Radiation Effects in Circular Electron Accelerators*

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The effects of the radiation emission on the motion of electrons in high-energy synchrotrons are analyzed. The damping rates and quantum excitation of the three principal modes of oscillation are derived for strong focusing and constant gradient accelerators. Methods for correcting the radiation effects for strong-focusing accelerators are discussed.

INTRODUCTION

THE general method of treating the motion of relativistic electrons in circular accelerators would include the interaction between the electron and radiation field by the general methods of quantum electrodynamics. For practical electron energies and magnetic fields, the nature of the radiation is accurately given by a classical calculation. Then the electron motion may be analyzed by determining the quantum states of the electron in the magnetic guide field and considering the classically calculated radiation as a perturbation coupling these states. This approach has been used by Sokolov and Ternov.[1] In practical accelerators the time associated with the emission of radiation quanta is short compared with the periods of the classical modes of oscillation of the particles, and the radiation effects may be analyzed as a damping and a driving force applied to these modes.

NATURE OF RADIATION LOSS

The instantaneous power radiated by a relativistic electron in a magnetic field has been calculated to be[2]

$$P_\gamma=\tfrac{2}{3}(e^4B^2/m^2c^3)(E/mc^2)^2. \tag{1}$$

E is the electron energy, and B is the magnetic field. This is obtained by a classical calculation using the relation $\frac{2}{3}(e^2a^2/c^3)$ in the rest system of the electron. Quantization of the radiation field would reduce the radiated power below this classical value at extremely high electron energy. This will occur as the energy of the radiated quanta becomes comparable to the total electron energy. At this energy direct pair production in the magnetic guide field would also become important.

The spectrum of the radiated energy has been calculated classically to be[2]

$$P(\epsilon)=(3^{5/2}/8\pi)(P_\gamma/\epsilon_c^2)\int_{\epsilon/\epsilon_c}^{\infty}K_{5/3}(\eta)d\eta, \tag{2}$$

where $P(\epsilon)$ is the frequency of emission of photons of energy ϵ per unit energy range; the critical energy ϵ_c is given by $\frac{3}{2}(\hbar c/r)(E/mc^2)^3$; r is the radius of curvature of the electron in the magnetic field; ϵ_c is approximately the maximum energy photons radiated. For a magnetic field of 10 kilogauss, the photon energy becomes comparable to the electron energy at an energy of about 10^{15} ev.

The radiation is emitted in a narrow cone about the instantaneous direction of motion, with an angular width given approximately by mc^2/E.

In order to conserve momentum in the radiation of a photon of energy ϵ, the magnetic guide field must take up a momentum equal to

$$\Delta p=[(E/c)^2-(mc)^2]^{\frac{1}{2}}-[(E-\epsilon)^2/c^2-(mc)^2]^{\frac{1}{2}}-\epsilon/t \approx\tfrac{1}{2}(\epsilon/c)(mc^2/E)^2. \tag{3}$$

Then for $E\gg mc^2$ the momentum transferred to the magnetic field may be neglected, and the average effect of the radiation loss is to produce a force on the electron equal to P_γ/c and directed opposite to the instantaneous velocity of the electron.

The path length associated with the emission of a photon is related to the angular width of the direction of the radiation. The arc associated with the emission of a photon is given approximately by mc^2/E. Then for $E\gg mc^2$, a photon may be considered to be emitted in a time very short compared with the periods of the modes of oscillation of an electron, which are comparable to the period of one revolution.

The average force P_γ/c of the radiation loss will produce damping effects on the various modes of oscillation, and the sudden emission of the individual photons will excite the various modes as a driving force.

DAMPING OF OSCILLATION AMPLITUDES BY RADIATION LOSS

The general method of describing the motion of a particle in a circular accelerator is to determine a principal orbit, and then analyze small deviations from the principal orbit as a linear combination of normal modes of oscillation. The principal orbit may be defined as a particle motion which is repeated identically in each complete period of the accelerator. For small deviations from the principal orbit, a transfer matrix for a complete period may be written relating initial to final deviations. This is usually done for radial and vertical displacements and velocities, and may be extended to a sixth order transfer matrix relating initial

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[1] A. A. Sokolov and I. M. Ternov, J. Exptl. Theoret. Phys. U.S.S.R. 28, 431 (1955) [translation: Soviet Phys. JETP 1, 227 (1955)].

[2] J. Schwinger, Phys. Rev. 75, 1912 (1949).

to final values of radial and vertical displacements and velocities, and also energy variation, and longitudinal displacement, from the values of a particle on the principal orbit. For this general transfer matrix, the complete periods of the accelerator are defined so as to be identical in both magnet structure and radio-frequency accelerating system.

The characteristics of the modes of oscillation are determined by solving for the principal values of the sixth order transfer matrix. If the particle motion is stable such that the particle oscillates about the ideal positon, and since the transfer matrix is real, the principal values will be three pairs of complex conjugate numbers, which determine the frequencies and damping rates of the three principal modes of oscillation. In most accelerators there is one mode in the vertical plane corresponding to the vertical betatron oscillations, and two modes in the radial plane, one having a lower frequency and large energy and phase variations called the synchronous oscillation, and the other having a higher frequency and only small energy and phase variation called the radial betatron oscillation.

