$E2$ ($\delta=+0.05$) if the 41-kev radiation is pure $E1$. If $M2$ mixing is allowed in the 41-kev transition, then reasonable fits can be obtained with two parameters. If, for example, the latter were 60% $M2$, the 140-kev transition would have to contain 2% $E2$ in order to fit the angular-correlation data.

The observed attenuation of the anisotropy of the 740–181 kev gamma-ray cascade as a function of the magnetic field applied perpendicularly to the plane of the gamma rays is shown in Fig. 2. The experimental curve has been fitted by a theoretical curve which takes account of the solid angle of the detectors. The fit is obtained by relative adjustment of the horizontal scales. The result for the gyromagnetic ratio of the 3.5×10^{-9}-sec state is $g=+1.5\pm0.2$ nuclear units. The sign of g was obtained from measurements at 135° and 225° with a field of 10 000 oersteds.

The measured gyromagnetic ratio and the assignment of spin $\frac{5}{2}$ lead to a magnetic moment $\mu=3.8\pm0.5$ nuclear magnetons. This value lies just below the upper Schmidt limit, suggesting positive parity for the state if the spin is indeed $\frac{5}{2}$. As our proposed spin scheme assigns negative parity, we would like to see more precise internal conversion coefficients for the 41-, 181-, and 740-kev gamma rays. This would provide a better test of the spin and parity assignments. If the $\frac{5}{2}-$ assignment for the 181-kev level is correct, the magnetic moment is surprisingly large.

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Polarization of Bremsstrahlung from Polarized Electrons

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We analyze the polarization of bremsstrahlung emitted from polarized electrons, in the Born approximation. Spins of the final electron are summed over. Differential cross sections are given, and are integrated over final electron directions. The results are valid for all electron energies, subject only to the limitations on the validity of the Born approximation. Screening is taken into account by using an exponential atomic form factor. Methods for relativistic spin description are discussed.

I. INTRODUCTION

BREIT[1] seems to have been the first to point out that polarized electrons can produce circularly polarized bremsstrahlung. This subject, however, became of interest only recently when it was discovered that electrons from β decay[2] and $\mu-e$ decay[3] are polarized. One of the first experiments to determine the polarization of these electrons[4] consisted in measuring the circular polarization of the bremsstrahlung emitted. The evaluation of this experiment was based on results of McVoy[5] who calculated in Born approximation the bremsstrahlung circular polarization for longitudinally polarized electrons and forward emission of the radiation. Calculations of this type have subsequently been extended: differential cross sections were published by Claesson[6] and Böbel,[7] and integrated cross sections for high-energy electrons by the authors[8] and by Olsen and Maximon[9]; the last reference is the only work which goes beyond the Born approximation and takes Coulomb corrections into account. In the following, we derive the differential cross section of bremsstrahlung for polarized initial electrons and polarized photons (Sec. III), and integrate it over final electron directions (Sec. IV). In Sec. II, we give a short summary of the methods used in the literature for describing the spin of relativistic particles, as well as a discussion of projection operators. In Sec. V, the polarizations of the emitted radiation are analyzed. This includes the linear polarization, which has been discussed previously by May[10] and by Gluckstern and Hull.[11] Whereas May limited his investigation to energetic electrons, the latter authors considered all electron energies, but introduced the effect of screening in an intuitive way which is reasonable for energies of a few Mev only. Our treatment gives a correct account of the screening, using an exponential atomic form factor.

As to experimental applications of our results, they might, on the one hand, provide a more accurate theory for use in the experiment of Goldhaber *et al.*[4] than does that of McVoy; on the other hand, they could represent a means for determining transverse electron polarization

[1] Gluckstern, Hull, and Breit, Phys. Rev. **90**, 1026 (1953).
[2] Frauenfelder, Bobone, von Goeler, Levine, Lewis, Peacock, Rossi, and De Pasquali, Phys. Rev. **106**, 386 (1957).
[3] Culligan, Frank, Holt, Kluyver, and Massam, Nature **180**, 751 (1957).
[4] Goldhaber, Grodzins, and Sunyar, Phys. Rev. **106**, 826 (1957).
[5] K. W. McVoy, Phys. Rev. **106**, 828 (1957); K. W. McVoy and F. J. Dyson, Phys. Rev. **106**, 1360 (1957).
[6] A. Claesson, Arkiv Fysik **12**, 569 (1957).
[7] G. Böbel, Nuovo cimento **6**, 1241 (1957).
[8] C. Fronsdal and H. Überall, Nuovo cimento (to be published).
[9] H. Olsen and L. C. Maximon, Proceedings of the Physics Seminar in Trondheim, 1958 (unpublished), Vol. 2.
[10] M. M. May, Phys. Rev. **84**, 265 (1951).
[11] R. L. Gluckstern and M. H. Hull, Phys. Rev. **90**, 1030 (1953)

by a polarization measurement of the radiation. Such a measurement of transverse spin components of electrons from decay processes[12] seems to be of interest as it could help to determine the exact form of Fermi couplings, detect terms which violate time-reversal invariance, and establish whether the electron polarization is complete, as is predicted by the two-component theory.

