\sum_{j=1}^{\infty}\left[\frac{\lambda^{j-1}(j-\xi)^{j-1}}{(j-1)!}-\frac{\lambda^{j}(j-\xi)^{j}}{j!}\right]e^{-(j-\xi)\lambda}. \tag{26}$$

Substituting (26) in (25), we obtain the desired form:

$$u(\xi)=\sum_{j=n+1}^{\infty}\left[\frac{\lambda^{j-1}(j-\xi)^{j-1}}{(j-1)!}-\frac{\lambda^{j}(j-\xi)^{j}}{j!}\right]e^{-(j-\xi)\lambda}, \quad n\leqslant\xi\leqslant n+1. \tag{27}$$

The series given by (27) is not oscillatory. Some numerical examples given at the end of this paper will demonstrate that the series is quite rapidly convergent —for all $\lambda\ll 1$. The expansion given in (26) is not proper for $\lambda>1$; however, the situation $\lambda=\gamma_p\alpha\delta>1$ is rarely, if ever, encountered in practice.

In order to obtain an asymptotic formula for $u(\xi)$ and at the same time recover the result of Miyoshi for the unit-pulse case, we wish to invert directly the Laplace transform given by (23). This may be done by the standard methods of the calculus of residues. We find

$$u(\xi)=\sum_{n=0}^{\infty}\left(\frac{p_n}{\lambda}\right)\frac{e^{p_n\xi}}{1-\lambda+p_n}, \tag{28}$$

where the p_n are the roots of the transcendental equation

$$\frac{p_n}{1-e^{-p_n}}=\lambda. \tag{29}$$

When $\lambda>0$, there is one positive real root and an infinite number of complex roots which occur in pairs

[5] L. P. Smith, *Mathematical Methods for Scientists and Engineers* (Prentice-Hall, Inc., New York, 1953), p. 184.

as complex conjugates. We label these roots as

$$p_0=\lambda-\lambda_0, \quad p_n=\lambda-x_n\mp iy_n, \quad n>0; \tag{30}$$

where the λ_0, x_n, y_n are all real positive. The roots are then given by the set of equations

$$\lambda_0 e^{-\lambda_0}=\lambda e^{-\lambda}, \quad x_n=y_n \cot y_n, \quad x_n e^{-x_n}=\lambda e^{-\lambda}\cos y_n. \tag{31}$$

Two cases may be distinguished, depending on the magnitude of λ. They are (a) $\lambda<1<\lambda_0$, and (b) $\lambda_0<1<\lambda$. As indicated previously, case (b) may be neglected since it is not physically interesting. For case (a) one can readily see on the basis of (31) that

$$\lambda<1; \quad 1<\lambda_0<x_1\cdots<x_n\cdots, \tag{32}$$

where the x_n have been sorted according to size. The important fact is that $|p_0|$ as defined in (30) is the smallest root and all roots have negative real parts. The quantity $u(\xi)$ decreases with time for $\lambda<1$ and its asymptotic behavior is given by the term containing p_0.

After some rearrangement, (28) becomes with the aid of (30)

$$\lambda u(\xi)=\left(\frac{\lambda_0-\lambda}{\lambda_0-1}\right)e^{-(\lambda_0-\lambda)\xi}+2\sum_{n=1}^{\infty}\frac{e^{-(x_n-\lambda)\xi}}{(x_n-1)^2+y_n^2} \times\{[(x_n-\lambda)(x_n-1)+y_n^2] \times\cos y_n\xi-(1-\lambda)y_n \sin y_n\xi\}, \tag{33}$$

where the pertinent roots are to be found from (31). The above expression has been derived for a pure γ_p secondary but is equally valid for a pure γ_i secondary upon the proper change in symbols. It is equivalent to Miyoshi's result[6] and is the unit-pulse analog to Davidson's original result for constant i_0.* For purposes of numerical computation over finite time spans, we have found the expression given by (27) somewhat more convenient than the equivalent form given by (33).

The asymptotic form follows from (33) and the previous discussion:

$$\lim_{\xi\to\infty}\lambda u(\xi)=\left(\frac{\lambda_0-\lambda}{\lambda_0-1}\right)e^{-(\lambda_0-\lambda)\xi}. \tag{34}$$

This is the unit-pulse analog of Davidson's original simple exponential expression. Numerical comparison of (34) with exact results will be given in the last section. The results of (27), (33), and (34), along with (18) and (19) provide the complete solution of the unit-pulse case.

The criterion for breakdown is readily derived from the asymptotic form of $g(\xi)$. We find, from (19) and (34),

$$\lim_{\xi\to\infty}g(\xi)=\left(\frac{\lambda_0-\lambda}{\lambda_0-1}\right)e^{(\lambda+\sigma-\lambda_0)\xi}. \tag{35}$$

When the exponent in (35) is negative, all currents decay to zero eventually; when the exponent is positive, the currents grow without bound and breakdown results. Threshold is obtained at

$$\lambda_0=\lambda+\sigma. \tag{36}$$

One can easily obtain on the basis of (31) that the above is equivalent to the familiar Townsend criterion

$$\gamma(e^{\alpha\delta}-1)=1, \tag{37}$$

where γ denotes either γ_p or γ_i. It is interesting to note that at threshold the unit pulse case leads to an asymptotic constant current, while immediately below or above threshold the current decays or grows with time.

V. CONSTANT ILLUMINATION

The results of the previous section allows us to find expressions for the cathode current under a variety of external conditions. We shall be particularly interested in the case of constant external illumination described by (17b). In general the relation in (15) states that

$$f(\xi)=\int_0^{\xi} i(\xi-\eta)g(\eta)d\eta. \tag{15}$$

In many instances the easiest way to evaluate this integral is to take the Laplace transform. Using the convolution theorem, we find

$$\tilde{f}(p)=\tilde{\imath}(p)\tilde{g}(p), \tag{38}$$

where the symbols denote Laplace transforms.

The Laplace transform is easily computed from previous results. It is proper, in this case, to use the convention that the Laplace transform of the delta function is unity. Thus

$$\tilde{g}(p)=1+\lambda\tilde{u}(p-\sigma) =\frac{p-\sigma}{p-\sigma-\lambda(1-e^{-(p-\sigma)})}. \tag{39}$$

If the Laplace transform of $i_0(\xi)$ exists, one can usually invert (38) with little labor and thereby find the asymptotic form.