Consider an element of the accelerator of infinitesimal length. The element may include both a magnetic guide field and a radio-frequency accelerating field. Let x, x', y, y' represent the variation of displacement and angular deviation in the radial and vertical planes. x and y are measured normal to the principal orbit at that point. ΔE and z represent the variation in energy and azimuthal position from the values of an ideal particle, as measured at the time the particle transverses the infinitesimal element. The sixth order transfer matrix for the infinitesimal element will have infinitesimal non-diagonal terms which are first order in the length of the element, and the diagonal terms will differ from unity by a quantity which is proportional to the infinitesimal length of the element. In order to determine damping, the determinant of the transfer matrix of the infinitesimal element is evaluated. The only terms in the determinant which will be first order in the length of the element will be due to the diagonal terms of the matrix, and all higher order terms may be neglected. The determinant of the transfer matrix is given by $1+\sum\delta_{nn}$, where δ_{nn} are the differences of the diagonal terms from

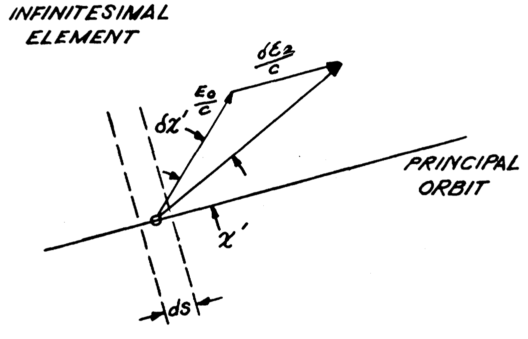

FIG. 1. Reduction in angular variation due to energy gain.

unity. The diagonal terms for x, y, z will not have a first order difference from unity as changes in x, y are only related to x', y' and changes in z are related to x for relativistic particles.

The diagonal term for ΔE may be determined from the characteristics of the radiation loss,

$$P_\gamma \propto E^2B^2 \propto [1+2(\Delta B/B_0)+2(\Delta E/E_0)] \qquad (4)$$

for small variations in E and B from the values for an ideal particle. B is only a function of position, then the diagonal term for ΔE due to radiation loss is $1-2\delta\epsilon_1/E_0$, with $\delta\epsilon_1$ the radiation loss for an ideal particle in the infinitesimal element. The energy gain from the radio-frequency system is not dependent on ΔE and contributes no change in the ΔE diagonal term.

The difference from unity of the x' and y' diagonal terms is determined from the energy gain from the radio-frequency system and is unaffected by radiation loss. The energy increase due to the radio-frequency system will add a momentum change parallel to the principal orbit, as shown in Fig. 1, and will reduce the angular variation. From Fig. 1, $\delta x' = -(\delta\epsilon_2/E_0)x'$ and the diagonal term for x' is $1-\delta\epsilon_2/E_0$, with $\delta\epsilon_2$ the energy gain from the radio-frequency system in the infinitesimal element for an ideal particle. Similarly the diagonal term for y' is $1-\delta\epsilon_2/E_0$. Then the determinant for the infinitesimal element is

$$1+\sum\delta_{nn}=1-2\delta\epsilon_1/E_0-2\delta\epsilon_2/E_0. \qquad (5)$$

The determinant of the transfer matrix for one complete period is the product of the transfer matrices of the infinitesimal elements of that period. Since the fractional radiation loss in one period is very small, only first order terms need be considered and the determinant of the transfer matrix for one period is given by

$$D=1-2\epsilon_{10}/E_0-2\epsilon_{20}/E_0, \qquad (6)$$

where ϵ_{10} and ϵ_{20} are the radiation loss and energy gain from the radio-frequency system in one period.

The characteristics of the principal modes of oscillation are determined by solving for the principal values of the transfer matrix for one complete period. If all modes are oscillatory the principal values will be of the form $e^{\gamma_i'}$,with the six values of γ_i' being three pairs of complex conjugates.

Then

$$\prod e^{\gamma_i'}=D=1-2\epsilon_{10}/E_0-2\epsilon_{20}/E_0. \qquad (7)$$

$\gamma_i'=\alpha_i'\pm i\beta_i'$, where α_i' is the fractional damping of a mode in one period of the accelerator. Then

$$\exp\sum 2\alpha_i'=1-2\epsilon_{10}/E_0-2\epsilon_{20}/E_0. \qquad (8)$$

For $|\alpha_i'|\ll 1$, then

$$\sum\alpha_i'=-\epsilon_{10}/E_0-\epsilon_{20}/E_0. \qquad (9)$$

For equilibrium conditions the radiation loss is equal to the energy gain from the radio-frequency system for one complete period, $\epsilon_{10}=\epsilon_{20}=\epsilon_0$. Then

$$\sum\alpha_i'=-2\epsilon_0/E_0. \qquad (10)$$

The sum of the fractional damping of the three modes in one period is thus $-2\epsilon_0/E_0$, with ϵ_0 the radiation loss in one period. The sum of the damping rates of the three modes is therefore given by

$$\sum\alpha_i=-2P_{\gamma 0}/E_0, \qquad (11)$$

where $P_{\gamma 0}$ is the average rate of radiation loss, and the amplitude of an oscillation varies as $e^{\alpha_i t}$. This is a general result for any type of electron accelerator if the average electron energy is constant. For a varying electron energy, adiabatic damping would be superimposed on the radiation damping.

