Velocity-Dependent Features of a Static Nucleon-Nucleon Potential*

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The consequences of assuming a pseudoscalar classical field with pseudoscalar coupling are considered in a static approximation. Virtual pair formation of nucleon-antinucleon pairs is not calculated but it is supposed that it gives rise to an attraction. The interaction is being studied mainly in order to see what features of commonly assumed static potentials may not apply. The quantitative features of the derived interaction are not expected to correspond to the physical one. The qualitative character of deviations from static potential behavior is believed, however, to have a bearing on the velocity-dependent features of the actual interaction. The work neglects some of the effects of the exclusion principle caused by the population of negative-energy states. Some of the qualitative nonstatic features are: (a) different magnitudes of the spin-orbit potential in even and odd states, (b) inadequacy of the usual procedure of the insertion of e^2/r in the wave equation for the calculation of phase shifts in p-p scattering; this inadequacy is connected with the fact that the standard form of the wave equation is obtained only after a transformation of a wave equation in which e^2/r has a direct significance as a potential energy, (c) velocity (energy) dependence of core radii and of magnitude of effective potentials, (d) occurrence of quadratic term in the tensor interaction in the effective potential even though in an earlier stage of the calculation only linear terms are present.

I. INTRODUCTION

A NUMBER of attempts have been made to work out the interaction between two nucleons, making use of the meson theory of nuclear forces. An account of the more important of these is given in the book by Bethe and de Hoffmann,[1] from which it is clear that a satisfactory treatment of the interaction of nucleons through the pseudoscalar field is not available. The present note does not pretend to provide a solution of this problem. Its object is to report on calculations which suggest the possibility of certain differences between the actual behavior of nucleons and their description by effective static potentials. The possibility that the interaction is velocity-dependent has been realized for a long time. Thus Breit, Thaxton, and Eisenbud[2] have calculated some of the effects on the nuclear-force range parameter following from experiment on the supposition that the depth parameter varies with the energy. Internal excitation of a nucleon has been studied by Breit and Yovits[3] as a model giving rise to such a velocity dependence. In the more purely field-theoretic studies the effects of internal excitation have been considered by Matsumoto, Hamada, and Sugawara,[4] who have considered the effects of the $(\frac{3}{2},\frac{3}{2})$ state in the static limit. Iwadare[5] is concerned with "nonstatic" corrections employing the quantized symmetrical pseudoscalar theory and making use of an expansion in powers of the coupling constant. The well-known difficulties of this procedure make it hard to arrive at definite results on such a basis[1] and the publication of some, in the main part, old calculations[6] showing somewhat related effects in a relatively simple manner appears justifiable. It will be seen that many of the qualitative velocity-dependent features of the potential appear to be not in contradiction with the analysis of nucleon-nucleon scattering data made by Marshak and Signell[7] and by Gammel and Thaler[8] in terms of static potentials. The interaction is considered by making use of pseudoscalar coupling to a symmetrical classical meson field. The employment of a nonquantized field simplifies the calculations and makes it possible to treat a part of the problem to within all powers of the interaction constant. The omission of quantization of the field is an approximation only. It doubtless causes serious errors at energies approaching the threshold for meson production. It is not clear, however, that this method is necessarily very inaccurate at the lower energies. In the case of the electromagnetic interaction, the larger effects in the interaction between charges follow from a nonquantized electromagnetic field theory, although the emission of photons as well as the Lamb shift and associated phenomena depend on quantization in an essential manner.

Another limitation of the present treatment is the omission of an explicit consideration of the effect of the exclusion principle in connection with the population of

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[1] H. A. Bethe and F. de Hoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston and White Plains, 1956), Vol. II, Sec. 47.

[2] Breit, Thaxton, and Eisenbud, Phys. Rev. **55**, 1018 (1939).

[3] G. Breit and M. C. Yovits, Phys. Rev. **81**, 416 (1951).

[4] Matsumoto, Hamada, and Sugawara, Progr. Theoret. Phys. Japan **10**, 199 (1953).

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[6] G. Breit, *Proceedings of the Fifth Annual Rochester Conference on High-Energy Physics* (Interscience Publishers, Inc., New York, 1955).

[7] R. Marshak, *Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, Inc., New York, 1957); P. S. Signell and R. Marshak, Bull. Am. Phys. Soc. Ser. II, **2**, 176 (1957); P. S. Signell and R. Marshak, Phys. Rev. **106**, 832 (1957); **109**, 1229 (1958).

[8] J. L. Gammel and R. M. Thaler, *Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, Inc., New York, 1957); Phys. Rev. **107**, 291, 1337 (1957).

negative-energy states. In the interaction between electrons a similar omission matters only in rather small corrections. In the present case the effect of the omission may be more serious because of the peculiarities of the pseudoscalar interaction. A discussion of the error introduced by this simplifying assumption is attempted in the fourth section of the present paper.

The treatment is entirely static in the sense of neglecting the time taken for the field produced at one nucleon to reach the other one. It includes no recoil effects of nucleons caused by meson emission. Judging by analogy to the electromagnetic case, one may suspect that some spin-orbit interaction effects have been omitted as a result of these approximations. This is probably the case. One of the omitted effects is considered in a very crude and phenomenologic manner in connection with a discussion of indications concerning the spin-orbit interactions. It appears possible that this effect originates in a Thomas-type correction to the attractive potential caused by virtual nucleon-antinucleon pair formation and that the effect of this term adds itself to the spin-orbit interaction term which arose in the absence of pair formation. In some respects the treatment of the spin-orbit interaction is related to the paper by Araki.[9] The main difference is that no attempt is being made here to include corrections for nucleon recoil. These corrections should eventually be included of course. It is doubtful, however, that one can do so consistently without bringing in effects of forces between nucleons caused by virtual pair formation. For this reason no attempt is being made here to treat the recoil effects. The large repulsive effects present in the static approximation are speculatively supposed to make the relativistic features less important than in a formal expansion in powers of the interaction constant. The recoil effects are therefore considered as giving rise to corrections which apply to the combined action of the static and pair-formation effects. The main emphasis in the present paper is on the deficiencies of the equivalent static potential concept and its object is modest, being mainly qualitative.

The work appeared worth reporting because it shows possibilities of the presence of the following effects. The spin-orbit interaction is expected to be different for triplet even and triplet odd states. Furthermore, the spin-orbit interaction which appears in the wave equation in the form of a term containing the usual $(\mathbf{L}\cdot\mathbf{S})$ as a factor is not the only interaction giving rise to effects which are most readily attributed to first order effects of an $(\mathbf{L}\cdot\mathbf{S})$ term in nucleon-nucleon scattering. In addition to the occurrence of the tensor interaction operator in the final radial wave equation, one finds also a term in $(S_{12})^2$. The occurrence of this term is a consequence of a velocity dependence of the effective interaction operator when the equation is reduced from the original 16-component to a 4-component form. Depending on the form in which the results are expressed, there appear tensor force operators containing one $\mathbf{r}$ replaced by relative momentum $\mathbf{p}$ or both $\mathbf{r}$ replaced by $\mathbf{p}$. The interaction potential is velocity dependent also in other ways. Thus the effective potential energy wells depend on the incident kinetic energy. The possible existence of these types of velocity dependence of nuclear forces has been considered before.[2,3] The present work furnishes a model of such a dependence. There appears in the calculations a soft repulsive core and possibilities of obtaining hard cores for some states. The presence of the cores appears to be approximately reconcilable with phenomenological requirements.

An additional general feature of the equation is that the radial equation containing an effective potential is not directly related to the four-component function obtained by reduction to 4 components, a transformation of the radial function being involved. The transformation is necessary in order to remove terms linear in d/dr which occur as a result of the velocity dependence of the effective potential. A consequence of this additional transformation is that it is not quite right to be making corrections for Coulomb effects by introducing the Coulomb energy e^2/r in the radial equation used for the calculation of phase shifts. Comparisons of p-p and p-n interactions made for the purpose of tests of charge independence need refinement on account of the presence of correction terms containing e^2 and arising through the introduction of the Coulomb energy e^2/r. Numerical estimates presented in the fourth section are not discouraging. The main object of the work was, however, to determine those qualitative features of static potential models which one should treat carefully, rather than to attempt the formulation of a quantitative theory.