For the case of constant illumination, we have

$$\tilde{\imath}_0(p)=i_0/p, \tag{40}$$

which may be combined with (39) and inverted by the

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[6] Our expression corresponds to Miyoshi's exact results. In his detailed discussion Miyoshi uses only the approximate form given by our asymptotic formula.

* *Note added in proof.*—The author has been advised (Davidson, private communication) that Davidson's original result (reference 1) contains the unit pulse solution as a special example.

calculus of residues to yield the final result

$$f(\xi)=i_0\left\{\frac{1}{1-\gamma(e^{\alpha\delta}-1)}+\left(\frac{\lambda_0-\lambda}{\lambda_0-1}\right)\frac{e^{(\lambda+\sigma-\lambda_0)\xi}}{\lambda+\sigma-\lambda_0}+\sum_{n=1}^{\infty}\left(\frac{p_n}{1-\lambda+p_n}\right)\left(\frac{e^{(p_n+\sigma)\xi}}{p_n+\sigma}\right)\right\}; \quad (41)$$

$$p_n=\lambda-x_n\mp iy_n.$$

The quantities λ_0, x_n, y_n, are the same as the ones discussed in the preceding section; the quantity γ represents either γ_p or γ_i, depending on which secondary is operative. If the complex roots p_n are used for computation, both pairs of complex conjugates must be included.

The asymptotic form can be obtained from (41) by the same arguments as used previously:

$$\lim_{\xi\to\infty}f(\xi)=i_0[A+Be^{at}],$$

$$A=[1-\gamma(e^{\alpha\delta}-1)]^{-1},$$

$$B=\left(\frac{\lambda_0-\lambda}{\lambda_0-1}\right)[\lambda+\sigma-\lambda_0]^{-1}, \quad (42)$$

$$a=\lambda+\sigma-\lambda_0.$$

The same breakdown criterion is obtained from (42) as in the unit-pulse case. When $a<0$, B is negative and a steady state is reached as given by A. This is the familiar Townsend result. When $a\geqslant 0$, B is positive and the current grows without bound until breakdown.

An alternative expression for $f(\xi)$ analogous to the series results of (25) and (27) in the unit pulse case may be obtained by expansion of the Laplace transform $\tilde{f}(p)$ before inversion. This result may also be derived from recurrence relations obtainable from the $u(\xi)$ difference equations of (20). We find

$$f(\xi)=i_0\left\{\frac{\sigma}{q}\left[\frac{1-(p/q)^{n+1}}{1-(p/q)}\right]+\frac{\lambda}{q}\sum_{j=0}^{n}(p/q)^j u(\xi-j)e^{\sigma(\xi-j)}\right\}, \quad n\leqslant\xi\leqslant n+1, \quad (43)$$

$$p=\lambda e^{\sigma},$$

$$q=\lambda+\sigma.$$

The expression given in (43) is an equivalent form of an expression derived previously for this case[7] and is the correct version of an expression proposed subsequently.[8] If tabulated values of the $u(\xi)$ functions are available, (43) may be simpler than (41) for computation purposes.

[7] P. L. Auer, Phys. Rev. **98**, 320 (1955), Eq. (3.14).

[8] P. L. Auer, Phys. Rev. **101**, 1243 (1956), Eq. (2).

VI. CURRENTS IN THE UNIT-PULSE CASE

The last two sections have been devoted to developing general methods for the calculation of the cathode electron current density. According to the results of the first three sections the formal solution to the transient analysis is complete once the cathode electron current density is known. In the present section we wish to apply these formal results to the calculation of the pertinent currents for the unit-pulse case. In order to save space we shall not attempt to generalize these calculations to other external cases. However, it is hoped that the exposition to follow will enable interested readers to extend these results to other cases of particular interest.

It has been our experience that the simplest consistent way to apply the formal analysis of the first three sections to the computation of currents is to use the convolution theorem of Laplace transforms in conjunction with the unit-pulse solutions of Sec. IV. Direct inversion can usually be accomplished readily by the method of residues. This has the advantage that the constants of integration are found automatically. Also, by picking out the dominant roots, one can obtain asymptotic relations with a minimum of labor. Convenient forms for short-time behavior, on the other hand, may be obtained by expanding the Laplace transform before inversion in the appropriate power series.

In the case of the unit pulse, (17a) gives the desired form for the cathode electron current density:

$$f(\xi)=i_0 g(\xi)=i_0[\delta(\xi)+\lambda u(\xi)e^{\sigma\xi}], \quad \xi>0. \quad (44)$$

The electron current density in the gap becomes

$$i(x,t)=f([\xi-x/\delta]\tau_1)e^{\alpha x}, \quad t\geqslant x/v. \quad (45)$$

The electronic contribution to the external current in the event of a pure γ_p secondary follows immediately from the original integral Eq. (9):

$$I_e(t)=(\beta\delta)^{-1}[f(t)-i_0(t)],$$

$$I_e(\xi)=i_0 u(\xi)e^{\sigma\xi}, \quad \xi=t/\tau_1>0. \quad (46)$$

The positive-ion current density at the cathode is given by the expression

$$j(0,t)=\alpha u\int_{t-\tau_3}^{t}e^{\alpha u(t-s)}f(s)ds; \quad (47)$$

and in the case of unit pulse with pure γ_p secondary this becomes

$$j(0,\xi)=i_0\mu\sigma e^{\mu\sigma\xi}\left\{1+\lambda\int_{\xi-\mu^{-1}}^{\xi}e^{(1-\mu)\sigma\eta}u(\eta)d\eta\right\},$$

$$\mu=u/v=w/(w+v). \quad (48)$$

In actual practice, $\mu\ll 1$, $\mu^{-1}\gg 1$, and the lower limit of integration remains zero for a large number of electron transit times.

TABLE I. Values of the unit-pulse function $u(\xi)$.