II. SPIN DEFINITION AND POLARIZATION OPERATORS

The state of a single Dirac particle with given momentum is completely specified by the eigenvalues of two commuting operators (in the following, we use a Euclidean metric, Hermitean γ matrices, $\hbar=c=1$, and the convention $a_4=-ia_0$):

$$(i/m)p\psi=\psi, \tag{1}$$

$$i\gamma_5 s\psi=\psi, \tag{2}$$

where s^μ, a covariant generalization of the nonrelativistic spin vector, represents the spin of the particle,[13] and $p\equiv p^\mu\gamma_\mu$. The choice of the vector s^μ is somewhat arbitrary, corresponding to the various possible spin directions, and is restricted only by the conditions

$$s^2=1, \tag{3}$$

$$s^\mu p_\mu=0, \tag{4}$$

which follow from (2) and from the commutativity of the two operators, respectively.

Besides this covariant way of describing the spin, there is another method which has been applied extensively in the literature,[14] namely the use of the spin in the rest system of the electron,

$$(s^\mu)_{\text{rest}}=(\boldsymbol{\zeta},0).$$

Condition (4) in the covariant method permits the introduction of a 3-vector $\mathbf{s}$ by

$$s^\mu=(\mathbf{s},\,-\mathbf{s}\cdot\mathbf{p}/E),$$

where, because of (3), $\mathbf{s}$ is not normalized to 1. To obtain $\boldsymbol{\zeta}$ from $\mathbf{s}$, one simply performs a Lorentz transformation connecting s^μ to $(s^\mu)_{\text{rest}}$, with the result

$$\boldsymbol{\zeta}=\mathbf{s}-\frac{\mathbf{p}\cdot\mathbf{s}}{E(E+m)}\mathbf{p}. \tag{5}$$

This can be written particularly simply in terms of the spin components perpendicular and parallel to $\mathbf{p}$:

$$\zeta_t=s_t,\quad \zeta_l=(m/E)s_l, \tag{6}$$

the subscripts meaning transverse and longitudinal, respectively. From (3) it follows that $\boldsymbol{\zeta}$ is normalized to 1.

Feynman showed[13] that, in the rest system, the spin direction defined by (2) reduces to the nonrelativistic spin vector. An alternative proof that Eq. (2) defines the spin may proceed as follows: generally, the spin is the eigenvalue of the operator

$$S=(1/4i)s^{\mu\nu}\gamma_\mu\gamma_\nu, \tag{7}$$

where $s^{\mu\nu}$ is the antisymmetric spin tensor.

Using (1), we can write

$$S\psi=(1/4m)p^\rho s^{\mu\nu}\gamma_\mu\gamma_\nu\gamma_\rho\psi. \tag{8}$$

As there exist simultaneous eigenfunctions of energy and spin, the two operators p and $s^{\mu\nu}\gamma_\mu\gamma_\nu$ commute, and only the part of $\gamma_\mu\gamma_\nu\gamma_\rho$ which is completely antisymmetric in (μ,ν,ρ) actually contributes to (8). However, this part is nothing else than the pseudovector $\gamma_5\gamma_\lambda$, where (μ,ν,ρ,λ) is an even permutation of (1,2,3,4).

Hence, S may be written

$$S=\tfrac{1}{2}i\gamma_5 s^\mu\gamma_\mu, \tag{9}$$

where $s^\lambda=(1/2mi)p^\rho s^{\mu\nu}\epsilon_{\rho\mu\nu\lambda}$ is a pseudovector which satisfies Eqs. (3) and (4). Thus, Eq. (2) does indeed specify the spin. In particular, in the rest system, the spin tensor has the form

$$s^{\mu\nu}=\delta_i{}^\mu\delta_j{}^\nu-\delta_j{}^\mu\delta_i{}^\nu,$$

where (i,j,k) is an even permutation of (1,2,3), if the spin points along the k axis. In this case, s^λ reduces to a unit space vector in the k direction.

Squared matrix elements between states of specified energy and spin are evaluated with the aid of energy-spin projection operators:

$$|M|^2=\mathrm{Tr}O^\dagger P_f O P_0,$$
$$P_i(p_i,s_i)=\psi_i(p_i,s_i)\bar{\psi}_i(p_i,s_i).$$

Because of the equivalence of (7) and (9), P_i may be written as[15]:

$$P_i=\frac{i}{2m}(p-im)\left(\tfrac{1}{2}+\frac{1}{4i}s^{\mu\nu}\gamma_\mu\gamma_\nu\right), \tag{10}$$

or

$$P_i=\frac{i}{2m}(p-im)\tfrac{1}{2}(1+i\gamma_5 s). \tag{10a}$$

With the help of (5), P_i can be expressed by the vector $\boldsymbol{\zeta}$, but this leads to a more complicated expression.[16] We preferred, nevertheless, to give our final

[12] In β decay: Jackson, Treiman, and Wyld, Phys. Rev. **106**, 517 (1957); G. Györgyi and H. Überall, Nuclear Phys. **5**, 405 (1958). In μ decay: H. Überall, Nuovo cimento **6**, 376 (1957); T. Kinoshita and A. Sirlin, Phys. Rev. **108**, 844 (1957).

[13] R. P. Feynman, California Institute of Technology lecture notes on Quantum Electrodynamics, 1953 (unpublished), p. 51; L. Michel and A. S. Wightman, Phys. Rev. **98**, 1190 (1955); C. Bouchiat and L. Michel, Compt. rend. **243**, 642 (1956); Nuclear Phys. **5**, 416 (1958).