II. DERIVATION OF THE EFFECTIVE POTENTIAL

Two nucleons a and b are coupled symmetrically to a pseudoscalar field φ. The Hamiltonian of the system is accordingly

$$H=H_a+H_b+\tfrac{1}{2}\int\sum_\alpha[(\partial\varphi_\alpha/\partial x)^2+(\partial\varphi_\alpha/\partial y)^2+(\partial\varphi_\alpha/\partial z)^2+(\partial\varphi_\alpha/c\partial t)^2+\mu^2\varphi_\alpha{}^2]d\mathbf{r}+(4\pi)^{\frac{1}{2}}f\sum_\alpha[\tau_\alpha{}^a\rho_2{}^a\varphi_\alpha(\mathbf{r}^a)+\tau_\alpha{}^b\rho_2{}^b\varphi_\alpha(\mathbf{r}^b)]. \quad (1)$$

Here α takes three values 1, 2, 3 which correspond to the three components of the isotopic spin vectors $\boldsymbol{\tau}^a$, $\boldsymbol{\tau}^b$. μ is a constant while the matrix ρ_2 is one of the three matrices ρ introduced by Dirac, ρ_1 entering $\boldsymbol{\alpha}=\rho_1\boldsymbol{\sigma}$, while $\beta=\rho_3$ and $\rho_3\rho_1=i\rho_2$; H_a and H_b are, respectively, the free-particle Hamiltonians for a and b. The equations of motion for the unquantized meson field yield

$$(\Delta-\mu^2-\partial^2/c^2\partial t^2)\varphi_\alpha=(4\pi)^{\frac{1}{2}}f[\tau_\alpha{}^a\rho_2{}^a\delta(\mathbf{r}-\mathbf{r}^a)+\tau_\alpha{}^b\rho_2{}^b\delta(\mathbf{r}-\mathbf{r}^b)], \quad (1.1)$$

[9] G. Araki, Progr. Theoret. Phys. Japan 6, 379 (1951).

where the δ functions are three-dimensional. Neglecting the motion of the nucleons,

$$\varphi_\alpha(\mathbf{r})=-(4\pi)^{-\frac{1}{2}}f\Big[\tau_\alpha{}^a\rho_2{}^a\frac{\exp(-\mu|\mathbf{r}-\mathbf{r}^a|)}{|\mathbf{r}-\mathbf{r}^a|}+\tau_\alpha{}^b\rho_2{}^b\frac{\exp(-\mu|\mathbf{r}-\mathbf{r}^b|)}{|\mathbf{r}-\mathbf{r}^b|}\Big]. \quad (1.2)$$

Substitution of (1.2) into (1) yields an infinite energy. This divergence is harmless, however, in the present essentially nonrelativistic theory because it can be removed by spreading the sources of the field in (1) in three dimensions through the insertion in (1) of an integral over the field coordinates $\mathbf{r}$ and of three-dimensional source functions $D_a(\mathbf{r}-\mathbf{r}^a)$, $D_b(r-\mathbf{r}^b)$. The change in H caused by the proximity of a and b can be obtained by surrounding these particles by small spheres and applying Green's theorem to the volume outside these surfaces and inside a sphere of infinite radius surrounding the system. One has

$$\int[(\nabla\varphi_\alpha)^2+\mu^2\varphi_\alpha{}^2]d\mathbf{r}=\int\varphi_\alpha(\partial\varphi_\alpha/\partial n)dS, \quad (1.3)$$

as a consequence of

$$(\Delta-\mu^2)\varphi_\alpha=0,$$

which holds in the space between the particles in the static approximation. The integration on the left side of (1.3) is over the previously mentioned volume, while on the right it is taken over the surface enclosing the volume with n standing for the outward-drawn normal. At the small sphere enclosing particle a, one may use the approximation

$$\frac{\partial}{\partial n}\Big[-(4\pi)^{-\frac{1}{2}}f\tau_\alpha{}^a\rho_2{}^a\frac{\exp(-\mu|\mathbf{r}-\mathbf{r}^a|)}{|\mathbf{r}-\mathbf{r}^a|}\Big]\cong-(4\pi)^{-\frac{1}{2}}\frac{\tau_\alpha{}^a\rho_2{}^a}{|\mathbf{r}-\mathbf{r}^a|^2}. \quad (1.4)$$

One has therefore

$$\tfrac{1}{2}\int[(\nabla\varphi_\alpha)^2+\mu^2\varphi_\alpha{}^2]d\mathbf{r}-\text{self energy}=-\tfrac{1}{2}H^{ab}, \quad (2)$$

where

$$H^{ab}=(4\pi)^{\frac{1}{2}}f\sum_\alpha[\varphi_\alpha{}^b(\mathbf{r}^a)\tau_\alpha{}^a\rho_2{}^a+\varphi_\alpha{}^a(\mathbf{r}^b)\tau_\alpha{}^b\rho_2{}^b], \quad (2.1)$$

the first and second terms in brackets having arisen, respectively, from integrations over the small spheres around a and b. For the first part, only the portion of the field having b as a source counts because the $\varphi_\alpha{}^a(r^a)$ contributes in this term only to the self energy; similarly in the second term with a and b interchanged. Making use of (1.2), one obtains

$$H^{ab}=-2f^2\sum_\alpha\tau_\alpha{}^a\rho_2{}^a\tau_\alpha{}^b\rho_2{}^b\times\{\exp(-\mu|\mathbf{r}^a-\mathbf{r}^b|)\}/|\mathbf{r}^a-\mathbf{r}^b|. \quad (2.2)$$

In Eq. (1) the last term can also be decomposed into a self-energy part and a part depending on particle proximity. Comparison with (2.1) shows that it is H^{ab}. Hence, from (1),

$$H=H^a+H^b+\text{self energy}+H_{ab}, \quad (3)$$

with

$$H_{ab}=\tfrac{1}{2}H^{ab}. \quad (3.1)$$

The self-energy part can be discarded since it amounts to a large but finite constant as long as the D-function distributions have finite extension. The same result can be obtained by keeping the D functions in the calculation without the employment of Green's theorem provided the limit of infinitely concentrated D functions is taken.

III. REDUCTION OF THE HAMILTONIAN

For two particles of equal masses, the Hamiltonian may therefore be taken to be

$$H=-c(\boldsymbol{\alpha}^a\cdot\mathbf{p}^a)-\beta^aMc^2-c(\boldsymbol{\alpha}^b\cdot\mathbf{p}^b)-\beta^bMc^2-f^2(\boldsymbol{\tau}^a\cdot\boldsymbol{\tau}^b)\rho_2{}^a\rho_2{}^be^{-\mu r}/r, \quad (4)$$

where

$$r=|\mathbf{r}^a-\mathbf{r}^b|, \quad (4.1)$$

the original Dirac choice of matrices $\boldsymbol{\alpha}$, β being used. The wave equation is

$$(H-E)\psi=0, \quad (4.2)$$

with ψ standing for a 16-component spinor. With the usual choice of matrices, one has

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\quad \beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

each of the entries in the matrices being a two-by-two matrix and the $\boldsymbol{\sigma}$ being the standard Pauli matrices. The 16 components arrange themselves into 4 sets with 4 components each, which in the nonrelativistic limit are large or small for a or b, respectively. These sets are here denoted as Ψ, χ^a, χ^b, Φ; Ψ is large for a and b, Φ is small for both, χ^a is small for a and χ^b is small for b. Equation (4.2) is equivalent to the four simultaneous equations

$$\begin{aligned}(E-2Mc^2)\Psi+\pi^a\chi^a+\pi^b\chi^b-F\Phi&=0,\\ \pi^a\Psi+E\chi^a+F\chi^b+\pi^b\Phi&=0,\\ \pi^b\Psi+F\chi^a+E\chi^b+\pi^a\Phi&=0,\\ -F\Psi+\pi^b\chi^a+\pi^a\chi^b+(E+2Mc^2)&=0,\end{aligned} \quad (4.3)$$

with

$$F=(f^2/r)(\boldsymbol{\tau}^a\cdot\boldsymbol{\tau}^b)e^{-\mu r},\quad \pi^a=c(\boldsymbol{\sigma}^a\cdot\mathbf{p}^a),\quad \pi^b=c(\boldsymbol{\sigma}^b\cdot\mathbf{p}^b). \quad (4.4)$$