(a) $\lambda=10^{-3}$, $\lambda_0=9.1191296$, $u^{\text{III}}=1.123042\times10^{3}e^{-9.1181296\xi}$

ξ	0.5	1.0	1.2	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
u^{I}	1.0005	1.000×10^{-3}	8.004×10^{-4}	5.008×10^{-4}	5.007×10^{-7}	1.255×10^{-7}	1.671×10^{-10}	2.10×10^{-11}	4×10^{-14}		
$(u^{\text{II}})_{(3)}$	1.0005	1.000×10^{-3}	8.004×10^{-4}	5.008×10^{-4}	5.007×10^{-7}	1.255×10^{-7}	1.671×10^{-10}	2.103×10^{-11}	4.188×10^{-14}	2.666×10^{-15}	8.413×10^{-18}
$(u^{\text{II}})_{(6)}$					5.007×10^{-7}				4.188×10^{-14}	2.666×10^{-15}	
u^{III}	11.76	123.2×10^{-3}		12.90×10^{-4}	135.1×10^{-7}		14.81×10^{-10}		16.24×10^{-14}	1.70×10^{-15}	17.81×10^{-18}

(b) $\lambda=10^{-2}$, $\lambda_0=6.48460$, $u^{\text{III}}=118.050e^{-6.47460\xi}$

ξ	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
u^{I}	1.005	1.005×10^{-2}	5.075×10^{-3}	5.067×10^{-5}	1.298×10^{-5}	1.702×10^{-7}	2.34×10^{-8}			
$(u^{\text{II}})_{(3)}$	1.005	1.005×10^{-2}	5.075×10^{-3}	5.067×10^{-5}	1.298×10^{-5}	1.702×10^{-7}	2.341×10^{-8}	4.387×10^{-10}	3.243×10^{-11}	9.147×10^{-13}
$(u^{\text{II}})_{(6)}$	1.005	1.005×10^{-2}	5.075×10^{-3}	5.067×10^{-5}	1.298×10^{-5}	1.702×10^{-7}		4.388×10^{-10}	3.245×10^{-11}	9.149×10^{-13}
u^{III}	4.636	18.20×10^{-2}	7.149×10^{-3}	28.07×10^{-5}	1.102×10^{-5}	4.329×10^{-7}	1.552×10^{-8}	6.676×10^{-10}		10.30×10^{-13}

(c) $\lambda=10^{-1}$, $\lambda_0=3.71495$, $u^{\text{III}}=13.31_5e^{-3.61495\xi}$

ξ	0.5	1.0	1.5	2.0	3.0	4.0	4.5	5.0
u^{I}	1.052	0.1052	5.800×10^{-2}	5.725×10^{-3}	2.184×10^{-4}	6.805×10^{-6}		
$(u^{\text{II}})_{(3)}$	1.045	0.1043	5.747×10^{-2}	5.645×10^{-3}	2.137×10^{-4}	6.523×10^{-6}	1.031×10^{-6}	1.757×10^{-7}
$(u^{\text{II}})_{(6)}$	1.051	0.1052	5.799×10^{-2}	5.714×10^{-3}	2.184×10^{-4}	6.800×10^{-6}	1.148×10^{-6}	1.877×10^{-7}
u^{III}	2.191	0.3243	5.881×10^{-2}	7.891×10^{-3}	2.597×10^{-4}	6.991×10^{-6}		1.882×10^{-7}

(d) $\lambda=0.5$, $\lambda_0=1.756432$, $u^{\text{III}}=3.32199e^{-1.256432\xi}$

ξ	0.5	1.0	1.2	1.5	2.0	3.0	4.0	5.0
u^{I}	0.992					7.558×10^{-2}	2.182×10^{-2}	6.213×10^{-3}
$(u^{\text{II}})_{(3)}$	0.9361	0.4285	0.4031	0.3341	0.1421	3.391×10^{-2}	0.675×10^{-2}	1.063×10^{-3}
$(u^{\text{II}})_{(6)}$		0.5504	0.5129	0.4332	0.1976	5.474×10^{-2}	1.346×10^{-2}	2.968×10^{-3}
$(u^{\text{II}})_{(10)}$					0.2268	6.715×10^{-2}	1.820×10^{-2}	4.680×10^{-3}
$(u^{\text{II}})_{(20)}$								5.945×10^{-3}
u^{III}	1.772	0.9457	0.7355	0.5046	0.2692	7.663×10^{-2}	2.181×10^{-2}	6.210×10^{-3}

When t is less than $\tau_3=\tau_1+\tau_2$, expressions already derived may be used for the quantity represented inside the curly brackets of (48). It is only necessary to substitute $(1-\mu)\sigma$ for σ in the expressions given by (41) to (43) of the previous section to obtain the correct solution for this quantity. For times greater than τ_3 we can obtain an expression for $j(0,t)$ by the Laplace transform method:

$$\bar{j}(p)=\int_0^\infty e^{-p\xi}j(0,\xi)d\xi$$
$$=\left(\frac{\mu\sigma}{p-\mu\sigma}\right)(1-e^{-(p-\mu\sigma)/\mu})\bar{f}(p). \tag{49}$$

The above expression follows from (47). In the case of the unit pulse (49) becomes

$$\bar{j}(p)=i_0\mu\sigma\left(\frac{p-\sigma}{p-\mu\sigma}\right)\left[\frac{1-e^{-(p-\mu\sigma)/\mu}}{p-\sigma-\lambda(1-e^{-(p-\sigma)})}\right]. \tag{50}$$

Note that the residue at $\mu\sigma$ disappears for $t>\tau_3$ and the simple exponential term $e^{\mu\sigma\xi}$ disappears with it. The expression in (50) may be readily inverted to yield the desired formula for $j(0,t)$,

$$j(0,\xi)=i_0\mu\sigma\left(\frac{\lambda_0-\lambda}{\lambda_0-1}\right)\left[\frac{1-e^{-[\lambda+(1-\mu)\sigma-\lambda_0]/\mu}}{\lambda+(1-\mu)\sigma-\lambda_0}\right]e^{(\lambda+\sigma-\lambda_0)\xi}$$
$$+i_0\mu\sigma\sum_{n=1}^{\infty}\left(\frac{p_n}{1-\lambda+p_n}\right)\times\left[\frac{1-e^{-[p_n+(1-\mu)\sigma]/\mu}}{p_n+(1-\mu)\sigma}\right]e^{(p_n+\sigma)\xi}, \tag{51}$$
$$p_n=\lambda-x_n\mp iy_n.$$

By arguments previously advanced, the asymptotic form is given by the first term in (51). Similar expressions can be derived for $j(x,t)$, but they are of little practical interest. The λ_0, x_n, y_n, appearing in (51) are the same as defined in Sec. IV.