For an accelerator in which there is no coupling between the radial and vertical planes, the transfer matrix will contain no nondiagonal terms between y or y' and the other coordinates x, x', ΔE, z. Then the second order matrix for the vertical plane may be diagonalized separately, and the product of the principal values will be equal to the determinant. The determinant of the second order matrix for the vertical plane of an infinitesimal element is given by $1-\delta\epsilon_2/E_0$, and for the complete period is $1-\epsilon_0/E_0$. Then $2\alpha_v'=-\epsilon_0/E_0$ and $\alpha_v'=-\epsilon_0/2E_0$. The damping rate for the vertical betatron oscillations is therefore

$$\alpha_v=-P_{\gamma 0}/2E_0, \qquad (12)$$

for any electron accelerator with no coupling between radial and vertical planes.

In order to determine the individual damping rates of the radial betatron and synchronous oscillations, it is convenient to calculate the damping rate of the synchronous oscillations and then determine the damping rate of the radial betatron oscillations, from the total damping rate.

It is assumed that the frequency of the synchronous oscillation is very low compared with the betatron frequency, such that a particle undergoing synchronous oscillation may be considered to travel on equilibrium orbits corresponding to adiabatically changing values of energy deviation.

The equations of synchronous oscillation may be written

$$dE/dN=\epsilon_2-\epsilon_1=\epsilon_{20}+(d\epsilon_2/dz)z-\epsilon_{10}-(d\epsilon_1/dE)\Delta E. \qquad (13)$$

It is assumed that the energy gain from the radio-frequency system in one complete period (ϵ_2), is only a function of the phase position. The effect of more general radio-frequency fields will be considered later.

Since the particle is assumed to travel on an equilibrium orbit corresponding to an energy variation the radiation loss in one period is only a function of energy.

For equilibrium conditions, the radiation loss for a particle on the principal orbit is equal to the energy gain from the radio-frequency system $\epsilon_{10}=\epsilon_{20}=\epsilon_0$.

Also since the particle is assumed to travel on an equilibrium orbit, the change in path length Δl is only a function of energy. For relativistic electrons the change in azimuthal position in one period is equal to the change in path length.

The equations of synchronous oscillation become

$$\begin{aligned} dE/dN&=(d\epsilon_2/dz)z-(d\epsilon_1/dE)\Delta E,\\ dz/dN&=-(dl/dE)\Delta E. \end{aligned} \qquad (14)$$

The solution of the equations has the form $e^{(\alpha_s'\pm i\beta_s')N}$ and the damping of the synchronous oscillation in one period is given by

$$\alpha_s'=-\tfrac{1}{2}(d\epsilon_1/dE). \qquad (15)$$

In the Cambridge type of accelerator, the principal orbit is on an isomagnetic line, except for straight sections. Then the radiation loss may be expressed as a first order expansion about the principal orbit.

$$\begin{aligned} \epsilon_1&=\frac{\epsilon_0}{l_0}\int\frac{E^2B^2(r_0+x)}{E_0^2B_0^2r_0}ds,\\ \epsilon_1&=\frac{\epsilon_0}{l_0}\int\left[1+2\frac{\Delta E}{E_0}+2\frac{\Delta B}{B_0}+\frac{x}{r_0}\right]ds,\\ \epsilon_1&=\epsilon_0+2\epsilon_0(\Delta E/E_0)\\ &\quad+2\frac{\epsilon_0}{l_0}\int(\Delta B/B_0)ds+\frac{\epsilon_0}{l_0}\int\left(\frac{x}{r_0}\right)ds. \end{aligned} \qquad (16)$$

The last term represents the increase in path length. The straight sections are excluded from the integral and from the path length l_0 as they do not contribute to the radiation loss.

The equation of motion of an electron in the magnetic guide field may be written

$$dx'/ds=-(1/r_0)[\Delta B/B_0+x/r_0-\Delta E/E_0]. \qquad (17)$$

For an equilibrium orbit the total change in x' in one complete period is zero.

$$\int\left(\frac{dx'}{ds}\right)ds=0=\int\left(\frac{\Delta B}{B_0}\right)ds+\int\left(\frac{x}{r_0}\right)ds-l_0\left(\frac{\Delta E}{E_0}\right). \qquad (18)$$

The straight sections are also excluded from the above integrals as they contribute no change in the angular variation.

The change in path length in one period is given by

$$\Delta l=\int\left(\frac{x}{r_0}\right)ds.$$

Then

$$\int\left(\frac{\Delta B}{B_0}\right)ds=l_0\left(\frac{\Delta E}{E_0}\right)-\Delta l, \qquad (19)$$

and

$$\epsilon_1=\epsilon_0+4\epsilon_0(\Delta E/E_0)-\epsilon_0(\Delta l/l_0). \quad (20)$$

The fractional increase in path length in the magnet is related to the fractional increase in energy (for relativistic particles), by a momentum compaction factor α. The momentum compaction factor, in this derivation, relates the change in path length to the fractional momentum change and the total path length in the magnets, excluding the straight sections:

$$\alpha=(E_0/l_0)(\Delta l/\Delta E).$$

Then

$$\begin{aligned} d\epsilon_1/dE&=(\epsilon_0/E_0)(4-\alpha),\\ \alpha_s'&=-(\epsilon_0/2E_0)(4-\alpha). \end{aligned} \quad (21)$$

The damping rate of the synchronous oscillations is given by

$$\alpha_s=-(P_{\gamma 0}/2E_0)(4-\alpha). \quad (22)$$

The damping of the radial betatron oscillations in one period is given by

$$\alpha_r'=-2\epsilon_0/E_0-\alpha_v'-\alpha_s'=(\epsilon_0/2E_0)(1-\alpha), \quad (23)$$

and the damping rate is given by

$$\alpha_r=(P_{\gamma 0}/2E_0)(1-\alpha). \quad (24)$$

For an alternating gradient accelerator of the Cambridge type, α is small compared to unity; then the radial betatron oscillations are antidamped by the radiation loss and grow exponentially with time.