Eliminating χ^a and χ^b one finds

$$\mathfrak{L}_1(\Psi+\Phi)=0, \quad \mathfrak{L}_2(\Psi-\Phi)=0, \tag{4.5}$$

with

$$\mathfrak{L}_1=E^2-F^2-4M^2c^4-(\pi^a-\pi^b)\frac{1}{E-F}(\pi^a-\pi^b)$$
$$\times(E-F)-(E+F)(\pi^a+\pi^b)\frac{1}{E+F}(\pi^a+\pi^b), \tag{4.6}$$

$$\mathfrak{L}_2=E^2-F^2-4M^2c^4-(E-F)(\pi^a-\pi^b)\frac{1}{E-F}$$
$$\times(\pi^a-\pi^b)-(\pi^a+\pi^b)\frac{1}{E+F}(\pi^a+\pi^b)(E+F), \tag{4.7}$$

and with the specialization in the last two equations to

$$(\mathbf{p}^a+\mathbf{p}^b)\psi=0, \tag{4.8}$$

which applies in the rest system of the particles. This specialization will be used henceforth. Calculation gives

$$\tfrac{1}{2}(\mathfrak{L}_1+\mathfrak{L}_2)$$
$$=E^2-F^2-4M^2c^4-4c^2p^2$$
$$-2c^2\hbar^2[\![\left[(\boldsymbol{\sigma}^a\boldsymbol{\sigma}^b)+(\boldsymbol{\sigma}^a\mathbf{r})(\boldsymbol{\sigma}^b\mathbf{r})\frac{1}{r}\frac{d}{dr}\right]\frac{(E/r)dF/dr}{E^2-F^2}]\!]$$
$$-2c^2\hbar^2[\![\left(3+i(\boldsymbol{\sigma}^a+\boldsymbol{\sigma}^b)\cdot[\boldsymbol{\nabla}\times\mathbf{r}]+r\frac{d}{dr}\right)$$
$$\times\frac{(F/r)dF/dr}{E^2-F^2}]\!], \tag{5}$$

where differential operators occurring inside the $[\![\ ]\!]$ are meant not to be applied outside these special brackets. One also finds

$$\tfrac{1}{2}(\mathfrak{L}_1-\mathfrak{L}_2)$$
$$=c^2\hbar^2\Big\{-2(\boldsymbol{\nabla}\cdot\mathbf{r})\frac{(E/r)dF/dr}{E^2-F^2}-2\frac{(E/r)dF/dr}{E^2-F^2}(\mathbf{r}\cdot\boldsymbol{\nabla})$$
$$-[(\boldsymbol{\sigma}^a\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}^b\cdot\mathbf{r})+(\boldsymbol{\sigma}^b\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}^a\cdot\mathbf{r})]$$
$$\times\frac{(F/r)dF/dr}{E^2-F^2}-\frac{(F/r)dF/dr}{E^2-F^2}$$
$$\times[(\boldsymbol{\sigma}^a\cdot\mathbf{r})(\boldsymbol{\sigma}^b\cdot\boldsymbol{\nabla})+(\boldsymbol{\sigma}^b\cdot\mathbf{r})(\boldsymbol{\sigma}^a\cdot\boldsymbol{\nabla})]\Big\}. \tag{5.1}$$

Here and below, we set

$$\mathbf{r}=\mathbf{r}^a-\mathbf{r}^b, \quad \mathbf{p}=\mathbf{p}^a=-\mathbf{p}^b. \tag{5.2}$$

Inspection shows that $(\mathfrak{L}_1+\mathfrak{L}_2)/2$ is Hermitean and that $(\mathfrak{L}_1-\mathfrak{L}_2)/2$ is anti-Hermitean. Hence, according to (4.5) neither $\Psi+\Phi$ nor $\Psi-\Phi$ satisfy equations with an equivalent Hermitean Hamiltonian. On the other hand, it follows from (4.5) that

$$[\mathfrak{L}_1+\mathfrak{L}_2-(\mathfrak{L}_1-\mathfrak{L}_2)(\mathfrak{L}_1+\mathfrak{L}_2)^{-1}(\mathfrak{L}_1-\mathfrak{L}_2)]\Psi=0, \tag{5.3}$$

and in this equation the operator to the left of Ψ is Hermitean. It would be complicated to work with (5.3), especially on account of the occurrence of the reciprocal of $\mathfrak{L}_1+\mathfrak{L}_2$. The effective Hamiltonian corresponding to this equation is velocity dependent in a very complicated way. In order to determine the phase shifts, however, it is not necessary to eliminate all components with the exception of Ψ. In fact, if $F=0$ in a part of the coordinate space, then in that part of the space Eq. (4.4) determines Φ in terms of Ψ and a phase shift in Ψ *is reproduced as an equal phase shift in* Φ. For this reason the reduction of the first of the two equations in (4.5) will be used. The starting point for further reduction is thus

$$\mathfrak{L}_1=E^2-F^2-4M^2c^4-4c^2p^2+c^2\hbar^2$$
$$\times\Big\{-2[\![\left[(\boldsymbol{\sigma}^a\cdot\boldsymbol{\sigma}^b)+(\boldsymbol{\sigma}^a\cdot\mathbf{r})(\boldsymbol{\sigma}^b\cdot\mathbf{r})\frac{1}{r}\frac{d}{dr}\right]\frac{(E/r)dF/dr}{E^2-F^2}]\!]$$
$$-2i[(\boldsymbol{\sigma}^a+\boldsymbol{\sigma}^b)\cdot(\boldsymbol{\nabla}\times\mathbf{r})]\frac{(F/r)dF/dr}{E^2-F^2}$$
$$-2[\![\left(3+r\frac{d}{dr}\right)\frac{(F/r)dF/dr}{E^2-F^2}]\!]$$
$$-2(\boldsymbol{\nabla}\cdot\mathbf{r})\frac{(E/r)dF/dr}{E^2-F^2}-\frac{2(E/r)dF/dr}{E^2-F^2}(\mathbf{r}\cdot\boldsymbol{\nabla})$$
$$-[(\boldsymbol{\sigma}^a\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}^b\cdot\mathbf{r})+(\boldsymbol{\sigma}^b\cdot\boldsymbol{\nabla})(\boldsymbol{\sigma}^a\cdot\mathbf{r})]$$
$$\times\frac{(F/r)dF/dr}{E^2-F^2}-\frac{(F/r)dF/dr}{E^2-F^2}$$
$$\times[(\boldsymbol{\sigma}^a\cdot\mathbf{r})(\boldsymbol{\sigma}^b\cdot\boldsymbol{\nabla})+(\boldsymbol{\sigma}^b\cdot\mathbf{r})(\boldsymbol{\sigma}^a\cdot\boldsymbol{\nabla})]\Big\}. \tag{5.4}$$

Further reduction can be made by specifying the value of the total spin which has the same value for Ψ and Φ. The truth of the last statement follows from the fact that all the operators occurring in (5) and (5.1) commute with

$$\mathbf{s}=\tfrac{1}{2}(\boldsymbol{\sigma}^a+\boldsymbol{\sigma}^b). \tag{5.5}$$

It is convenient to introduce the operators

$$T_{ab}{}^{+}=3[(\mathbf{s}\cdot\mathbf{r})(\mathbf{s}\cdot\mathbf{p})+(\mathbf{s}\cdot\mathbf{p})(\mathbf{s}\cdot\mathbf{r})]$$
$$-\mathbf{s}^2[(\mathbf{r}\cdot\mathbf{p})+(\mathbf{p}\cdot\mathbf{r})], \tag{6}$$

$$T_{ab}{}^{-}=3[(\mathbf{s}\cdot\mathbf{r})(\mathbf{s}\cdot\mathbf{p})-(\mathbf{s}\cdot\mathbf{p})(\mathbf{s}\cdot\mathbf{r})]$$
$$-\mathbf{s}^2[(\mathbf{r}\cdot\mathbf{p})-(\mathbf{p}\cdot\mathbf{r})]=3ih(\mathbf{L}\cdot\mathbf{s}), \tag{6.1}$$

with

$$\hbar\mathbf{L}=[\mathbf{r}\times\mathbf{p}], \tag{6.2}$$

in terms of which

$$[(\sigma^a+\sigma^b)\cdot\mathbf{r}][(\sigma^a+\sigma^b)\cdot\mathbf{p}] = \tfrac{2}{3}(T_{ab}{}^{+}+T_{ab}{}^{-})+\tfrac{4}{3}\mathbf{s}^2(\mathbf{r}\cdot\mathbf{p}), \quad (6.3)$$

$$[(\sigma^a-\sigma^b)\cdot\mathbf{r}][(\sigma^a-\sigma^b)\cdot p] = -\tfrac{2}{3}(T_{ab}{}^{+}+T_{ab}{}^{-}) +4(1-\tfrac{1}{3}\mathbf{s}^2)(\mathbf{r}\cdot\mathbf{p})+4i\hbar(\mathbf{s}\cdot\mathbf{L}). \quad (6.4)$$