[14] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949), second edition, p. 76; H. A. Tolhoek, Revs. Modern Phys. **28**, 277 (1956); G. W. Ford and C. J. Mullin, Phys. Rev. **108**, 477 (1957), for electrodynamic processes; reference 12 for β decay and μ-meson decay.

[15] C. Fronsdal, University of California thesis, 1957 (unpublished); and Nuovo cimento (to be published). See also Michel and Wightman, and Bouchiat and Michel, reference 13.

[16] F. W. Lipps and H. A. Tolhoek, Physica **20**, 85 (1954).

results in terms of $\boldsymbol{\zeta}$, in order to agree with the literature on production of polarized electrons.

The emitted photons will in general not be in a pure state, especially as the final electron variables are summed over. The polarization state of the radiation is then analyzed by a density matrix

$$\rho=\tfrac{1}{2}(1+\boldsymbol{\xi}\cdot\boldsymbol{\sigma}),$$

where the components of $\boldsymbol{\sigma}$ are the Pauli matrices, rather than by a vector potential $\mathbf{A}$. The three components of $\boldsymbol{\xi}$, called Stokes' parameters, determine the polarization completely; they can be taken as the probability of circular polarization, and two probabilities of linear polarization whose reference planes form an azimuth of $\pi/4$ [see Tolhoek, reference 14]. Alternatively, the polarization state could be described by an unpolarized and an elliptically polarized component, which is obtained by a diagonalization of the density matrix.

III. DIFFERENTIAL CROSS SECTIONS

We derived the differential cross sections by using the trace method described in Sec. II; the spins of the final electrons were summed over. The following variables are used:

Initial electron energy and momentum: E_1, $\mathbf{p}_1$;
Final electron energy and momentum: E_2, $\mathbf{p}_2$;
Photon momentum: $\mathbf{k}$, with $k=E_1-E_2$;
Recoil momentum of nucleus: $\mathbf{q}=\mathbf{p}_2-\mathbf{p}_1+\mathbf{k}$;
Nuclear charge: Z; initial electron spin: $\mathbf{s}$ or $\boldsymbol{\zeta}$;
Angle $(\mathbf{k},\mathbf{p}_1)$: θ_1; $\Delta_1=E_1-p_1\cos\theta_1$;
Angle $(\mathbf{k},\mathbf{p}_2)$: θ_2; $\Delta_2=E_2-p_2\cos\theta_2$;
Azimuth between $(\mathbf{p}_1,\mathbf{k})$ plane and $(\mathbf{p}_1,\boldsymbol{\xi})$ plane (i.e., the angle between $\mathbf{p}_1\times\boldsymbol{\zeta}$ and $\mathbf{p}_1\times\mathbf{k}$): ψ_1.

All energies and momenta are measured in units of mc^2 and mc. Exponential screening was assumed, with a screening radius $\beta=111Z^{-\frac{1}{3}}$ (see Schiff[17]).

The unit vector in the direction of the vector potential $\mathbf{A}$ of the photon is called $\mathbf{e}$. If it is set perpendicular to $\mathbf{k}$ and assumed real (linear polarization), we obtain, starting from the matrix element of bremsstrahlung given, e.g., in Heitler,[18] a cross section

$$\sigma=\frac{Z^2e^6}{(2\pi)^2}\frac{p_2}{p_1}\frac{dk}{k}\frac{d\Omega_1 d\Omega_2}{(\beta^{-2}+q^2)^2}\left\{\frac{4E_1^2-q^2}{\Delta_2^2}(\mathbf{p}_2\cdot\mathbf{e})^2+\frac{4E_2^2-q^2}{\Delta_1^2}(\mathbf{p}_1\cdot\mathbf{e})^2-2\frac{4E_1E_2-q^2}{\Delta_1\Delta_2}(\mathbf{p}_1\cdot\mathbf{e})(\mathbf{p}_2\cdot\mathbf{e})+k^2\left(\frac{q^2}{\Delta_1\Delta_2}+2-\frac{\Delta_2}{\Delta_1}-\frac{\Delta_1}{\Delta_2}\right)\right\}. \quad (11)$$

[17] L. I. Schiff, Phys. Rev. 83, 252 (1951).
[18] W. Heitler, *The Quantum Theory of Radiation* (Clarendon Press, Oxford, 1954), third edition, p. 243. See also Feynman,[13] p. 95.

This formula was given by May[10] and by Gluckstern *et al.*[19] It reduces to the Bethe-Heitler cross section σ_{BH} if summed over two vectors $\mathbf{e}$ perpendicular to each other. Even though in the evaluation of the traces, initial electron spin projectors were used, (11) does not depend on the electron polarization. Thus the linear polarization of bremsstrahlung is the same for polarized and unpolarized initial electrons.

If $\mathbf{e}$ is chosen parallel ($\mathbf{e}=\mathbf{e}_{\mathrm{I}}$) or perpendicular ($\mathbf{e}=\mathbf{e}_{\mathrm{II}}$) to $\mathbf{p}_1\times\mathbf{k}$, call the corresponding cross sections $\sigma_{\perp}$ and $\sigma_{\parallel}$. Then the linear polarization is given by

$$P_{\mathrm{lin}}^{(1)}=(\sigma_{\perp}-\sigma_{\parallel})/\sigma_{\mathrm{BH}}. \quad (12)$$

Another independent type of linear polarization, $P_{\mathrm{lin}}^{(2)}$, is obtained if the two basis vectors $\mathbf{e}$ are taken as $(1/\sqrt{2})(\mathbf{e}_{\mathrm{I}}\pm\mathbf{e}_{\mathrm{II}})$. Then $P_{\mathrm{lin}}^{(1)}$ and $P_{\mathrm{lin}}^{(2)}$ together with the circular polarization P_c describe the polarization state of the radiation uniquely, as pointed out in Sec. II.