For a constant gradient accelerator, $\alpha=1/(1-n)$, we have

$$\begin{aligned} \alpha_v&=-P_{\gamma 0}/2E_0,\\ \alpha_s&=-[(3-4n)/(1-n)](P_{\gamma 0}/2E_0),\\ \alpha_r&=-[n/(1-n)](P_{\gamma 0}/2E_0). \end{aligned} \quad (25)$$

These results are valid for a constant gradient accelerator with straight sections, as the straight sections are excluded from the definition of α used in the previous derivation. The results for the constant gradient accelerator have been derived by several authors.[3–6]

In a magnet structure of the FFAG (fixed frequency alternating gradient) type,[7] the equilibrium orbits change in size, but maintain the same shape for different energies. In this type of accelerator at every angular position $B\propto E/r$ for equilibrium orbits of different energy. In this case, the momentum compaction factor α is defined as $\Delta r/r_0=\alpha(\Delta E/E_0)$; then for each element of the accelerator $\Delta B/B_0=\Delta E/E_0-\Delta r/r_0$, and the radiation loss in each element is

$$\begin{aligned} \epsilon_1&=\epsilon_0[1+(4-2\alpha)(\Delta E/E_0)+\alpha(\Delta E/E_0)],\\ \epsilon_1&=\epsilon_0[1+(4-\alpha)(\Delta E/E_0)]. \end{aligned} \quad (26)$$

Thus the radiation damping rates of the oscillation modes in the FFAG accelerator, in terms of a momentum compaction factor α, are identical to the alternating gradient accelerator of the Cambridge type. In this case, α relates the change in path length to the fractional momentum change, and the total circumference including straight sections.

OSCILLATION AMPLITUDES DUE TO RADIATION EFFECTS

The amplitude of oscillation of the principal modes is determined by the damping rates of the modes, the initial amplitudes, and the excitation by the quantum emission of radiation. The radiation of a photon of energy ϵ by an electron causes a shift in the equilibrium orbit by an average amount $r_0\alpha(\epsilon/E_0)$. The radiation of the photon takes place in a time which is short compared with the period of betatron oscillation. This sudden radial shift in equilibrium orbit will cause radial betatron oscillation of amplitude approximately $r_0\alpha(\epsilon/E_0)$ about the displaced equilibrium orbit; also the equilibrium orbit will now oscillate about the principal orbit as a synchronous oscillation of amplitude $r_0\alpha(\epsilon/E_0)$.

Each photon will then add an oscillation of amplitude $r_0\alpha(\epsilon/E_0)$ to both radial betatron and synchronous oscillation. The phases of these incremental amplitudes will be random, so the increase in mean square amplitude due to the radiation of a large number of photons will be $(r_0\alpha/E_0)^2\Sigma\epsilon_i^2$. The rate of change of the mean square amplitude of betatron oscillations is given by

$$\frac{d(x^2)}{dt}=\left(\frac{r_0\alpha}{E_0}\right)^2\int_0^\infty \epsilon^2P(\epsilon)d\epsilon+2\alpha_r(x^2)-\frac{1}{E_0}\left(\frac{dE_0}{dt}\right)(x^2). \quad (27)$$

The last term is due to adiabatic damping of betatron oscillations with increasing particle energy.

The integral $\int_0^\infty \epsilon^2P(\epsilon)d\epsilon$ has been evaluated[4] to be $(55/2^3 3^{\frac{3}{2}})P_{\gamma 0}\epsilon_c$. Therefore

$$d(x^2)/dt=(55/2^3 3^{\frac{3}{2}})(r_0\alpha/E_0)^2P_{\gamma 0}\epsilon_c+(1-\alpha)(P_{\gamma 0}/E_0)(x^2)-(1/E_0)(dE_0/dt)(x^2), \quad (28)$$

and

$$(x^2)=(x_1^2)\frac{E_1}{E_0}\exp\left[\int_{E_1}^{E_0}(1-\alpha)\frac{P_{\gamma 0}'}{E_0'}dt'\right]+\frac{55}{2^3 3^{\frac{3}{2}}}\left(\frac{r_0\alpha}{E_0}\right)^2\int_{E_1}^{E_0}E_0'P_{\gamma 0}'\epsilon_c\times\exp\left[\int_{E_0'}^{E_0}(1-\alpha)\frac{P_{\gamma 0}''}{E_0''}dt''\right]dt'. \quad (29)$$

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[3] D. Bohm, Phys. Rev. **70**, 249 (1946).
[4] M. Sands, Phys. Rev. **97**, 470 (1955).
[5] I. Henry, Phys. Rev. **106**, 1057 (1957).
[6] A. A. Kolomenskii and A. N. Lebedev, *Proceedings of the CERN Conference on High-Energy Accelerators, Geneva* (European Organization of Nuclear Research, Geneva, 1956), Vol. 1, p. 447.
[7] K. Symon *et al.*, Phys. Rev. **103**, 1837 (1956).