These operators give

$$(\sigma^a\cdot\nabla)(\sigma^b\cdot\mathbf{r})+(\sigma^b\cdot\nabla)(\sigma^a\cdot\mathbf{r}) = (i/\hbar)\{\tfrac{2}{3}T_{ab}{}^{+}+[\tfrac{4}{3}S(S+1)-2](\mathbf{p}\cdot\mathbf{r})\}, \quad (6.5)$$

$$(\sigma^a\cdot\mathbf{r})(\sigma^b\cdot\nabla)+(\sigma^b\cdot\mathbf{r})(\sigma^a\cdot\nabla) = (i/\hbar)\{\tfrac{2}{3}T_{ab}{}^{+}+[\tfrac{4}{3}S(S+1)-2](\mathbf{r}\cdot\mathbf{p})\}, \quad (6.6)$$

where

$$[\mathbf{s}^2-S(S+1)]\Psi=0, \quad (6.7)$$

so that S has the values 0 and 1 for singlets and triplets, respectively. It may be shown that the elements of $T_{ab}{}^{+}$ are related to those of the tensor operator

$$S_{ab}=3(\sigma^a\cdot\mathbf{r})(\sigma^b\cdot\mathbf{r})/r^2-(\sigma^a\cdot\sigma^b), \quad (7)$$

in the following way:

$$(L-2|T_{ab}{}^{+}|L)/(L-2|S_{ab}|L) = (\hbar r/i)[d/dr+(L+1)/r], \quad (7.1)$$

$$(L+2|T_{ab}{}^{+}|L)/(L+2|S_{ab}|L) = (\hbar r/i)[d/dr-L/r], \quad (7.2)$$

$$(L|T_{ab}{}^{+}|L)/(L|S_{ab}|L)=(\hbar r/i)[d/dr+3/(2r)]. \quad (7.3)$$

Here the designation of magnetic quantum numbers is omitted since it is unessential in the application. These quantum numbers are the same for the elements of $T_{ab}{}^{+}$ and S_{ab} whose quotient is being considered.

Employing these formulas together with (6.5), (6.6), and (5.4), one finds

$$\begin{aligned}\mathfrak{L}_1=&E^2-F^2-4M^2c^4-4c^2p^2\\ &+c^2\hbar^2\Big\{-2\Big[\Big[(2S(S+1)-3)\\ &+\tfrac{1}{3}(S_{ab}+2S(S+1)-3)r\frac{d}{dr}\Big]\frac{(E/r)dF/dr}{E^2-F^2}\Big]\\ &-(6+4(\mathbf{L}\cdot\mathbf{s}))\frac{(F/r)dF/dr}{E^2-F^2}\\ &-2\Big[r\frac{d}{dr}\frac{(F/r)dF/dr}{E^2-F^2}\Big]-2\Big(3+r\frac{d}{dr}\Big)\frac{(E/r)dF/dr}{E^2-F^2}\\ &-\frac{2(E/r)dF/dr}{E^2-F^2}r\frac{d}{dr}-\Big[\Big(\tfrac{2}{3}r\frac{d}{dr}+1\Big)S_{ab}\\ &+\Big(\frac{4}{3}S(S+1)-2\Big)\Big(r\frac{d}{dr}+3\Big)\Big]\frac{(F/r)dF/dr}{E^2-F^2}\\ &-\frac{(F/r)dF/dr}{E^2-F^2}\Big[\Big(\tfrac{2}{3}r\frac{d}{dr}+1\Big)S_{ab}\\ &+\Big(\frac{4}{3}S(S+1)-2\Big)r\frac{d}{dr}\Big]\Big\}\ \text{(no coupling)}, \quad (8)\end{aligned}$$

where only the case of states with sharp values of L is explicitly considered. In this equation S_{ab} is a c number equal to the element of S_{ab} diagonal in L and S. The occurrence in this equation of terms containing the first power of d/dr shows that an additional transformation is needed before an equivalent potential can be introduced. For singlets the equation simplifies to

$$\Big\{E^2-F^2-4M^2c^4-4c^2p^2+(4c^2\hbar/i)\frac{r^{-1}dF/dr}{E+F}(\mathbf{r}\cdot\mathbf{p})\Big\} \times(\Psi+\Phi)=0, \quad (S=0). \quad (9)$$

Setting

$$\Psi+\Phi={}^1R_L{}^{+}(r)Y_{L,M_L}(\theta,\varphi), \quad (9.1)$$

and

$${}^1R_L{}^{+}(r)={}^1u^{+}(r)(E+F)^{\frac{1}{2}}/r, \quad (9.2)$$

there results the radial equation

$$\Big\{\frac{d^2}{dr^2}-\frac{L(L+1)}{r^2}+\frac{M}{\hbar^2}(E-2Mc^2-{}^1V_{\text{eff}})\Big\}\,{}^1u^{+}(r)=0, \quad (9.3)$$

with

$${}^1V_{\text{eff}}=-\frac{(E-2Mc^2)^2}{4Mc^2}+\frac{F^2}{4Mc^2}+\frac{m_\pi}{M}(m_\pi c^2) \times\Big[\tfrac{3}{4}\Big(1+\frac{1}{x}\Big)^2\Big(\frac{F}{E+F}\Big)^2-\frac{F}{2(E+F)}\Big], \quad (9.4)$$

where

$$x=\mu r, \quad \mu=m_\pi c/h, \quad (9.5)$$

with m_π standing for the meson mass. In this equation the first term is a relativistic correction to the kinetic energy. The second term gives a repulsive potential which becomes large at short distances and may be described as a soft core in the same terminology which refers to an infinite repulsive potential as a hard core. The last term shows a slight velocity dependence through the entrance of E. The shorter-range part of this term is always repulsive and is qualitatively similar to $F^2/(4Mc^2)$. The longer-range part of the last term is attractive for isotopic triplets ($T=1$) and repulsive for isotopic singlets ($T=0$). The ${}^1V_{\text{eff}}$ is not expected to represent the whole effective potential because of the omission of the effect of nucleon pair formation and other approximations.

For triplets ($S=1$), Eq. (8) can be reduced to a radial equation with an effective potential by means of the substitution

$$\Psi+\Phi=Y_{L,M_L}(\theta,\varphi)\,{}^3u^{+}(r) \times[(E+F)/(E-F)]^{\frac{1}{4}}(E^2-F^2)^{-(1+S_{12})/12}, \quad (10)$$

and in this case

$$\left\{\frac{d^2}{dr^2}-\frac{L(L+1)}{r^2}+\frac{1}{4c^2\hbar^2}(E^2-F^2-4M^2c^4)+\frac{1}{E^2-F^2}\right.$$
$$\times\left[-\tfrac{1}{6}(1+S_{ab})E\frac{d^2F}{dr^2}-\frac{Fd^2F}{2dr^2}+\tfrac{1}{6}(S_{ab}-2)\frac{EdF}{rdr}\right.$$
$$\left.-(1+(\mathbf{L}\cdot\mathbf{S}))\frac{FdF}{rdr}-\tfrac{1}{2}\left(\frac{dF}{dr}\right)^2\right]$$
$$+\frac{(dF/dr)^2}{(E^2-F^2)^2}\left[-\tfrac{1}{2}(1+S_{ab})EF-\tfrac{1}{4}E^2-F^2\right.$$
$$\left.\left.-\frac{1}{36}(1+S_{ab})^2F^2\right]\right\}{}^3u^+(r)=0. \quad (10.1)$$

As in (8), the S_{ab} in this equation is the value of the diagonal element of the operator S_{ab}. The effective potential corresponding to this equation can be inferred by noting that its first two terms are the same as in the singlet case and that the remaining terms are obtained from the factors multiplying ${}^3u^+(r)$ by the factor $-\hbar^2/M$. Among the terms there is the contribution to the spin-orbit interaction

$$V_{(\mathrm{L}\cdot\mathrm{S})}=\frac{\hbar^2(F/r)dF/dr}{M(E^2-F^2)}(\mathbf{L}\cdot\mathbf{S})$$
$$\sim\frac{\hbar^2}{2M^2c^2}\frac{d(F^2/4Mc^2)}{rdr}(\mathbf{L}\cdot\mathbf{S}). \quad (10.2)$$

This term has its origin in the second term in curly braces in (5.4) and can be inferred directly from the latter on noting that

$$E^2-4M^2c^4=\frac{1}{4Mc^2}\left\{E-2Mc^2+\frac{(E-2Mc^2)^2}{4Mc^2}\right\} \quad (10.3)$$

and solving for the first of the two terms in the curly braces. In the asymptotic form applying at long distances according to the last expression in (10.2), this spin-orbit interaction is somewhat reminiscent of the Thomas term which would arise from the repulsive core potential $F^2/(4Mc^2)$. It differs from the extension of the Thomas term to two-body potentials[10] only by its sign. In the region of large r it thus has the sign corresponding to shell model requirements.[11] It may be called an *anti-Thomas term.* The origin of the change of sign is the pseudoscalar rather than scalar character of the field. For distances at which $F^2<E^2$, the sign of the spin-orbit interaction changes. At $E=F$ there is a simple pole in the dependence of $V_{(\mathrm{L}\cdot\mathrm{S})}$ on r and this singularity does not introduce divergence troubles.