The probability of circular polarization is obtained by setting $\mathbf{e}=(1/\sqrt{2})(\mathbf{e}_{\mathrm{I}}+i\delta\mathbf{e}_{\mathrm{II}})$ in the matrix element, where $\delta=+1$ (-1) for right- (left-) handed photon polarization; $\mathbf{e}_{\mathrm{I}}$, $\mathbf{e}_{\mathrm{II}}$, and $\mathbf{k}$ must be chosen to form a right-handed reference system. We obtain

$$\sigma=\tfrac{1}{2}\sigma_{\mathrm{BH}}+\tfrac{1}{2}\delta\sigma_c, \quad (13)$$

$$\sigma_c=-\frac{Z^2e^6}{(2\pi)^2}\frac{p_2}{p_1}\frac{dk}{k}\frac{d\Omega_1 d\Omega_2}{(\beta^{-2}+q^2)^2}\left\{(\mathbf{s}\cdot\mathbf{p}_1)\frac{k}{E_1}\times\left[\frac{4E_1^2-q^2}{\Delta_2^2}-\frac{4E_2^2-q^2}{\Delta_1^2}+4(E_1+E_2)\left(\frac{1}{\Delta_1}-\frac{1}{\Delta_2}\right)\right]+(\mathbf{s}\cdot\mathbf{k})\left[\frac{4E_1^2-q^2}{\Delta_2^2}+\frac{4E_2^2-q^2}{\Delta_1^2}-2\frac{4E_1E_2+q^2}{\Delta_1\Delta_2}\right]-\mathbf{s}\cdot(\mathbf{p}_1+\mathbf{p}_2+\mathbf{k})\frac{2k}{\Delta_2}\left(\frac{2E_1-\Delta_1}{\Delta_2}-\frac{2E_2-\Delta_2}{\Delta_1}\right)\right\}. \quad (13a)$$

The circular polarization is given by

$$P_c=\sigma_c/\sigma_{\mathrm{BH}}. \quad (14)$$

We see that it is proportional to the degree of electron polarization, $|\boldsymbol{\zeta}|$. Hence, as predicted by Gluckstern *et al.*,[1] unpolarized electrons cannot emit circularly polarized bremsstrahlung.

IV. INTEGRATED CROSS SECTIONS

In practice, outgoing electrons are not observed, and an integration of (11) and (13) over $d\Omega_2$, the final electron directions, is necessary. The only remaining variables are then $\mathbf{p}_1$, $\mathbf{k}$, $\boldsymbol{\zeta}$, and δ. In (13), σ_c is multiplied by

[19] Gluckstern, Hull, and Breit, Science 114, 480 (1951).

δ and must therefore be a pseudoscalar of the form

$$\sigma_c = a_l \frac{\boldsymbol{\zeta}\cdot\mathbf{p}_1}{p_1} + a_t \frac{\boldsymbol{\zeta}\cdot\mathbf{n}\times\mathbf{p}_1}{|\mathbf{n}\times\mathbf{p}_1|} \equiv \sigma_{c,\,l} + \sigma_{c,\,t},$$

with $\mathbf{n} = \mathbf{p}_1 \times \mathbf{k}$. The first term is proportional to the longitudinal spin component ζ_l of the electron, the second one to the transverse component ζ_t projected into the $(\mathbf{p}_1, \mathbf{k})$ plane, i.e., $\zeta_t \cos\psi_1$.

The integration of (11) and (13) is greatly facilitated by using the relation

$$\frac{1}{(\beta^{-2}+q^2)^2} = -\frac{\partial}{\partial(\beta^{-2})}\left(\frac{1}{\beta^{-2}+q^2}\right).$$

In this way, the following expression is obtained:

$$\sigma_i = \frac{Z^2e^6}{2\pi}\frac{dk}{k}\frac{d\Omega_1}{p_1}F_i\frac{\partial}{\partial(\beta^{-2})}\left\{\frac{2p_2 g}{c}C_1{}^i + \frac{L_1}{c^{\frac{3}{2}}}(cC_2{}^i - bC_1{}^i) + \frac{L_2}{a^{\frac{3}{2}}}(aC_3{}^i - bC_4{}^i) - \beta^{-2}C^iL_3\right\} \quad (15)$$

(the actual differentiation with respect to β^{-2} is trivial). Here, σ_i stands for σ_{BH}, $\sigma_\perp$, $\sigma_{c,\,l}$, and $\sigma_{c,\,t}$ where $\sigma_\perp$ is again the cross section for emission of radiation polarized in the direction $\mathbf{n}$, and σ_c was decomposed according to the longitudinal and transverse components of $\boldsymbol{\zeta}$, as indicated above.