The first term is due to amplitudes at injection, and the last term is due to quantum fluctuations in the radiation loss.

The parameters of the Cambridge accelerator are $r_0 = 25.8$ m, $\alpha = 0.042$, and $E_1 = 20$ Mev. The magnet is excited with $\frac{1}{2}$ sine wave, and $\frac{1}{2}$ dc bias. At magnet frequencies of 15 and 30 cps, the amplitude of radial betatron oscillations at 6 Bev are

$$(x^2) = (7.7x_1)^2 + (14.9 \text{ cm})^2 \quad \text{at 15 cps},$$
$$(x^2) = (0.67x_1)^2 + (1.63 \text{ cm})^2 \quad \text{at 30 cps}.$$

It is seen that the antidamping of radial betatron oscillations may produce large radial oscillations and result in loss of particles at 6-Bev operation.

The radial amplitude of the synchronous oscillations is determined by a similar equation with α_r replaced by α_s. The factor α_s is negative for the alternating gradient accelerator and large compared with the reciprocal of the accelerating time. The radial amplitude will then approach an equilibrium value due to the radiation damping and quantum effects, which is given by

$$(x^2) = (55/2^3 3^{\frac{3}{2}})(r_0\alpha)^2(\epsilon_c/E_0)[1/(4-\alpha)]. \tag{30}$$

This mean square radial amplitude is quite small for the Cambridge accelerator, but there will be large phase oscillations associated with the radial amplitudes, which increase the radio-frequency voltage required to maintain the particles in a phase stable position.

The damping rates for the radial betatron and synchronous oscillations will then not be observed directly due to the increase in oscillation amplitudes produced by the quantum fluctuations, unless the amplitudes are much larger than the increase due to quantum fluctuations in one damping or antidamping period.

The vertical betatron oscillations are damped by the radiation loss, and not excited by sudden changes of energy, and will have a very small amplitude at high energy. The radiation will excite very small vertical oscillations due to the transverse momentum transferred to the electron when a photon is radiated at a small angle with the direction of motion.

CORRECTION OF RADIATION EFFECTS

In order to achieve optimum performance of an alternating gradient electron accelerator, it may be necessary to use correcting devices to redistribute the total damping rate of $2P_{\gamma 0}/E_0$ among the principal modes. In order to minimize the radio-frequency power required, which is related to the magnitude of phase oscillations, it is desirable to make the damping rate of the synchronous oscillation as large as possible, while also controlling the damping rates of the radial and vertical betatron oscillations so as to keep the electrons within the proper region of the vacuum chamber.

It is shown in Appendix I that the damping rates of the individual modes are independent of the nature of the radio-frequency accelerating system, and hence cannot be changed by an external electromagnetic field, unless the external field is sufficiently strong to make an appreciable change in the electron trajectories in the magnet structure.

Radio-frequency fields of this magnitude are usually impractical to attain.

There are two possible methods of correcting the damping rates of the individual modes. The first method uses correcting devices which utilize a force not due to an external electromagnetic field, such as energy-loss foils, or radiation loss in a magnetic field.

The only practical method of this type is to use the radiation loss in a magnetic field. The use of foils would introduce excessive fluctuations in loss due to bremsstrahlung radiation. The damping rates were derived in terms of the momentum compaction factor α, for a magnet structure in which the principal orbit is on an isomagnetic line except for straight sections. By changing the magnet structure so that the principal orbit is not on an isomagnetic line, it is possible to change the damping rates without changing α appreciably. This could be done by redesigning the magnetic structure such as adding quadrupole lenses or making the field strength different in focusing and defocusing sectors. A correcting device which may be added to the Cambridge type of magnet structure to reduce the antidamping of the radial betatron oscillations is a magnet with a large n value (n'), such that the radiation loss decreases with increasing radius. This device is made with close-spaced alternations in direction of magnetic field such that the phase angle associated with each magnet is very small. It will then produce very little effect on the electron motion. The radiation loss caused by this structure may be written

$$P_\gamma' = P_{\gamma 0}'(E/E_0)^2(B/B_0)^2$$
$$= P_{\gamma 0}'[1 + 2(\Delta E/E_0) + 2(\Delta B/B_0)]$$
$$\Delta B/B_0 = -n'(x/r_0) = -n'\alpha(\Delta E/E_0);$$

then

$$P_\gamma' = P_{\gamma 0}'[1 - 2(n'\alpha - 1)(\Delta E/E_0)]. \tag{31}$$

From Eqs. (15) and (11), the change in damping rates of the synchronous and radial betatron oscillations are

$$\begin{aligned} \Delta\alpha_s &= (n'\alpha - 1)(P_{\gamma 0}'/E_0), \\ \Delta\alpha_r &= -(n'\alpha + \tfrac{1}{2})(P_{\gamma 0}'/E_0). \end{aligned} \tag{32}$$

In order to eliminate the anti-damping of the radial betatron oscillations by this method requires

$$\tfrac{1}{2}(P_{\gamma 0}/E_0)(1-\alpha) = (n'\alpha + \tfrac{1}{2})(P_{\gamma 0}'/E_0). \tag{33}$$

The other method of correcting damping rates is to use an external field which couples modes of oscillation strongly, such that the new principal modes of oscillation are radically changed from the original modes, with correspondingly changed damping rates. It is not practical to couple the synchronous oscillations strongly with the betatron oscillations with a device which is small compared with the magnet structure. This is due to the different characteristics of the betatron and synchronous oscillations. However, if the vertical and radial betatron oscillations have nearly identical frequencies, it will be possible to couple the modes strongly with a relatively small device. This has the effect of producing new modes of betatron oscillation with reduced antidamping and damping rates, and has the desirable characteristic of not reducing the damping rate of the synchronous oscillation.