A possibility of this kind of difficulty exists in the group of terms in (10.1) which contain the factor $1/(E^2-F^2)^2$ and occur last in the formula. If these terms amount to a repulsive potential close to the singularity, no difficulty arises because at most such a term enforces the vanishing of ${}^3u^+(r)$ at the singularity, in which case it *simulates a hard core potential.* The last term in curly braces in (10.1) can be expressed as

$$\{-\tfrac{1}{4}[E+(1+S_{ab})F]^2+(2/9)(1+S_{ab})^2F^2-F^2\}$$
$$\times\frac{(dF/dr)^2}{(E^2-F^2)^2}. \quad (10.4)$$

A sufficient condition for the corresponding potential to be repulsive is

$$1>(2/9)(1+S_{ab})^2.$$

For $J=L$ this condition is not satisfied because in this case $S_{ab}=2$. However, in this case the first of the two factors in (10.4) is

$$-\tfrac{1}{4}(E+F)(E+5F).$$

If $F>0$, the potential corresponding to (10.4) is therefore always repulsive and there is no difficulty regarding the existence of an acceptable solution. If $F<0$, there is also no difficulty, because at the value $r=r_0$ for which $E+F=0$, the singularity in the potential is of the type $1/(r-r_0)$. This potential energy term changes sign at $r=r_0$. Taken literally, it calls for a special consideration at $r=r_0$, being similar to the Coulomb function at $r=0$ for $L=0$. For the Coulomb function, one has solutions one of which is regular and the other irregular at $r=0$. The irregular solution behaves at small r like

$$\text{const}\times[1+(2r/a)\ln(2r/a)],$$

where a is the Bohr length, the dependence being written here for the case of repulsive fields only. In the case of Coulomb functions, the three-dimensional wave function corresponding to the above form is infinite since it is obtained from the usual Coulomb functions as F_0/r or G_0/r. In the present case, however, r of the Coulomb function is replaced by $r-r_0$ and one has to divide essentially by r_0. There are thus two acceptable linearly independent solutions for $r>r_0$ and for $r<r_0$ which can be matched.

Equations (8) and (10.1) have been written for the uncoupled case; i.e., for $J=L$. If there is coupling between states with different L, two equations instead of one have to be used. These can be written in the form of a matrix containing differential operators, the matrix multiplying a two-component wave function. The diagonal terms of this matrix determine the properties of the radial functions if one uncouples the functions by setting the off-diagonal elements equal to zero. The diagonal terms contain in this sense equivalent po-

[10] G. Breit, Phys. Rev. **51**, 248, 778 (1937); **53**, 153 (1938).
[11] M. G. Mayer, Phys. Rev. **75**, 1969 (1949); **78**, 16 (1950); Haxel, Jensen, and Suess, Phys. Rev. **75**, 1766 (1949); Z. Physik **128**, 295 (1950).

tentials in the decoupled form. These equivalent potentials are reproduced by Eqs. (8) and (10.1), only (7.3) entering in this case. For these potentials one has in the $({}^3L)_{L+1}$ case

$$\begin{aligned} &1+S_{ab}=1-2L/(2L+3)<1,\\ &(2/9)(1+S_{ab})^9<1. \end{aligned} \qquad ((^3L)_{L+1}) \quad (10.5)$$

The expression in (10.4) gives rise to a repulsive potential. No difficulty involving too much attraction at $r=r_0$ arises in this case.

For $({}^3L)_{L-1}$ states,

$$1+S_{ab}=-3/(2L-1), \quad ((^3L)_{L-1}). \qquad (10.6)$$

The absolute value of the right-hand side is less than 1 for $L>1$ and the $1/(r-r_0)^2$ terms are repulsive. For $L=1$, $1+S_{ab}=-3$ and the quantity in curly braces in Eq. (10.4) is

$$\{\ \}=-\tfrac{1}{4}(E-F)(E-5F), \quad ({}^3P_0). \qquad (10.7)$$

If $F<0$, this is negative for all r and one deals with a repulsive potential. If $F>0$, there is a $1/(E-F)$ singularity so that no divergence difficulty arises. The ${}^3P_0$ state corresponds only to $T=1$ so that for it $F>0$. The hard core cases arising under some conditions are likely to need modification in a more complete theory.

IV. PROPERTIES OF THE POTENTIAL

The potential differs from those ordinarily dealt with in the following respects.

(a) It is related to the four-component function Ψ only indirectly. For the triplet case the function ${}^3u^+(r)$ is connected with $\Psi+\Phi$ by means of Eq. (10). For the singlet case the analogous relationship is given by (9.1) and (9.2). The significance of the potential is somewhat different, therefore, from the usual one which refers either to ψ or to Ψ. The entrance of the Coulomb potential in the final equations can also be expected to be somewhat different from the usual one because e^2/r occurs with E in (4.2) and (4.3) while E occurs in various places in (9), (9.4), and (10.1). If e^2/r were treated as part of E, the latter would have to be treated as a function of r and differentiations of $E+e^2/r$ would have to take place. *Tests of charge independence* by comparing *p-p* and *p-n* interactions *may be expected to be affected* by the occurrence of additional terms having their origin in e^2/r.

(b) The effective potentials entering (9) and (10.1) contain some *velocity dependence*, the total energy E entering in various terms. The presence of $1/(E+F)$, $1/(E^2-F^2)$, $1/(E^2-F^2)^2$ in the equations gives strong variations of the effective potential whenever the denominator in one of these expressions is close to zero. Some of this velocity dependence originates in the operators $T_{ab}{}^+$, $T_{ab}{}^-$ which enter $\mathfrak{L}_1$ through (6.3), (6.4), (6.5) and (6.6). The $(\mathbf{L}\cdot\mathbf{S})$ terms originate in $T_{ab}{}^-$ and their velocity dependence needs no special discussion so far as this operator is concerned. The operator $T_{ab}{}^+$ is seen to have a structure related to that of the familiar tensor interaction S_{ab}. It contains however a $\mathbf{p}$ in place of one of the $\mathbf{r}$. A term in the Hamiltonian containing $T_{ab}{}^+$ directly would destroy symmetry under time reversal if it had a real coefficient. Actually $T_{ab}{}^+$ does not occur directly in the Hamiltonian and enters the calculations with a pure imaginary coefficient as in (6.5), (6.6). No contradiction to time-reversal invariance is involved therefore.

The occurrence of different velocity-dependent operators depends on the way in which the result is expressed. Thus, referring to (4.6) and (4.7), there occurs also the operator

$$T_{ab}{}^p=6(\mathbf{s}\cdot\mathbf{p})^2-2s^2p^2, \qquad (11)$$

which enters these formulas for $\mathfrak{L}_1$ and $\mathfrak{L}_2$ through

$$\begin{aligned} &-(\pi^a+\pi^b)\frac{1}{E+F}(\pi^a+\pi^b)\\ &=-\frac{c^2}{E+F}[(4-\tfrac{4}{3}\mathbf{s}^2)p^2-\tfrac{2}{3}T_{ab}{}^p]\\ &\qquad+\frac{c^2\hbar}{i}\frac{r^{-1}dF/dr}{(E+F)^2}[-\tfrac{2}{3}T_{ab}{}^+-\tfrac{2}{3}T_{ab}{}^-\\ &\qquad\qquad+4(1-\tfrac{1}{3}\mathbf{s}^2)(\mathbf{r}\cdot\mathbf{p})+4i\hbar(\mathbf{s}\cdot\mathbf{L})], \end{aligned} \qquad (11.1)$$

$$\begin{aligned} &(\pi^a-\pi^b)\frac{1}{E-F}(\pi^a-\pi^b)\\ &=\frac{c^2}{E-F}[\tfrac{4}{3}\mathbf{s}^2p^2+\tfrac{2}{3}T_{ab}{}^p]+\frac{c^2\hbar r^{-1}dF/dr}{i(E-F)^2}\\ &\qquad\times[\tfrac{2}{3}T_{ab}{}^+-\tfrac{2}{3}T_{ab}{}^--\tfrac{4}{3}\mathbf{s}^2(\mathbf{r}\cdot\mathbf{p})]. \end{aligned}$$

The operators $T_{ab}{}^p$ were mentioned in a preliminary account of this work.[6] They enter the coupled equations between Ψ and Φ, and in this form $T_{ab}{}^+$ is not needed. Velocity-dependent potentials of related types have been considered by Moshinsky[12] and by Marshak and Okubo,[13] although the reasons for their consideration were only those of consistency with general invariance requirements.