In terms of quantities already defined, the ionic contribution to the external current becomes

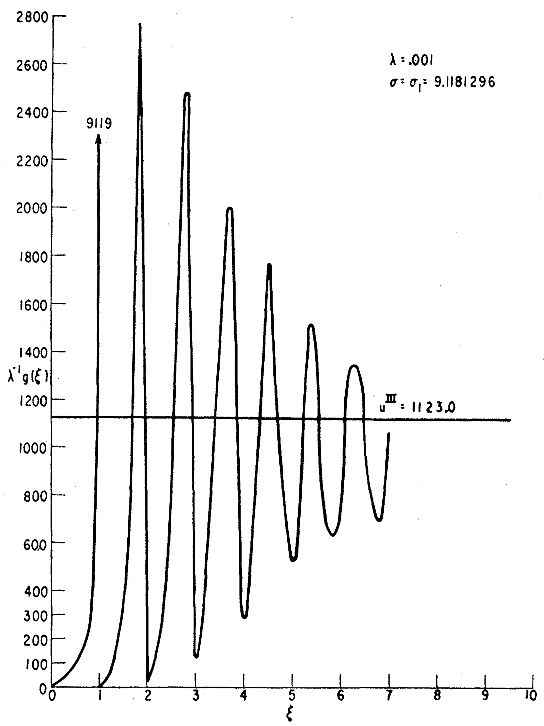
$$I_p(\xi)=\left(\frac{\mu}{1-\mu}\right)I_e(\xi)-\frac{\sigma^{-1}}{1-\mu}j(0,\xi)+\left(\frac{\mu}{1-\mu}\right)e^{\sigma}\int_{\xi-\mu^{-1}}^{\xi-1}f(\eta)d\eta. \tag{52}$$

When $t<\tau_1$, the integral in (52) vanishes. For $\tau_1<t<\tau_3$, the lower bound of the integral is zero and its value for the unit-pulse case has already been computed; it is only necessary to substitute $\xi-1$ for ξ in the expressions given by (41) to (43) to evaluate this integral. For times greater than τ_3, the integral may be written as the difference of two integrals both starting at zero, and the results of (41) to (43) may be used once more to evaluate it.

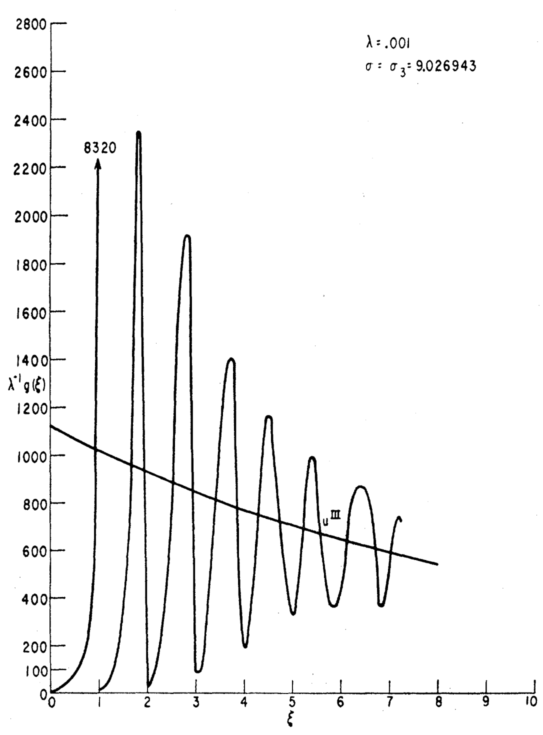
From results previously obtained, asymptotic expressions for the external current can be obtained readily for the unit-pulse case:

$$\lim_{\xi\to\infty}I_e(\xi)=i_0\lambda^{-1}\left(\frac{\lambda_0-\lambda}{\lambda_0-1}\right)e^{(\lambda+\sigma-\lambda_0)\xi},$$
$$\lim_{\xi\to\infty}I_p(\xi)=i_0\left(\frac{\mu}{1-\mu}\right)\lambda^{-1}K\left(\frac{\lambda_0-\lambda}{\lambda_0-1}\right)e^{(\lambda+\sigma-\lambda_0)\xi},$$
$$K=1+\lambda e^{\sigma}\left[\frac{e^{-(\lambda+\sigma-\lambda_0)}-e^{-(\lambda+\sigma-\lambda_0)/\mu}}{\lambda+\sigma-\lambda_0}\right]-\lambda\left[\frac{1-e^{-[\lambda+(1-\mu)\sigma-\lambda_0]/\mu}}{\lambda+(1-\mu)\sigma-\lambda_0}\right]. \tag{53}$$

In the next section we shall present the results of numerical computations based on formulas obtained in Secs. IV–VI.



(a)



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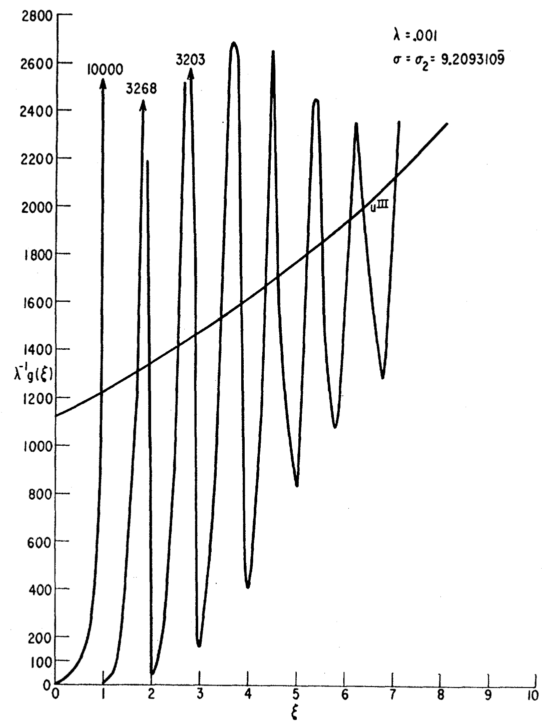


(c)

FIG. 1. Unit-pulse solution for $\lambda=0.001$, (a) threshold; (b) undervoltage; (c) overvoltage.