A practical method of achieving this coupling uses a magnetic quadrupole lens which is rotated 45° as compared with the usual type of quadrupole lens used for correction of betatron frequency.

By proper choice of coordinate system, the transfer matrix for radial betatron oscillations for one complete revolution is taken to be of the form

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\theta_x & \sin\theta_x \\ -\sin\theta_x & \cos\theta_x \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}. \tag{34}$$

The transfer matrix for radial and vertical betatron oscillations, including the effect of radiation damping is

$$\begin{bmatrix} e^{\alpha_1}\cos\theta_x & e^{\alpha_1}\sin\theta_x & 0 & 0 \\ -e^{\alpha_1}\sin\theta_x & e^{\alpha_1}\cos\theta_x & 0 & 0 \\ 0 & 0 & e^{-\alpha_1}\cos\theta_y & e^{-\alpha_1}\sin\theta_y \\ 0 & 0 & -e^{-\alpha_1}\sin\theta_y & e^{-\alpha_1}\cos\theta_y \end{bmatrix}, \tag{35}$$

where α_1 is the radiation damping in one revolution. It is assumed for simplicity that the radial anti-damping is equal to the vertical damping.

The transfer matrix of the rotated quadrupole lens will be of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & q & 0 \\ 0 & 0 & 1 & 0 \\ q & 0 & 0 & 1 \end{bmatrix}. \tag{36}$$

The damping of the new modes of oscillation are determined by solving for the principal values λ of the resulting transfer matrix.

The secular equation is

$$\begin{vmatrix} e^{\alpha_1}\cos\theta_x-\lambda & e^{\alpha_1}\sin\theta_x & qe^{\alpha_1}\sin\theta_x & 0 \\ -e^{\alpha_1}\sin\theta_x & e^{\alpha_1}\cos\theta_x-\lambda & qe^{\alpha_1}\cos\theta_x & 0 \\ qe^{-\alpha_1}\sin\theta_y & 0 & e^{-\alpha_1}\cos\theta_y-\lambda & e^{-\alpha_1}\sin\theta_y \\ qe^{-\alpha_1}\cos\theta_y & 0 & -e^{-\alpha_1}\sin\theta_y & e^{-\alpha_1}\cos\theta_y-\lambda \end{vmatrix} = 0. \tag{37}$$

By reducing the determinant and substituting

$$\lambda = e^{\pm i\theta}(1+\gamma_2),$$
$$\theta_x = \theta,$$
$$\theta_y = \theta+\Delta\theta,$$

and retaining terms to second order in γ_2, α_1, $\Delta\theta$, q, the result is

$$4(\gamma_2-\alpha_1)(\gamma_2+\alpha_1\pm i\Delta\theta)+q^2=0. \tag{38}$$

By substituting $\gamma_2=\alpha_2+i\beta_2$ and eliminating β_2, one obtains

$$4\alpha_1^2-4\alpha_2^2+(\alpha_1/\alpha_2)^2(\Delta\theta)^2-(\Delta\theta)^2=q^2, \tag{39}$$

where α_2 is the damping and antidamping of the new modes of oscillation in one revolution. α_1 and α_2 will probably be very small compared with the difference in phase $\Delta\theta$ in one revolution. Then

$$\alpha_2/\alpha_1 = [(q/\Delta\theta)^2+1]^{-\frac{1}{2}}. \tag{40}$$

This coupling has the effect of rotating the principal planes of oscillation such that they are no longer radial and vertical. This coupling could not be used at injection as it would rotate the large radial oscillations partially into the vertical plane, and would cause loss of particles due to the small vertical aperture. The coupling would be turned on when the oscillation amplitudes have been reduced by adiabatic damping.

The radial and vertical betatron oscillations could also be coupled by applying a magnetic field parallel to the direction of motion. This method requires the frequency difference between the two modes to be very much smaller, in order to couple them with a device of practical size.

A combination of the two correcting methods, coupling of radial and vertical betatron oscillations and the magnetic radiation loss device, is probably the best way of eliminating the antidamping of the radial betatron oscillations, and obtaining maximum damping of the synchronous oscillations.

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APPENDIX I

We prove here that the damping of the individual modes is not changed by a general radio-frequency field.

A radio-frequency cavity is considered in which there may be both radial and azimuthal forces on the electrons, which vary with the deviation of the particle from the position on the principal orbit.