The notations are as follows:

$$L_1 = \ln\left(\frac{2\Delta_1(E_1E_2-1)+\beta^{-2}E_2+p_2c^{\frac{1}{2}}}{2\Delta_1(E_1E_2-1)+\beta^{-2}E_2-p_2c^{\frac{1}{2}}}\right),$$

$$L_2 = \ln\left(\frac{\beta^{-2}+(T+p_2)^2}{\beta^{-2}+(T-p_2)^2}\right), \quad (15a)$$

$$L_3 = \ln\left(\frac{E_2+p_2}{E_2-p_2}\right);$$

$$a = 4T^2,$$
$$b = 4[\tfrac{1}{2}g(E_2-\Delta_1) - E_2p_1{}^2\sin^2\theta_1],$$
$$c = g^2 + 4p_1{}^2\sin^2\theta_1, \quad (15b)$$
$$\mathbf{T} = \mathbf{p}_1 - \mathbf{k},$$
$$g = \beta^{-2} + 2(E_1\Delta_1 - 1).$$

The individual coefficients are

(1) for σ_{BH}:

$$F = 1,$$
$$C_1 = 4E_1{}^2 + \beta^{-2},$$
$$C_2 = 2\Delta_1{}^{-1}[-4E_1E_2 + k^2\Delta_1{}^2 + \beta^{-2}(p_1{}^2 + E_2{}^2 + k\Delta_1 + \tfrac{1}{2}\beta^{-2})], \quad (16a)$$
$$C_3 = \Delta_1{}^{-2}[4E_2{}^2 + \beta^{-2}(1 - 2k\Delta_1)],$$
$$C_4 = 2k^2\Delta_1{}^{-1},$$
$$C = \Delta_1{}^{-1};$$

(2) for $\sigma_\perp$:

$$F = 1,$$
$$C_1 = 0,$$
$$C_2 = \Delta_1[k^2 - 4E_1{}^2(E_1E_2 - 1 + k\Delta_1)/p_1{}^2\sin^2\theta_1] + \beta^{-2}[\tfrac{1}{2}(4E_1{}^2+g)(E_2-\Delta_1)/p_1{}^2\sin^2\theta_1 - E_2 + k^2\Delta_1{}^{-1}], \quad (16b)$$
$$C_3 = -2k^2 + T^2(4E_1{}^2 + \beta^{-2})/p_1{}^2\sin^2\theta_1,$$
$$C_4 = k^2\Delta_1{}^{-1},$$
$$C = \tfrac{1}{2}(E_2-\Delta_1)/p_1{}^2\sin^2\theta_1;$$

(giving agreement with the formulas of Gluckstern and Hull[11] for $Z=0$).

(3) for $\sigma_{c,\,l}$:

$$F = \zeta_l k/p_1,$$
$$C_1 = 4E_1(E_1 - \Delta_1) - \beta^{-2},$$
$$C_2 = 2\Delta_1{}^{-1}[-4E_1E_2 + 2(E_1+E_2)\Delta_1 - kE_1\Delta_1{}^2 + \beta^{-2}E_1(E_1+E_2-\Delta_1)], \quad (16c)$$
$$C_3 = \Delta_1{}^{-2}[4E_2(E_2-\Delta_1) + \beta^{-2}(1-2E_1\Delta_1)],$$
$$C_4 = 2E_1k\Delta_1{}^{-1},$$
$$C = 0;$$

(4) for $\sigma_{c,\,t}$:

$$F = \zeta_t k\cos\psi_1/p_1{}^2\sin\theta_1,$$
$$C_1 = 2p_1{}^2\sin^2\theta_1 + g(p_1{}^2 + E_1{}^2 - E_1\Delta_1),$$
$$C_2 = 2\Delta_1{}^{-1}[4E_1E_2 - 2\Delta_1(2E_1{}^2E_2 + E_1 + E_2) + \Delta_1{}^2(2E_1{}^2 + E_1E_2 + 1 - k\Delta_1) + \beta^{-2}(p_1{}^2\sin^2\theta_1 - E_1E_2 + E_2\Delta_1)], \quad (16d)$$
$$C_3 = \Delta_1{}^{-2}[-4E_2{}^2 + 4E_2\Delta_1(E_1E_2+1) + 2\Delta_1{}^2(1 - E_1{}^2 - 2E_1E_2) + \beta^{-2}(p_1{}^2\sin^2\theta_1 + E_1\Delta_1 - \Delta_1{}^2)],$$
$$C_4 = 2\Delta_1{}^{-1}(E_1E_2 - 1 + k\Delta_1),$$
$$C = 0.$$

The three photon polarization states are specified by:

$$P_{\mathrm{lin}}{}^{(1)} \equiv P_{\mathrm{lin}} = (2\sigma_\perp/\sigma_{\mathrm{BH}} - 1),$$
$$P_{\mathrm{lin}}{}^{(2)} \equiv 0,$$
$$P_c = (\sigma_{c,\,l} + \sigma_{c,\,t})/\sigma_{\mathrm{BH}},$$

or

$$P_{c,\,l} = \sigma_{c,\,l}/\sigma_{\mathrm{BH}}, \quad P_{c,\,t} = \sigma_{c,\,t}/\sigma_{\mathrm{BH}}.$$

The second type of linear polarization integrates to zero.