Before presenting the numerical results a few remarks about pure ion secondary and mixed ion-photon secondary mechanisms may be appropriate. The case of pure ion secondary mechanism is probably of no practical interest. The results of the preceding presentation are readily applicable for this case, however. For pure ion secondary the notation becomes

$$\lambda=\omega\delta=\gamma_i\sigma, \quad \xi=t/\tau_3, \quad \mu=\tau_1/\tau_3, \quad i_0=q_0/\tau_3, \tag{54}$$

where i_0 designates the externally induced current for the unit pulse case.

The electron and ion current densities at the cathode are, respectively,

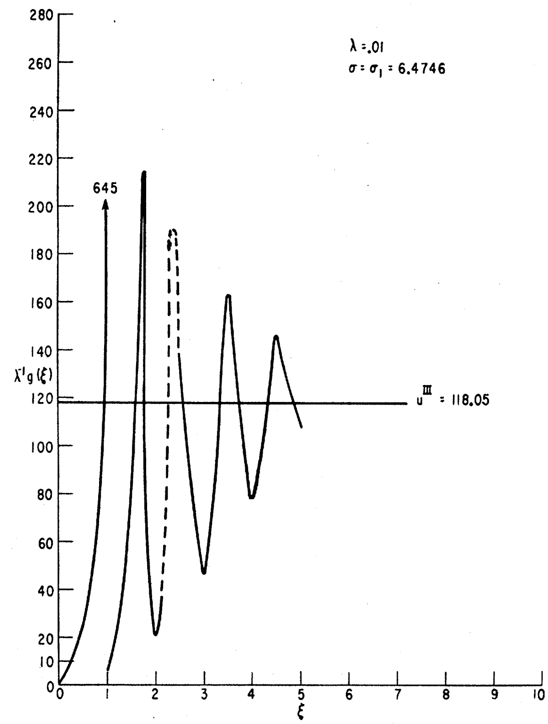
$$\begin{aligned} i(0,\xi)&=i_0[\delta(\xi)+\lambda u(\xi)e^{\sigma\xi}],\\ j(0,\xi)&=i_0\sigma u(\xi)e^{\sigma\xi}. \end{aligned} \tag{55}$$

The contribution of the electron and ion currents to the total external current may be written

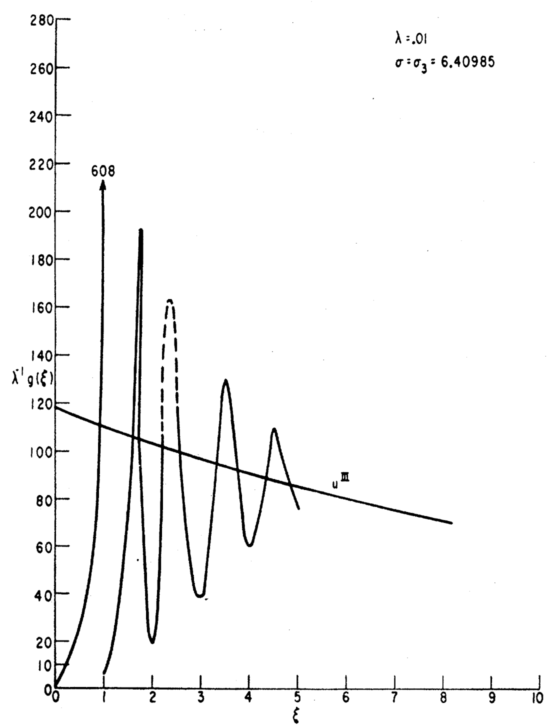
$$\begin{aligned} I_e(\xi)&=(i_0/\mu)\int_{\xi-\mu}^{\xi} e^{\sigma(\xi-\eta)/\mu}g(\eta)d\eta,\\ I_p(\xi)&=\left(\frac{\mu}{1-\mu}\right)I_e(\xi)-\frac{\sigma^{-1}}{1-\mu}j(0,\xi)\\ &\quad+i_0\left(\frac{e^{\sigma}}{1-\mu}\right)\int_{\xi-1}^{\xi-\mu} g(\eta)d\eta. \end{aligned} \tag{56}$$

The above formulas apply to the unit-pulse case.

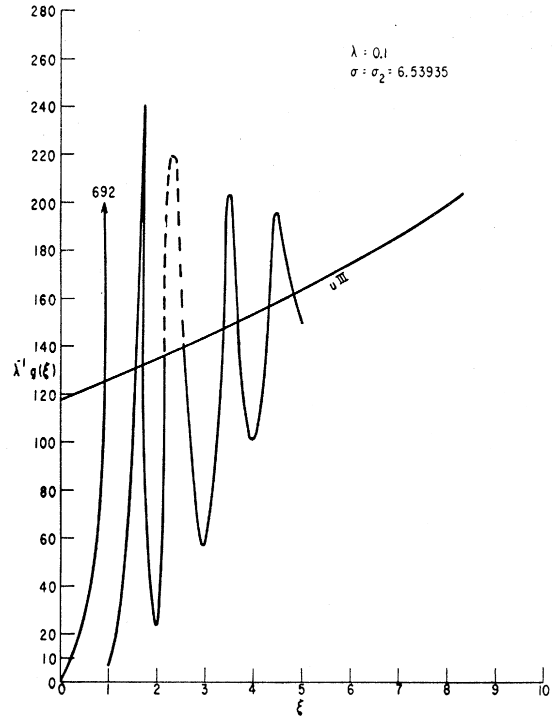
Considerable simplification can be obtained for the case of a pure ion secondary over time intervals less than τ_3. From previous results we obtain, for $t \leqslant \tau_3$,



(a)



(b)



(c)

FIG. 2. Unit-pulse solution for $\lambda=0.01$, (a) threshold; (b) undervoltage; (c) overvoltage.