It will be noted from (10.2) that $V_{(\mathbf{L}\cdot\mathbf{S})}$ is also velocity dependent, there being the factor E^2-F^2 in its denominator. Locally the velocity dependence is pronounced close to the value of r for which $E^2=F^2$. This does not mean, however, that the approximation by a velocity-independent $V_{(\mathbf{L}\cdot\mathbf{S})}$ term is necessarily poor. Similarly the other terms in the effective potentials which involve $1/(E^2-F^2)$ and $1/(E^2-F^2)^2$, while formally showing large energy dependence close to $E=F$, need not necessarily cause a strong velocity dependence of a phenomenologically dependent potential. The local

[12] M. Moshinsky, Phys. Rev. **106**, 117 (1957).
[13] R. E. Marshak and S. Okubo, Bull. Am. Phys. Soc. Ser. II, **3**, 11 (1958).

velocity dependence caused by such terms is doubtless exaggerated, some rounding off of F at $r=0$ being probable as previously mentioned.

(c) The effective potentials appearing in (9.4) and (10.1) contain a repulsive potential

$$F^2/(4Mc^2),$$

which is large at small r. It provides a contribution to a *soft core* for the interaction. This term increases at small r as $1/r^2$. The increase is not strong enough to outweigh the strong attraction resulting from virtual pair formation which appears in the Lévy[14] type calculation. As pointed out by Brueckner and Watson,[15] there are reasons for believing that the effect of virtual pair formation should be suppressed. In the present note, no attempt is being made to calculate the effect of virtual pair formation. Adopting the general viewpoint of Brueckner and Watson, one would expect, however, that virtual pair formation is suppressed inside the soft core. The soft core may be supposed then to remain a permanent part of the effective potential. This view is, of course, somewhat speculative and conclusions reached by means of it cannot be precise, but if one adopts it an easy explanation of the cores, which are desirable phenomenologically, is seen to be arrived at. According to (4.3), χ^a and χ^b are for large F of the order of $(1/F)$ when expressed in terms of Ψ and Φ. The probability of finding the nucleons close together thus depends on the latter quantities. It has not been shown rigorously that the probability of finding the nucleons within the cores is negligible in this approximation, but it appears possible that this is the case if $F^2/(4Mc^2)$ is large enough; i.e., if the interaction constant f^2 is made sufficiently large. The repulsive potential $F^2/(4Mc^2)$ becomes then similar at small r to a repulsive centrifugal potential with a sufficiently large L, and the functions are then like r^{L+1} at small r. If the probability of finding the nucleons close together is small as a result of the largeness of the core, then virtual pair formation inside the cores may be expected to be small also. If this is the case, the attraction caused by virtual pair formation may be supposed to be small in comparison with the repulsive potential so that the cores would retain their repulsive character.

(d) The effective potential contains the previously referred to *spin-orbit interaction terms* as in Eq. (10.2). For $|F|<E$, these terms have the same sign as indicated by nuclear shell structure theory. In this case the approximate form in (10.2) applies. The coefficient of $(\mathbf{L}\cdot\mathbf{S})$ is therefore negative, making terms with large j fall lower. This sign of $V_{(\mathrm{L}\cdot\mathrm{S})}$ is such that small phase shifts are larger for larger j. The spin-orbit interaction listed in (10.2) changes sign at $E=|F|$. For energies that are not too large, this means for $|F|\cong 2Mc^2$ which corresponds to a repulsive core height $F^2/4Mc^2\cong Mc^2$. The point at which reversal of the sign of $V_{(\mathrm{L}\cdot\mathrm{S})}$ takes place may be expected therefore to lie in the range of values of r which are shielded by the repulsive core, and the sign to be such as corresponds to (10.2) for large r. The absolute magnitude $|V_{(\mathrm{L}\cdot\mathrm{S})}|$ at large r depends on F^2 and is thus proportional to

$$\begin{aligned} |(\tau^a\cdot\tau^b)|^2=1, \quad (\tau^a\cdot\tau^b)=1, \quad T=1, \\ =9, \quad (\tau^a\cdot\tau^b)=-3, \quad T=0. \end{aligned} \tag{11.2}$$

This spin-orbit interaction should therefore be larger in absolute value for triplet-even than for triplet-odd states. The latter correspond to p-p scattering and are the ones for which the presence of $V_{(\mathrm{L}\cdot\mathrm{S})}$ is especially suggested by experimental data. Evidence for spin-orbit interaction in triplet even states does not seem to be clearly established but will be discussed later on.

The spin-orbit interaction term obtained in the approximation considered so far is not likely to be the whole spin-orbit interaction, because the attractive potential which must exist in addition to that calculated above is probably associated with an additional spin-orbit interaction. Since the additional potential is attractive, its parts for larger r give the inverted order indicated by shell theory provided one assumes that the Thomas-term sign holds. This assignment of the sign corresponds in the case of two-particle interactions[10] to the second-order interaction through a scalar field. The virtual pair production may perhaps behave relativistically as though each of the particles were producing a scalar field which is absorbed by the other nucleon. If the sign of the additional $(\mathbf{L}\cdot\mathbf{S})$ terms should be as just supposed, there are two sources of the $(\mathbf{L}\cdot\mathbf{S})$ interaction combining to give effects in the same direction and capable perhaps of making it large enough. If, however, the virtual pair potential turns out to have transformation properties similar to those caused by electromagnetic field interactions, the effect of the potential (10.2) would be opposed and there might be difficulty in obtaining the large $V_{(\mathrm{L}\cdot\mathrm{S})}$ potential suggested by shell theory. A similarity to the transformation property of the electromagnetic field appears unlikely, however. Therefore, in a purely speculative manner, some of the numbers involved will be considered on the supposition that the potential in (10.2) is either helped by an additional Thomas term arising from the pair formation term or is at least not counteracted by additional terms. It will be noted that the range constant of the $(\mathbf{L}\cdot\mathbf{S})$ potential is expected to be $1/(2\mu)$ rather than $1/\mu$. In (10.2) this follows from the occurrence of F^2. For the virtual pair formation potential the range may be expected to be $\frac{1}{2}$ of that of the second-order part. In both cases, the range constant is seen to be in agreement with expectation[16] from dispersion relations. The asymptotic

[14] M. Lévy, Phys. Rev. **88**, 725 (1952); the phenomenologic introduction of hard and soft cores was advocated by H. A. Bethe and H. A. Kramers, Ann Arbor Summer Physics Symposium, 1935 (unpublished), and in connection with high-energy data it was used first by R. Jastrow, Phys. Rev. **79**, 389 (1950); **81**, 165 (1951).

[15] K. Brueckner and K. Watson, Phys. Rev. **92**, 1023 (1953).

[16] Goldberger, Nambu, and Oehme, Ann. Phys. **2**, 226 (1957).

rate of decrease of the spin-orbit potential at $r=\infty$ is seen to be steeper than that of Marshak and Signell[7] and to resemble the potential obtained by Gammel and Thaler[8] somewhat more in this respect, as will be discussed more fully presently.

Regarding, for the present, the fits obtained by Signell and Marshak as furnishing a phenomenologic potential and considering the soft repulsive core in the singlet-even case as furnishing $F^2/(4Mc^2)$, one obtains on reproducing the Gartenhaus[17] potential at $x=\mu r=0.1$ the approximate value

$$F^2/4Mc^2=17(e^{-2x}/x^2)\text{ Mev},\quad (x=0.1,\ \mathrm{T}=1),\quad (12)$$

which when used in (10.2) gives

$$V_{(\mathrm{L}\cdot\mathrm{S})}\cong 0.4\left(\frac{1}{x^2}+\frac{1}{x}\right)e^{-2x}/x^2\text{ Mev}. \quad (12.1)$$

A fit to the spin-orbit potential used by Marshak and Signell in the region $x=0.2$ to 0.4, where the potential is large, can be obtained by changing the 0.4 Mev in (12.1) to 4 Mev. This way of trying to fit the $(\mathbf{L}\cdot\mathbf{S})$ interaction is unsuccessful, giving an interaction of $\sim\frac{1}{10}$ of the empirically required magnitude. The disagreement is not significant, however, for the following reasons. In the first place, the identification of the term $F^2/(4Mc^2)$ by comparison with the singlet-even potential cannot be carried out uniquely. A larger value for $F^2/(4Mc^2)$ would have been obtained had one postulated the existence of an attractive pair-formation potential. Similarly, part of the phenomenologic potential may be caused by the Thomas term of the pair-formation potential. In addition to these causes, the factor $(1-F^2/E^2)^{-1}$ present in (10.2) and neglected in (12) is estimated to be 2.26 for $x=0.3$ and 1.34 at $x=0.4$.