Equation (15) simplifies considerably in the high-energy limit, where one can neglect terms of order $1/E_1$,

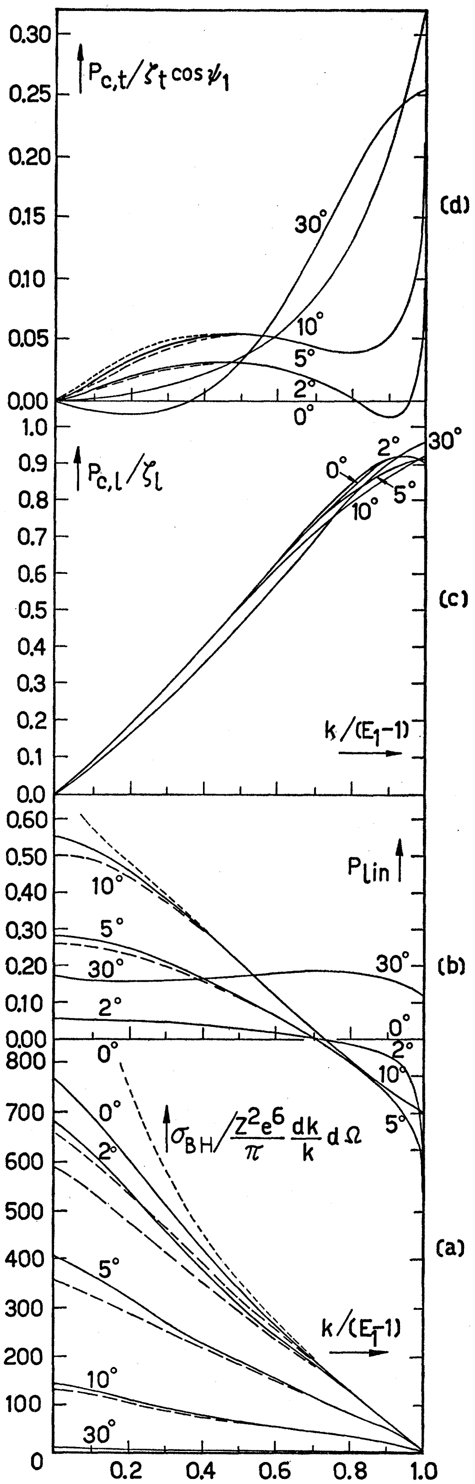

FIG. 1. Bremsstrahlung cross section (σ_{BH}), linear polarization (P_{lin}), and circular polarization (for 100% longitudinally or transversely polarized electrons: $P_{c,l}$, $P_{c,t}$) for initial electron energy $E_1=6$ (≈ 2.5 Mev) *vs* fractional photon energy $k/(E_1-1)$, for photon emission angles $\theta_1=0°$, 2°, 5°, 10°, and 30°. The solid curves are for $Z=26$ (Fe), with screening; the broken curves are for $Z=82$ (Pb), with screening; the effect of no screening ($Z=0$) is indicated by a dotted curve for one representative angle in each figure.

$1/E_2$. We obtain

$$\sigma_{\mathrm{BH}}=Z^2e^6\frac{dx}{x}\frac{4UdU}{(1+U^2)^2}\frac{d\psi_1}{2\pi}\times\left\{[1+(1-x)^2]\left[\ln\left(\frac{(1+U^2)^2}{\alpha}\right)-1\right]\right.$$
$$\left.-2(1-x)\left[\frac{2U^2}{(1+U^2)^2}\left[\ln\left(\frac{(1+U^2)^2}{\alpha}\right)-4\right]+1\right]\right\}, \quad (17a)$$

$$\sigma_{\perp}=Z^2e^6\frac{dx}{x}\frac{4U\mathrm{d}U}{(1+U^2)^2}\frac{d\psi_1}{2\pi}\frac{1}{2}\left\{[1+(1-x)^2]\right.$$
$$\left.\times\left[\ln\left(\frac{(1+U^2)^2}{\alpha}\right)-1\right]-2(1-x)\right\}, \quad (17b)$$

$$\sigma_{c,\,l}=Z^2e^6dx\frac{4U\mathrm{d}U}{(1+U^2)^2}\frac{d\psi_1}{2\pi}\zeta_l\left\{(2-x)\left[\ln\left(\frac{(1+U^2)^2}{\alpha}\right)-1\right]\right.$$
$$\left.-2(1-x)\left[\frac{2U^2}{(1+U^2)^2}\left[\ln\left(\frac{(1+U^2)^2}{\alpha}\right)-4\right]+1\right]\right\}, \quad (17c)$$

$$\sigma_{c,\,t}=Z^2e^6dx\frac{4UdU}{(1+U^2)^2}\frac{d\psi_1}{2\pi}\zeta_t\cos\psi_1(1-x)$$
$$\times\frac{2U(1-U^2)}{(1+U^2)^2}\left[\ln\left(\frac{(1+U^2)^2}{\alpha}\right)-4\right], \quad (17d)$$

in agreement with previously published results.[8,10] New notations are $x=k/E_1$, $U=E_1\theta_1$, and

$$\alpha=\beta^{-2}+\left[\frac{1}{2E_1}\left(\frac{x}{1-x}\right)\right]^2(1+U^2)^2.$$

As is well known,[20] the screening is very important at these energies; it contributes, however, only to the arguments of the logarithmic terms.