$$g(\xi)=\delta(\xi)+\lambda u(\xi)e^{\sigma\xi}$$
$$=\delta(\xi)+\lambda e^{(\lambda+\sigma)\xi}. \qquad (57)$$

Substituting (57) into (56) we obtain, for $0 \leqslant t \leqslant \tau_1$,

$$I_e(\xi)=\frac{i_0/\mu}{\sigma-\mu(\lambda+\sigma)}\left[(1-\mu)\sigma e^{\sigma\xi/\mu}-\mu\lambda e^{(\lambda+\sigma)\xi}\right]; \qquad (58a)$$

for $\tau_1 < t \leqslant \tau_3$, we obtain

$$I_e(\xi)=i_0\frac{\lambda e^{(\lambda+\sigma)\xi}}{\sigma-\mu(\lambda+\sigma)}\left[e^{\sigma-\mu(\lambda+\sigma)}-1\right]. \qquad (58b)$$

There is a discontinuity at $t=\tau_1$ in $I_e(\xi)$ equal to $(i_0/\mu)e^{\sigma}$ due to the fact that the initial pulse is lost to the avalanche system at this point.

For times less than τ_1, the integral in the expression (56) for $I_p(\xi)$ is zero; and for times $\tau_1 < t \leqslant \tau_3$, it becomes

$$i_0\left(\frac{e^{\sigma}}{1-\mu}\right)\left[1+\frac{\lambda}{\lambda+\sigma}\left(e^{(\lambda+\sigma)(\xi-\mu)}-1\right)\right]. \qquad (59)$$

The above expression combined with (58), (57), and (56) gives the required result for $I_p(\xi)$.

In the event that both a photon and ion secondary are active, the appropriate notation becomes

$$\lambda=\beta\delta=\gamma_p\sigma, \quad \xi=t/\tau_1, \quad \mu=\tau_1/\tau_3, \quad \nu=\omega/\beta=\gamma_i/\gamma_p. \qquad (60)$$

The fundamental integral equation (8) for the cathode electron current takes the form

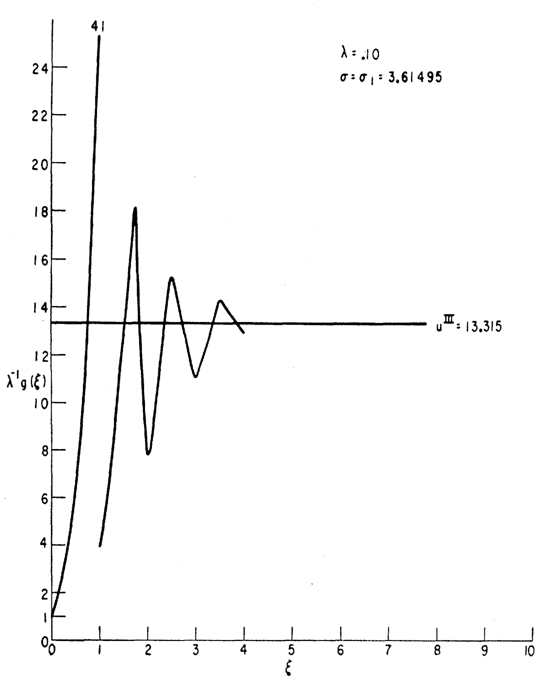
$$f(\xi)=i_0(\xi)+\lambda\int_{\xi-1}^{\xi} e^{\sigma(\xi-\eta)}f(\eta)d\eta + \nu\mu\lambda\int_{\xi-\mu^{-1}}^{\xi} e^{\mu\sigma(\xi-\eta)}f(\eta)d\eta. \quad (61)$$

There are no conceptual difficulties in obtaining a solution for Eq. (61). The methods presented in Secs. IV and V are directly applicable. However, the solutions of (61) are considerably more complicated than the cases treated previously because of the additional parameters μ and ν. A further objection to the practical use of (61) arises from the fact that the experimentalist rarely has precise values of ν available. One ordinarily observes the sum of γ_i and γ_p, not their ratio. We have not considered it practical to discuss the general solution of (61) because of these mathematical and experimental inconveniences.

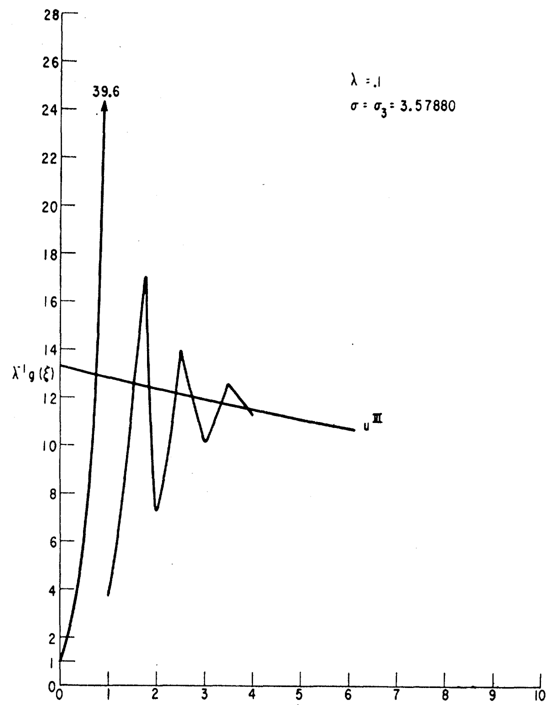
VII. NUMERICAL EXAMPLES

Example I

The unit-pulse function $u(\xi)$ is tabulated for four different λ values in Tables I(a)–(d). With appropriate notation, the results are equally valid for pure γ_i or γ_p mechanisms. The quantity marked u^{I} is calculated from (25); the quantity $(u^{\mathrm{II}})_{(k)}$ is calculated from (27) using the first k terms of the series. When significant-figure difficulties occur in the use of (25), the number of significant figures in u^{I} and u^{II} will differ for given ξ



(a)