Another view on the comparison is obtained by adjusting $F^2/(4Mc^2)$ to be 10 times larger than the value in (12) and thus more than accounting for the desired spin-orbit interaction. This gives

$$F\cong 800(e^{-x}/x)\text{ Mev},\quad (\mathbf{L}\cdot\mathbf{S}). \quad (13)$$

The Lévy potential[14] in the modification of Blatt and Kalos[18] gives a second-order part,

$$V^{(2)}=(m_\pi c^2/3)(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(g^2/4\pi)\left(\frac{m_\pi}{2M}\right)^2 \times\left[(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+S_{12}\left(1+\frac{3}{x}+\frac{3}{x^2}\right)\right]\frac{e^{-x}}{x}, \quad (13.1)$$

which when compared with the corresponding term contained in (5) gives

$$g^2/4\pi=f^2/\hbar c. \quad (13.2)$$

Employing the low-energy fits of Blatt and Kalos, it appears reasonable to use $(g^2/4\pi)=9$ which gives

$$F\cong 1300(e^{-x}/x)\text{ Mev},\quad (\text{Blatt and Kalos}). \quad (13.3)$$

With this adjustment the proportionality constant in $V_{(\mathrm{L}\cdot\mathrm{S})}$ is more than accounted for, the factor to spare being at least $(1300/800)^2=2.6$.

For large r, the spin-orbit interaction arising as the anti-Thomas term of $F^2/4Mc^2$ is 9 times larger for $\mathrm{T}=0$ than for $\mathrm{T}=1$. There appears to be no phenomenological evidence for considering the whole triplet-even ($\mathrm{T}=0$) interaction to be stronger than the triplet-odd ($\mathrm{T}=1$). However, for $\mathrm{T}=0$ the value of r at which the spin-orbit term changes sign is larger than for $\mathrm{T}=1$. Thus, if one uses the rough phenomenologic value of F in Eq. (13), the sign reversal for $\mathrm{T}=0$ takes place when $e^{-x}/x=2Mc^2/(2400\text{ Mev})=0.78$ which corresponds to $x\cong 0.7$, while for $\mathrm{T}=1$ the sign reversal takes place for $e^{-x}/x=2.35$ which corresponds to $x\cong 0.3$. The much larger x for $\mathrm{T}=0$ should decrease appreciably the effectiveness of the $V_{(\mathrm{L}\cdot\mathrm{S})}$. The fact that $x=0.3$ is close to the usual core radii appears to fit in with the speculative origin of cores in the expressions of the type considered in (10.4). The presence in (10.1) of terms quadratic in S_{ab} indicates that usual calculations may give incorrect conclusions regarding $V_{(\mathrm{L}\cdot\mathrm{S})}$. In the usual considerations the nucleon-spin polarization produced by scattering owes its origin mainly to $V_{(\mathrm{L}\cdot\mathrm{S})}$. The term in S_{ab}^2 produces an effect on phase-shift differences which is neither of a pure S_{ab} nor of a pure $V_{(\mathrm{L}\cdot\mathrm{S})}$ type. In principle these terms make phenomenological conclusions regarding $V_{(\mathrm{L}\cdot\mathrm{S})}$ more difficult, and effects of this type make conclusions arrived at phenomenologically regarding $V_{(\mathrm{L}\cdot\mathrm{S})}$ more questionable than in a theory without such terms. The effect under discussion is caused by the transition from a wave equation containing d/dr linearly and quadratically to one without the linear term and is thus a typical velocity-dependence effect.

V. ERRORS CAUSED BY NEGLECTING THE POPULATION OF NEGATIVE-ENERGY STATES

The calculations in this paper have been made as though there were only two nucleons in the physical system. This procedure neglects the fact that negative-energy states of protons must be considered as occupied by protons forming a Dirac sea of particles in negative-energy states and that a similar Dirac sea must be pictured for neutrons. The problem is somewhat similar to that of considering the interaction of two electrons in the presence of two closed electron shells. Some of the essential features of the problem are manageable with the aid of the Bethe-Salpeter[19] equation, which should furnish relativistic answers provided one neglects the emission of real pions. The work involved in reductions

[7] S. Gartenhaus, Phys. Rev. **100**, 900 (1955).
[8] J. M. Blatt and M. H. Kalos, Phys. Rev. **92**, 1563 (1953).
[19] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).

of the Bethe-Salpeter equation is laborious in the case of the nucleon-nucleon problem, however. Since the potential obtained provides repulsive cores, a relativistic treatment does not appear to be imperative for bombarding energies below the pion production threshold. Nonrelativistically there is a similarity to the ordinary spectroscopic treatment of electron shells.

The problem of taking into account the filled states is simple if the states are known. In the present case, however, the states occupied are themselves determined by the interaction. This circumstance can be seen simply in the case of the interaction of an electron with a proton. In the limit of infinite proton mass the electron moves in a time-independent field. One can consider the problem in terms of plane-wave states which will be denoted for brevity by

$$u^m(\mathbf{r},\mu). \tag{14}$$

Here m labels the state, μ is the Dirac spin index, and $\mathbf{r}$ is the electron's position vector. The vacuum polarization is well known to be directly obtainable from the distorted wave functions

$$v^m(\mathbf{r},\mu), \tag{14.1}$$

the solutions of Dirac's equation in the field of a point charge. The functions v^m rather than the plane wave functions u^m are in this limit the simpler to deal with. They can similarly be used for the description of a state with total charge $-e$, not counting the proton charge $+e$. Writing in the Dirac notation

$$v^m(\mathbf{r},\mu)\equiv|m\rangle, \tag{14.2}$$

the antisymmetric wave function for which all negative-energy states m_1^-, m_2^-, $\cdots$ as well as a positive-energy state m_0^+ are filled may be represented as

$$(\ |m_0^+\rangle;\ |m_1^-\rangle,\ |m_2^-\rangle,\ \cdots)\equiv\|m_0^+\rangle\rangle, \tag{14.3}$$

the functions separated by commas in parentheses being used for the formation of a normalized Slater determinant. The state $\|m_0^+\rangle\rangle$ describes a condition in which one electron is physically present in state m_0^+ and for which there exists, besides, the vacuum polarization charge. The interaction of the vacuum polarization charge with $|m_0^+\rangle$ is not taken into account in (14.3) and radiative effects are not considered in it either. The correctness of the above statement may be seen if one calculates contributions to particle density, which are then seen to arise from the states $|m_0^+\rangle$, $|m_1^-\rangle$, $|m_2^-\rangle$ separately. The result of the consideration after taking into account the symmetry of density contributions for positive- and negative-energy states for the field-free functions is

$$\rho=\rho(m_0^+)+\rho(\text{vac. pol.}) \tag{14.4}$$

For many problems one could, in this limiting case, forget the population of the negative-energy states and use only the distorted functions v^m. It would be wrong to remove from $|m_0^+\rangle$ its projection on the subspace of all the plane wave states u^m having negative energies. The error in such a procedure lies in the fact that the negative-energy states u^m are themselves distorted by the proton field. If the proton field is switched on adiabatically, the orthonormal set u^m changes to the orthonormal set v^m which makes the direct construction of the Slater determinant (14.3) possible. It is essential for the simplicity of the result that the v^m are orthonormal.

For a finite but large proton mass, the problem of mass motion effects has been treated for bound states by means of the Bethe-Salpeter equation by Salpeter.[20] This treatment is characterized by its essential symmetry in space-time. In the stage called by Salpeter "the instantaneous interaction," this method of making the calculation has the appearance of yielding a very different result from that which would be obtained by considering just two particles. In this "instantaneous approximation" there appears a projection operator

$$\Lambda_+{}^a\Lambda_+{}^b-\Lambda_-{}^a\Lambda_-{}^b.$$

Here $\Lambda_+{}^a$ is the projection operator on the positive-energy subspace for particle a with the plane wave rather than the v^m meaning of energy states. At this stage, in the plane wave sense, the whole wave function contains only the components $\varphi_{++}(\mathbf{p})$ and $\varphi_{--}(\mathbf{p})$ in the center-of-mass system.