The effect of the cavity as observed in the rest system of an electron may be written

$$\Delta p_z^* = -ex^* \int \left(\frac{\partial E_z^*}{\partial x^*}\right) dt^* - ez^* \int \left(\frac{\partial E_z^*}{\partial z^*}\right) dt^*,$$

$$\Delta p_x^* = -ex^* \int \left(\frac{\partial E_x^*}{\partial x^*}\right) dt^* - ez^* \int \left(\frac{\partial E_x^*}{\partial z^*}\right) dt^*, \qquad (41)$$

$$\int \left(\frac{\partial E_x^*}{\partial z^*}\right) dt^* - \int \left(\frac{\partial E_z^*}{\partial x^*}\right) dt^* = -\int \left(\frac{\partial B_y^*}{\partial t^*}\right) dt^*.$$

For a complete transversal of the cavity, $\int (\partial B_y^*/\partial t) dt^* = 0$. Therefore

$$(\partial p_x^*/\partial z^*) = (\partial p_z^*/\partial x^*). \qquad (42)$$

In the laboratory system the effect of the radio-frequency cavity is to introduce an angular deviation and energy change, relative to an ideal particle,

$$x_1' = cp_x^*/E_0, \quad \epsilon_2 = cp_z^*(E_0/mc^2);$$

then

$$\begin{aligned}(\partial x_1'/\partial z) &= (c/E_0)(\partial p_x^*/\partial z) \\ &= (c/mc^2)(\partial p_x^*/\partial z^*) = (c/mc^2)(\partial p_z^*/\partial x^*), \qquad (43)\end{aligned}$$

$$(\partial x_1'/\partial z) = (1/E_0)(\partial \epsilon_2/\partial x).$$

The fourth order transfer matrix of the magnet structure for one complete revolution, relating initial to final values of radial displacement, radial angular variation, energy variation, and azimuthal displacement, will be of the form:

$$\begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ 0 & 0 & 1 & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}. \qquad (44)$$

The radio-frequency system and radiation loss are neglected in this transfer matrix.

The basic relationships in the magnet structure are

$$\begin{aligned}dz/ds &= -x/r, \quad dx/ds = x', \\ dx'/ds &= (1/r)(\Delta E/E_0) + xf(s), \qquad (45)\end{aligned}$$

where s is the position on the principal orbit and r is the radius of curvature of the principal orbit.

By writing these relationships in terms of fixed initial conditions at one point, the following equations are obtained:

$$\begin{aligned}dx/ds &= \gamma_{11}'x_0 + \gamma_{12}'x_0' + \gamma_{13}'(\Delta E/E_0) \\ &= \gamma_{21}x_0 + \gamma_{22}x_0' + \gamma_{23}(\Delta E/E_0), \\ dz/ds &= \gamma_{41}'x_0 + \gamma_{42}'x_0' + \gamma_{43}'(\Delta E/E_0) \\ &= -(1/r)[\gamma_{11}x_0 + \gamma_{12}x_0' + \gamma_{13}(\Delta E/E_0)], \qquad (46) \\ dx'/ds &= \gamma_{21}'x_0 + \gamma_{22}'x_0' + \gamma_{23}'(\Delta E/E_0) \\ &= f(s)\gamma_{11}x_0 + f(s)\gamma_{12}x_0' \\ &\quad + [f(s)\gamma_{13} + (1/r)](\Delta E/E_0).\end{aligned}$$

Since the initial conditions are independent, their coefficients must be independently equal for each equation. Then the following relationships may be derived:

$$\begin{aligned}&\frac{d}{ds}(\gamma_{11}\gamma_{22} - \gamma_{21}\gamma_{12}) = 0, \\ &\frac{d}{ds}(\gamma_{23} - \gamma_{42}\gamma_{21} + \gamma_{41}\gamma_{22}) = (\gamma_{13} - \gamma_{42}\gamma_{11} + \gamma_{41}\gamma_{12})f(s), \qquad (47) \\ &\frac{d}{ds}(\gamma_{13} - \gamma_{42}\gamma_{11} + \gamma_{41}\gamma_{12}) = (\gamma_{23} - \gamma_{42}\gamma_{21} + \gamma_{41}\gamma_{22}).\end{aligned}$$

Then the well-known relationship follows

$$\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} = 1.$$

Also

$$\begin{aligned}\gamma_{13} - \gamma_{42}\gamma_{11} + \gamma_{41}\gamma_{12} &= 0, \\ \gamma_{23} - \gamma_{42}\gamma_{21} + \gamma_{41}\gamma_{22} &= 0.\end{aligned} \qquad (48)$$

To determine the effect of the cavity on the synchronous oscillation, the change in the equilibrium orbit is analyzed.

The radial displacement at the cavity of an equilibrium orbit, for an energy variation ΔE is given by

$$\begin{aligned}x_1 &= \left(\begin{vmatrix} \gamma_{13} & -\gamma_{12} \\ \gamma_{23} & 1-\gamma_{22} \end{vmatrix} \Big/ \begin{vmatrix} 1-\gamma_{11} & -\gamma_{12} \\ -\gamma_{21} & 1-\gamma_{22} \end{vmatrix} \right) \frac{\Delta E}{E_0} \\ &= \frac{\gamma_{23}\gamma_{12} + \gamma_{13}(1-\gamma_{22})}{2-\gamma_{11}-\gamma_{22}} \frac{\Delta E}{E_0}. \qquad (49)\end{aligned}$$

The change in length of an equilibrium orbit with an angular deflection x_1' at the cavity is given by

$$\Delta l = -\Delta z = -\frac{\gamma_{41}\gamma_{12} + \gamma_{42}(1-\gamma_{11})}{2-\gamma_{11}-\gamma_{22}} x_1'. \qquad (50)$$