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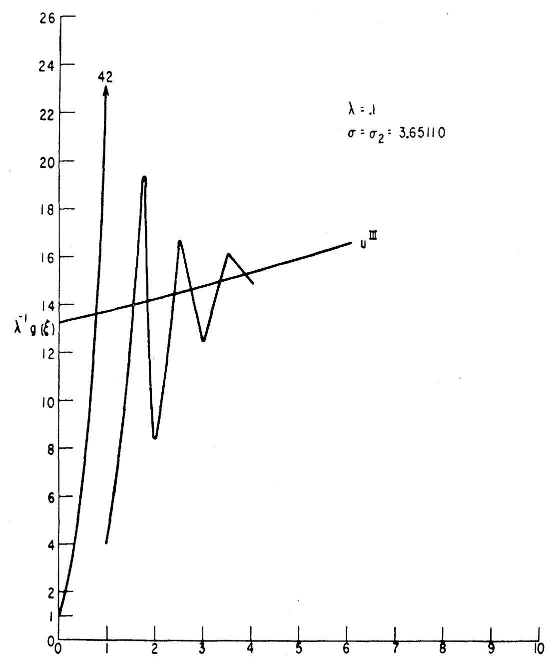


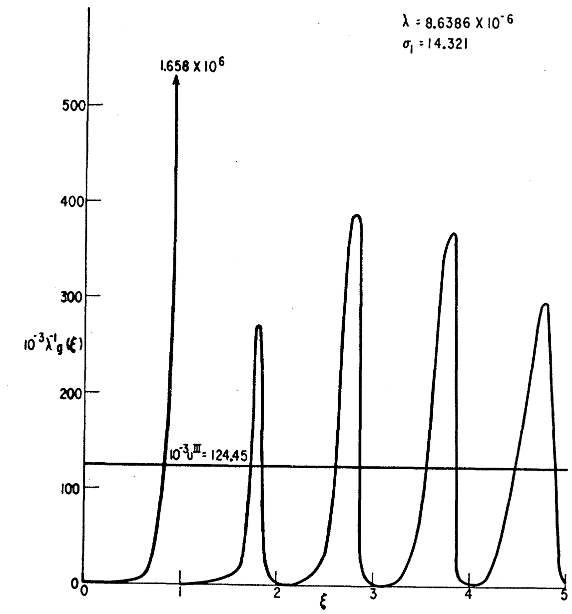
(c)

FIG. 3. Unit-pulse solution for $\lambda=0.1$, (a) threshold; (b) undervoltage; (c) overvoltage.

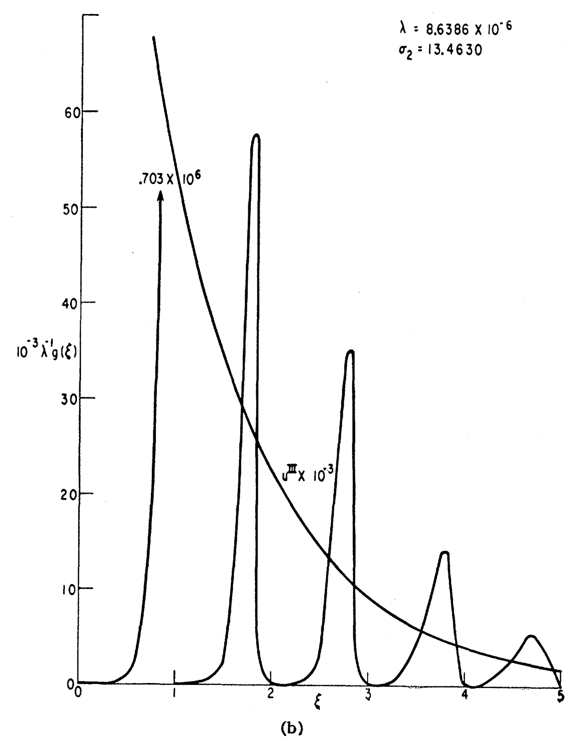
values. The quantity u^{III} is the asymptotic formula of (34).

It will be noted that for $\lambda<0.1$, the convergence of the

u^{II} series is quite rapid. Since in practice one usually finds $\lambda < 10^{-3}$, we may conclude that the u^{II} series is most useful for calculation of short-time behavior. When $\lambda \geqslant 10^{-1}$, the u^{II} series does not converge rapidly enough for convenient computation. However, in this region the simple asymptotic expression u^{III} rapidly becomes a very good approximation to the true value of u.



(a)



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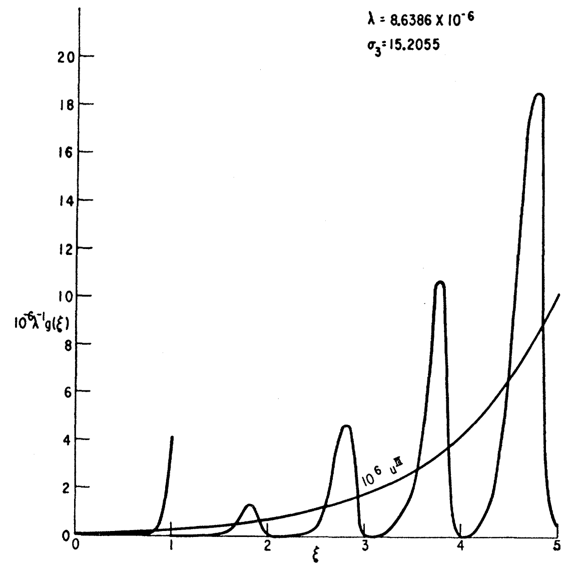

(c)

FIG. 4. Unit-pulse solution using Miyoshi's breakdown parameters, (a) threshold; (b) undervoltage 1%; (c) overvoltage 1%.

Example II

In Figs. 1–3 we have drawn the quantity $\lambda^{-1}g(\xi) = u(\xi)e^{\sigma\xi}$ as a function of ξ for three different λ values. The oscillatory curves represent g functions obtained from true u values; the smooth curves labeled u^{III} represent the asymptotic formula for g obtained from (34). For a given λ value three different σ values were chosen. In each case, figures marked (a) represent threshold conditions where $\alpha\delta \equiv \sigma_1 = \lambda_0 - \lambda$; the figures marked (b) represent conditions of slight undervoltage with $\alpha\delta \equiv \sigma_3 = 0.99\sigma_1$; and figures marked (c) represent conditions of slight overvoltage conditions with $\alpha\delta \equiv \sigma_2 = 1.01\sigma_1$.