Were one to use the instantaneous interaction approximation in the above sense for the nucleon-nucleon interaction problem, the results would be entirely different from those obtained in the preceding sections. The equation in momentum space can then be transformed to coordinate space only in a symbolic sense, there being no simple coordinate space indicated by the problem itself. The transformation to coordinate space carried out by the usual Fourier transform procedure leads to a complicated integral equation. Approximating in it quantities such as $(M^2c^4+p^2)^{\frac{1}{2}}$ by the first two terms and thus introducing usual differential operator forms, one does not obtain the interaction forms which have followed from the consideration of two particles without pair (hole) theory, there being some changes in the sign of the effective coordinate space interactions. However, the "instantaneous interaction" approximation does not lead directly to the static wave function solutions (14.2) or (14.3) in the proton-electron problem. To obtain these solutions it is necessary to include the effect of the $G_{cc}{}^{(2)}$ diagram of Salpeter's paper,[20] as shown by Salpeter. At this stage of the calculation he obtains also a correction term for the motion of the proton which is not obvious from more elementary considerations. The correction term is small in the hydrogen problem. The fact that $G_{cc}{}^{(2)}$ has to be included shows that the "instantaneous interaction" approximation cannot be used for the derivation of the effective interaction potential in the nucleon-nucleon problem unless it is

[20] E. E. Salpeter, Phys. Rev. **87**, 328 (1952).

supplemented by the calculation of higher approximations. These would have to include terms in $\Lambda_+{}^a\Lambda_+{}^b$, $\Lambda_+{}^a\Lambda_-{}^b$, $\Lambda_-{}^a\Lambda_-{}^b$ in addition to the $\Lambda_-{}^a\Lambda_+{}^b$ combination in Salpeter's Eq. (34). The results of such calculations would give terms which would have to be supplemented by further terms in the expansion in powers of the interaction constant. The value of such work would not be clear, however, because the starting point of the calculation involves approximations of uncertain final significance. Since, on the other hand, the considerations of Eqs. (14) to (14.4) show that in the limit in which one mass becomes very great the elimination of the plane wave negative-energy parts is incorrect and since the Bethe-Salpeter equation leads to a similar result in the hydrogen problem if terms beyond the "instantaneous approximation" are included, there appears so far no special reason for doubting the *qualitative* aspects of conclusions obtained neglecting the difference between the interaction using pair theory and the interaction in the absence of the population of negative-energy states. To be sure, the pseudoscalar interaction differs considerably from the electromagnetic one with respect to the possibility of obtaining a static interaction. Pseudoscalar coupling between particles of unequal masses M, m gives no interaction in the relativistic limit for $M/m\to\infty$ if the coupling constant is kept fixed. But one obtains a nonvanishing interaction if F^2/M is kept at a constant value in the limit of $M\to\infty$. The wave function distortion described for the electromagnetic case in connection with Eqs. (14), (14.4) is present also for the PS interaction with the above understanding. The limit just discussed does not correspond to actual conditions and does not directly prove the reality of the terms derived by the naive procedure of the preceding section. It indicates only that the difference between the pseudoscalar and electromagnetic interaction does not destroy the similarity of treatment of $E<0$ states in the limit of one mass becoming infinite, provided the limit is taken in the manner described.

It is believed, therefore, that the omission of the consideration of the occupation of $E<0$ states does not prevent the employment of Eq. (4) from having a meaning as furnishing an interaction which can be improved on through the inclusion of virtual pair formation, field quantization, and other omitted effects. The occurrence of repulsive cores and of the spin-orbit interaction is in qualitative agreement with experiment. The correspondence of empirical indications with suggestions arising from the model is difficult to discuss quantitatively, not only because of the incompleteness of the theory but also because it is not clear whether such fits to experimental data as have been obtained[7,9] by means of potentials are the only ones. The potentials used are not the same and they differ in different editions of the same work. Thus the potential used in the work on deuteron photodisintegration by de Swart, Bilhorn, and Marshak[21] does not contain a spin-orbit interaction for triplet-even states, this change having given better agreement with the photodisintegration data. A fit of this type is described in a preprint of the longer paper by Marshak and Signell.[7] The possibility of using no spin-orbit interaction in the triplet-even states (T=0) appears at first sight to disagree with (10.2) because according to (11.3) it should be stronger for triplet even states. This conclusion does not necessarily follow, however. The reversal of sign for the spin-orbit potential occurs at larger r when $\tau=0$ and it may therefore become effectively weaker, even though for sufficiently large r it might be stronger. According to the estimates at the end of the preceding section, the reversal of sign might occur at $x=\mu r\cong 0.7$ for $\tau=0$ and at a much smaller value for $\tau=1$. If the sign reversal may be considered as taking place at these large values, the spin-orbit interaction appears to be in qualitative agreement with the phenomenological analyses. Thus, according to Fig. 7 of Signell and Marshak, the phenomenologic $(\mathbf{L}\cdot\mathbf{S})$ potentials of the Rochester as well as the Los Alamos workers are much smaller for $x>0.7$ than for $x<0.7$. The centrifugal barrier for $L=1$ at $x=0.7$ is ~ 90 Mev and this region is thus well accessible to the particles in the experimental range which has been mainly considered in making the phenomenological fits. The Gammel-Thaler spin-orbit interaction potential is stronger in odd than in even states by a factor of ~ 1.4. This difference in the interactions is in the direction just discussed and may be accountable for by the sign reversal just discussed.

According to the considerations in the present paper as well as the dispersion theoretic considerations of Goldberger, Nambu, and Oehme,[16] the asymptotic form of $V_{(L\cdot S)}$ must be $e^{-2\mu r}$. The potentials compare as follows regarding the exponential factors determining their asymptotic behavior:

Marshak-Signell	$\exp(-0.93r)$,
Gammel-Thaler	$\exp(-3.7r)$,
Theory	$\exp(-1.46r)$,

where r is expressed in fermis (1 fermi$\equiv 1f\equiv 10^{-13}$ cm). The theoretical form has a meaning only asymptotically and the above comparison may not be too meaningful therefore. It appears to be in reasonable agreement with the empirical indications, however. The Signell-Marshak fit is concerned with adjustment of ^3P wave effects more than with effects of higher L. The Gammel-Thaler fit, being concerned with higher energies, is relatively more sensitive to higher L. The short-range character of the spin-orbit interaction in their fit presumably has been caused by the desirability to have small effects of $V_{(L\cdot S)}$ on the F waves and has considerable chance of being right. The fact that the

[21] de Swart, Bilhorn, and Marshak, Bull. Am. Phys. Soc. Ser. II, 3, 48 (1958).

theoretical dependence is intermediate between the two empirical ones appears to be not accidental therefore. The adjustment of phenomenologic spin-orbit potentials depends on the central and tensor potentials used with them and is therefore not unique even apart from the question of choice of phase shifts. The very small value of the range of the $V_{(L\cdot S)}$ potential of Gammel and Thaler may accordingly have a number of explanations without implying a real disagreement with the theoretical asymptotic form. There is of course a slight chance that the leading term in the dispersion theoretic form cancels or is accidentally small and that the Gammel-Thaler result should be interpreted as $\exp(-2.92r)$. There is also the possibility that the $V_{(L\cdot S)}$ is not a truly meaningful concept. This view is partially supported by there being a difference in these potentials for even and odd states in the formulas of the present paper. Some indication of this is also furnished by the occurrence of S_{ab}^2 terms in Eq. (10.1). A physically complete theory may make the literal employment of $V_{(L\cdot S)}$ even less meaningful. The introduction of $V_{(L\cdot S)}$ has produced useful effects mainly on P waves. It is possible therefore that its physical character has more to do with the production of P-wave phase shifts, which produce effects somewhat like those caused by $(\mathbf{L}\cdot\mathbf{S})$ but have no such interpretation in other states. The experimental evidence has not been shown to point to anything more specific regarding $V_{(L\cdot S)}$ so far, although at least a slight deviation from the pure tensor pattern of phase shifts for ^3F terms is probably necessary.

The fact that the Gammel-Thaler fit gives a wholly repulsive $^1V^-$ while their $^1V^+$ has an attractive part appears to be not in contradiction with the larger value of F^2 for $\mathrm{T}=0$. Although the Gartenhaus potential has a deep attractive $^1V^-$ part, the second paper of Signell and Marshak shows that a bound $^1P_1$ state is the result of this attraction, and it is therefore necessary to modify the Gartenhaus potential as already modified by the Case-Pais $(\mathbf{L}\cdot\mathbf{S})$ term by the inclusion of an additional repulsion such as a repulsive core.

Evidence from the $^3V_c^+$, $^3V_c^-$ (c here stands for central) appears undecisive. The Rochester and the Los Alamos groups agree in giving mostly repulsion for $^3V_c^-$ which corresponds to $\mathrm{T}=1$. Strong repulsion in the model of the present paper is characteristic of $\mathrm{T}=0$, however. Figure 3 of the second paper of Signell and Marshak[7] shows that a strong repulsion may be used for $^3V_c^+(\mathrm{T}=0)$, the upward trend beginning at about $x=0.5$ and increasing toward small x. For $^3V^-$ Gammel and Thaler used a hard core radius with $x\approx 0.3$ and no further interaction. There is thus no apparent contradiction with the expectation of a greater value of the repulsive $F^2/4Mc^2$ term in the $\mathrm{T}=0$ state. This comparison is complicated by the presence of spin dependence such as in (13.1) and the strong effect which the tensor interaction may have on the binding energy of the deuteron. There appears therefore no special reason for doubting the potential calculated here on the basis of the comparison of $^3V_c^+$ and $^3V_c^-$. This comparison may hardly be considered as giving special support to the view, however, the number of possible adjustments being too great to be convincing.

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