By using Eq. (48), one obtains

$$\begin{aligned}[\gamma_{23}\gamma_{12} + \gamma_{13}(1-\gamma_{22})] &= -[\gamma_{41}\gamma_{12} + \gamma_{42}(1-\gamma_{11})], \\ E_0(x_1/\Delta E) &= \Delta l/x_1'.\end{aligned} \qquad (51)$$

For particles traveling on an equilibrium orbit there is a linear relation between x_1, z, and ΔE. Therefore z and ΔE may be taken as the independent variables in writing the equations of synchronous oscillation:

$$\begin{aligned}dE/dN &= (\partial \epsilon_2/\partial z)z + (\partial \epsilon_2/\partial E)\Delta E \\ &\quad - (\partial \epsilon_1/\partial z)z - (\partial \epsilon_1/\partial E)\Delta E, \qquad (52) \\ dz/dN &= -(\partial l/\partial E)\Delta E - (\partial l/\partial z)z.\end{aligned}$$

The damping of the solution is given by

$$\alpha_s' = -\tfrac{1}{2}[(\partial \epsilon_1/\partial E) - (\partial \epsilon_2/\partial E) + (\partial l/\partial z)]. \qquad (53)$$

The fields of the generalized radio-frequency cavity are assumed to be small compared to the magnetic guide fields, such that the equilibrium orbit corresponding to an energy variation is not appreciably changed by the cavity.

Then $(\partial\epsilon_1/\partial E)_z$ is equal to the value of $d\epsilon_1/dE$ without the general radiofrequency cavity.

Also

$$(\partial l/\partial z)_E=(\partial l/\partial x_1')_E(\partial x_1'/\partial z)_x,$$
$$(\partial\epsilon_2/\partial E)_z=(\partial\epsilon_2/\partial x_1)_z(\partial x_1/\partial E)_z. \quad (54)$$

From Eq. (51), one finds

$$(\partial l/\partial x_1')_E=E_0(\partial x_1/\partial E)_z. \quad (55)$$

From Eq. (43), one finds

$$(\partial x_1'/\partial z)_x=(1/E_0)(\partial\epsilon_2/\partial x_1)_z. \quad (56)$$

Therefore

$$(\partial l/\partial z)_E=(\partial\epsilon_2/\partial E)_z, \quad (57)$$

$$\alpha_s'=-\tfrac{1}{2}(\partial\epsilon_1/\partial E)_z=-\tfrac{1}{2}(d\epsilon_1/dE). \quad (58)$$

Thus, if the fields are not sufficiently strong to change the form of the equilibrium orbit appreciably, the damping rate of the synchronous oscillation is not changed by any form of radio-frequency fields.

Since the total damping rate is invariant, the damping rate of the radial betatron oscillations is also not changed by a generalized radio-frequency field.

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Thermal Diffusion Factors from Column Operation*†

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Extending the measurements of Corbett and Watson, the performance of a carefully-constructed, all-metal, hot-wire thermal diffusion column has been determined for isotope separations in neon, argon, krypton, and xenon. The same quantitative agreement with theory for normal argon gas at the low wall-temperature ratio of 2 is again found. This is probably fortuitous, for in general there is a discrepancy between the calculated and observed separation factors, with a trend in the data indicating that the assumptions of the theory that (1) the molecules are Maxwellian and (2) the thermal diffusion factors are constant, independent of temperature, are at fault.

INTRODUCTION

The hot-wire thermal diffusion column has proven to be of great value in the isotopic enrichment of certain gaseous compounds. Many workers have compared the experimental performance of such columns with the theory of Jones and Furry,[1] finding that the agreement is in general only qualitative. Corbett and one of the present authors,[2] however, using a carefully constructed all-metal column, reported very good agreement between experiment and theory, without the inclusion of any parasitic remixing term, when the column was operated with normal argon gas at a temperature ratio of 2 between the hot wire and the cold wall.

This result raises the question as to whether under such conditions one can determine thermal diffusion factors α from measurements of column performance. To investigate this possibility further, we have studied the performance of this same column for other noble gases. Our results show that, in general, accurate values of α may not be so determined, and for the reasons discussed at the end of this report.

Although the Jones and Furry theory considers the gas as a binary isotopic mixture of Maxwellian molecules with a constant thermal diffusion factor to be evaluated at the cold-wall temperature, it is easy to generalize to the case of a multi-isotopic mixture. According to this theory the equilibrium separation factor q_{ij} for two isotopes i and j, defined as the ratio c_i/c_j of the concentrations at the upper end to that at the lower end, is given by the expression

$$q_{ij}=e^{2A_{ij}l}, \quad (1)$$

where

$$A_{ij}=H_{ij}/(K_c+K_d), \quad (2)$$

and l is the length of the column. The thermal diffusion factor α_{ij} is contained in the transport factor H_{ij}. A reduced thermal diffusion factor α_0 may be defined from

$$\alpha=\alpha_0(m_i-m_j)/\bar{m}, \quad (3)$$

where the m_i and m_j are the masses of any two isotopes in the multicomponent mixture and $\bar{m}$ is the average of the two. To the sum of the two remixing factors K_c from convection and K_d from axial diffusion is usually added of necessity a third, K_p, from parasitic effects originating in nonuniformities of construction, azimuthal temperature asymmetries, etc. Only in our first

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[1] R. Clark Jones and W. H. Furry, Revs. Modern Phys. **18**, 151 (1946).

[2] J. W. Corbett and W. W. Watson, Phys. Rev. **101**, 519 (1956).