It will be noted that the unit pulse solution $g(\xi)$ oscillates inside an envelope whose breadth decreases in time. The asymptotic formula for g runs approximately down the middle of this envelope. As time increases the amplitudes of oscillations decrease and eventually the true solution coincides with the asymptotic solution.

For a pure γ_p mechanism the quantity $\lambda^{-1}g(\xi)$ represents $I_e(\xi)/i_0$ according to (46) and also $\lambda^{-1}i(0,\xi)/i_0$ according to (44). For a pure γ_i mechanism the quantity $\lambda^{-1}g(\xi)$ represents $\sigma^{-1}j(0,\xi)/i_0$.

Example III

In order to compare numerically our results with those of Miyoshi,[4] we have taken his expressions for α and computed $\lambda^{-1}g(\xi)$ for his three cases of threshold, 1% undervoltage, and 1% overvoltage. The results are given in Figs. 4 (a)–(c), respectively, and are to be compared directly with Miyoshi's Fig. 9. Miyoshi plots only the initial transit time and does not mention the discontinuity. There appears to be a slight discrepancy

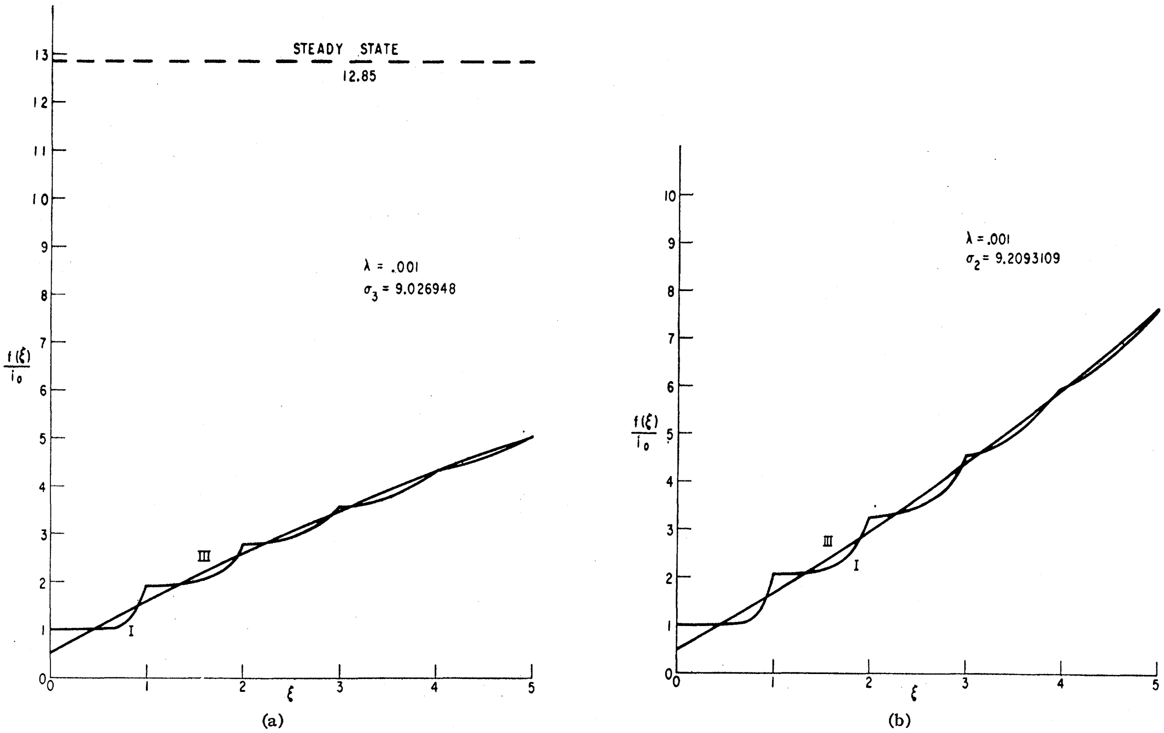

FIG. 5. Constant illumination for $\lambda=0.001$, (a) undervoltage; (b) overvoltage.

between our numerical results and Miyoshi's for the initial transit time. We are not certain whether this discrepancy is due to the fact that Miyoshi is using approximate expressions for his computation or whether there is an arithmetic error in his calculations.

The curves labeled $u^{\rm III}$ denote the asymptotic formula for $g(\xi)$ and are drawn in for comparison. Because of the scale chosen, the lower bound to the envelope is distorted in Fig. 4. For the sake of reference we quote Miyoshi's α values:

$$\alpha/p=A(E/p-B)^2,$$

where $A=1.048\times10^{-4}$ (cm mm Hg/v), $B=27.38$ (v/cm mm Hg), $p=760$ (mm Hg), $\delta=1$ cm, and $V_s=31$ kv. The constant value of γ is obtained from the threshold condition $\gamma(e^{\alpha\delta}-1)=1$ with the V_s value of α.

Example IV

In Figs. 5(a) and (b) we present the computational results for the quantity $f(\xi)/i_0$ for the cases of undervoltage and overvoltage, respectively, corresponding to the unit-pulse functions shown in Figs. 1(b) and (c). The curve labeled I represents the true value of $f(\xi)$ as obtained from (43), while the curve labeled III represents the asymptotic formula according to (42). The quantity $f(\xi)$ may be interpreted as either $i(0,\xi)$ for constant illumination or as $j(0,\xi)/\mu\sigma e^{\mu\sigma}$ for the unit pulse case with γ_p mechanism. [Strictly speaking the quantity $f(\xi)$ represents $j(0,\xi)$ only if we substitute $(1-\mu)\sigma$ for σ in the $f(\xi)$ formula; see (48).] It will be noted that the oscillations in $f(\xi)$ are very small compared to $g(\xi)$ and the discontinuity at $\xi=1$ appears only in the derivative of $f(\xi)$. The asymptotic formula is always a reasonably good approximation to the true function except in the initial interval.

ACKNOWLEDGMENT

We wish to express our sincerest thanks to Miss Ann Warner for the detailed calculations which appear in Part VII of this report.