At high energies, the radiation is emitted mostly forward, and it makes sense to consider the "average" polarization of the entire bremsstrahlung cone. We therefore integrate (17) over the angles U. Then $\sigma_\perp$ and $\sigma_{c,\,t}$ vanish on grounds of symmetry, and we obtain

$$\sigma_{\mathrm{BH}}=2Z^2e^6\frac{dx}{x}\{[1+(1-x)^2]A(\beta,q_m)$$
$$-\tfrac{2}{3}(1-x)B(\beta,q_m)\}, \quad (18a)$$

$$\sigma_{c,\,l}=2Z^2e^6dx\zeta_l\{(2-x)A(\beta,q_m)$$
$$-\tfrac{2}{3}(1-x)B(\beta,q_m)\}, \quad (18b)$$

$$q_m=\frac{1}{2E_1}\left(\frac{x}{1-x}\right),$$

[20] See, e.g., reference 18, p. 248.

$$A(\beta,q_m)=1+\ln\beta^2-\ln(1+q_m^2\beta^2)-2q_m\beta\tan^{-1}\left(\frac{1}{q_m\beta}\right),$$
$$B(\beta,q_m)=\tfrac{2}{3}+\ln\beta^2-\ln(1+q_m^2\beta^2)+4q_m^2\beta^2$$
$$-3q_m^2\beta^2\ln\left(1+\frac{1}{q_m^2\beta^2}\right)-4q_m^3\beta^3\tan^{-1}\left(\frac{1}{q_m\beta}\right).$$

Only the *longitudinal* electron polarization contributes to the average circular photon polarization. These formulas can be generalized to a nonexponential form factor, using the functions $\Phi_{12}(\gamma)$, defined by Bethe[21] and tabulated by Bethe and Heitler[22] for the Thomas-Fermi atom, with the result

$$A(\beta,q_m)=\tfrac{1}{2}[\Phi_1(\gamma)-\tfrac{4}{3}\ln Z],$$
$$B(\beta,q_m)=\tfrac{1}{2}[\Phi_2(\gamma)-\tfrac{4}{3}\ln Z], \tag{19}$$
$$\gamma=200Z^{-\frac{1}{3}}q_m.$$

The validity of Eqs. (15), (17), and (18) is determined by the Born approximation limits,

$$Ze^2/\hbar v_{12}\ll 1,$$

which prohibits too slow electrons or too hard spectral components. There is no limitation for soft spectral components, as the screening is taken care of correctly.

All corresponding formulas for pair production are obtained from the bremsstrahlung formulas by the substitution:

$$E_1\rightarrow E_-,\qquad \mathbf{p}_1\rightarrow\mathbf{p}_-,\qquad \mathbf{s}\rightarrow\mathbf{s}_-,$$
$$E_2\rightarrow -E_+,\qquad \mathbf{p}_2\rightarrow -\mathbf{p}_+,$$

and by an additional factor

$$p_-^2dE_+/k^2dk$$

representing the difference in phase spaces.

V. DISCUSSION

Equations (15) were evaluated numerically on the Paris IBM-704 electronic computer,[23] for $E_1=2$ and $E_1=6$; and $Z=0$, 26 (Fe), and 82 (Pb). Curves of the total cross section, linear and circular polarization are shown in Fig. 1 for initial energy $E_1=6$, or ≈ 2.5 Mev, and various emission angles. The solid lines are for $Z=26$, broken lines for $Z=82$, and some dotted curves for $Z=0$ (no screening).

The spectrum is given in part (a) of the figure. It is characterized by a rapid decrease in magnitude for increasing angles. The screening effect is not very pronounced at this low energy, and the tentative screening correction of Gluckstern and Hull[11] is more or less correct here. Part (b) gives the linear polarization which agrees with Gluckstern and Hull's result. Its magnitude is quite considerable (up to 60%), it is positive and large for soft spectral components, but small in the high spectral region and even negative for not too large angles. Screening is again of little importance, and then only for soft photons.

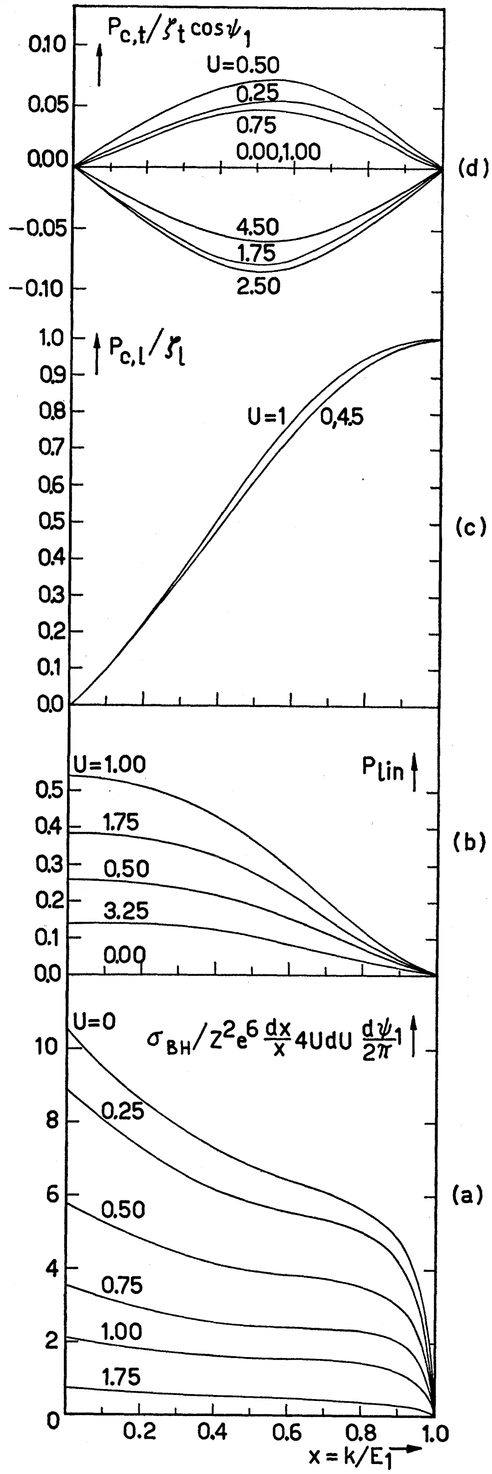

FIG. 2. Bremsstrahlung cross section, linear and circular polarization for initial electron energy $E_1=103.25$ ($=\frac{1}{2}m_\mu=52.75$ Mev), *vs* fractional photon energy for various photon emission angles $U=E_1\theta_1$, with screening, for $Z=26$ (Fe).

[21] H. A. Bethe, Proc. Cambridge Phil. Soc. **30**, 524 (1934).

[22] H. A. Bethe and W. Heitler, Proc. Roy. Soc. (London) **A146**, 83 (1934); see also M. Stearns, Phys. Rev. **76**, 836 (1949).

[23] A program for all energies and angles is available.

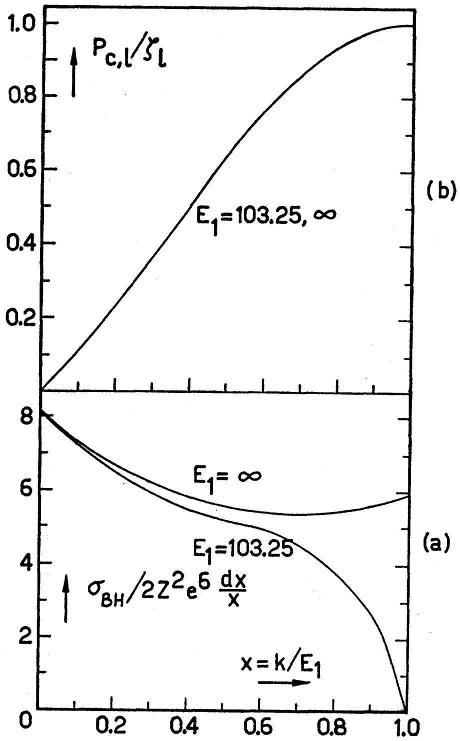

FIG. 3. Bremsstrahlung cross section and circular polarization, integrated over photon emission angles, for initial electron energies $E_1=103.25$ and ∞, *vs* fractional photon energy $x=k/E_1$, with screening, for $Z=26$.

For 100% longitudinally polarized electrons, the bremsstrahlung has the circular polarization given in part (c). Its value goes from zero for soft radiation to almost 100% at the high spectral end. There is little variation with angle, and practically no screening effect.

For $\theta_1=0°$, a similar curve was given by McVoy,[5] which does not entirely agree with our curve for $P_{\rm lin}$.[24] Finally, the circular polarization for 100% transversely polarized electrons and radiation emitted in the upper $(\mathbf{p}_1,\boldsymbol{\zeta})$ plane is shown in (d); it is mostly positive and of the order of 5–10%, but can reach up to 30% for larger angles and high spectral components.

Figure 2 represents the cross section and polarizations according to Eq. (17) for $E_1=103.25$, the maximum energy of electrons from $\mu-e$ decay. Screening is very important at this energy, and the curves are plotted for $Z=26$. The spectrum (a) was given by Schiff,[17] and the linear polarization (b) by May.[10] In $P_{\rm lin}$, the crossover to negative values which was found for low electron energies, does not occur. $P_{c,\,l}$ is similar to that for low energies, except that it now reaches entirely up to 100% for the hardest photons. $P_{c,\,t}$ has changed considerably; it is identically zero for $U=1$, smaller than 10% throughout, positive for $U<1$ and negative for $U>1$. Comparing with Fig. 1, we see that the large polarizations for hard quanta have disappeared, and that the negative trend for larger angles, exhibited by the 30° curve, is now dominating.

Figure 3 presents the intensity and $P_{c,\,l}$ for the entire high-energy bremsstrahlung cone, according to Eq. (18), for $E_1=103.25$ and $E_1=\infty$ (complete screening).

If the incident electrons have arbitrary polarizations, with longitudinal and transverse components ζ_l and ζ_t, the curves (c) and (d) have to be multiplied with ζ_l and $\zeta_t \cos\psi_1$, respectively, and added, in order to give the actual circular photon polarization. Because of the $\cos\psi_1$ dependence of $P_{c,\,t}$, an experiment to detect transverse electron polarizations is best performed by selecting an optimal emission angle and spectral component and then observing a periodic variation of the degree of circular polarization when the detector is rotated azimuthally around the direction of the incident electrons.

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[24] The disagreement seems to be due to an error in sign in McVoy's integrated cross sections, given in an unpublished note (while his differential cross section is correct). Use of the corrected cross section leads to better agreement with the experimental points in Fig. 2 of reference 4.