from the value of $\epsilon_1+\epsilon_2$ inferred from π decay, because the momentum transferred in K decay is considerably larger and higher powers of P^2/M^2 in the expansion of the functions h_i of Eq. (6) may become important. However, there exists the possibility that the nonlocal effects discussed here may increase the probability of the electron mode of decay. As an illustration of this point, if we make the naive assumption that the functions h_i are independent of P^2/M^2 and we use the values of $\epsilon_1+\epsilon_2$ determined from π decay and a reasonable value for $\bar{\beta}$ [we use the value of Eq. (13)] we get $1.3\times10^{-3}\leq R_K\leq 3.6\times10^{-3}$ which is smaller than the present upper limit $R_{K\,\mathrm{exp}}\leq 0.02$,[26] but considerably larger than the prediction of the local theory.

The nonlocal effects discussed here do not significantly affect the ratio $(K\rightarrow e+\pi+\nu)/(K\rightarrow\mu+\pi+\nu)$ because in these three-body decays the contribution of the local theory is relatively large [see discussion after Eq. (9b)].

In conclusion, we would like to mention that these nonlocal effects may change somewhat the predictions of local theories on the lifetime of such processes as $\Sigma\rightarrow p+e+\nu$. For example, if the function $f(P^2/M^2)$ can still be represented by the two terms of Eq. (7a) for the range of energies involved in Σ decay and the value of Eq. (13) for $\bar{\beta}$ is used, then it is easily seen that the prediction for the lifetime, as compared to the lifetime obtained in calculations which treat the weak interaction locally, is increased by a factor $\approx\frac{5}{2}$.

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[26] M. Gell-Mann and A. H. Rosenfeld, *Annual Review of Nuclear Physics* (Annual Reviews, Inc., Stanford, 1957), Vol. 7, p. 407.

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Quantization Process for Massless Particles*

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This paper gives the quantization of a recently proposed theory for particles of arbitrary spin and zero mass. An interesting result is that there is a connection between the spin and the statistics of the particles. It is found that the spinor components of a boson/fermion field with integral/half-integral spin commute/anticommute off of the light cone whereas the spinor components of a boson/fermion field with half-integral/integral spin do not. As in the unquantized theory, the two-component neutrino and the photon are special cases.

I. INTRODUCTION

RECENTLY a wave equation for massless particles was proposed[1] in which the Hamiltonian is[2]

$$H=(c/s)\mathbf{p}\cdot\mathbf{s} \tag{1}$$

($\mathbf{p}$ being the operator $-i\hbar\nabla$ and $\mathbf{s}$ being the angular momentum matrices for arbitrary spin s), and in which the wave function ϕ is related to the spin or components ψ of the field by

$$\psi=|H/c|^{s-\frac{1}{2}}\phi. \tag{2}$$

Also, as an auxiliary condition, only solutions with spin parallel or antiparallel to the momentum are retained. The purpose of this paper is to give the quantization of the theory. Since a uniform treatment of all spins is made, it is of interest to see how the spin and statistics are related and to find operator assignments for the number of particles, energy, momentum, and angular momentum.

The quantization process can be carried out in a straightforward way, using the coefficients of an expansion in plane waves. It is found that the spinor components of a boson/fermion field with integral/half-integral spin commute/anticommute off of the light cone, whereas the spinor components of a boson/fermion field with half-integral/integral spin do not. Also the different statistics lead to different expressions for the operators when they are written in terms of the wave function. For example, with Fermi-Dirac statistics the quantized Hamiltonian corresponds to the expectation value of the unquantized Hamiltonian, whereas with Bose-Einstein statistics it corresponds to the expectation value of the unquantized energy operator.

The equations for the spinor components ψ are a special case of the general Dirac-Pauli-Fierz theory,[3]

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[1] C. L. Hammer and R. H. Good, Jr., Phys. Rev. **108**, 882 (1957).

[2] The notation throughout the paper is the same as in reference 1.

[3] See, for example, H. Umezawa, *Quantum Field Theory* (Interscience Publishers, Inc., New York, 1956), Chap. IV, Sec. 3.

for which the quantization process was given by Fierz.[4] However, the connection between that general theory and the one considered here is complicated for arbitrary spin. Therefore, it is easier below to take advantage of the existence of a wave function and to quantize by using the coefficients of an expansion in the plane wave solutions. The connection between spin and statistics given in Sec. III is similar to the one given by Pauli[5] for bosons, but his argument for fermions does not apply here since the energy is positive definite for all spins.

II. COMMUTATION RULES

A convenient starting point is the expansion into plane waves of the solution of the wave equation and the auxiliary condition:

$$\phi(x)=(2\pi\hbar)^{-\frac{3}{2}}\int d\mathbf{p}\; a_+(\mathbf{p})u_+(\mathbf{p})\,\exp[i\hbar^{-1}(\mathbf{p}\cdot\mathbf{x}-cpt)]$$
$$+(2\pi\hbar)^{-\frac{3}{2}}\int d\mathbf{p}\; a_-^*(\mathbf{p})u_-(\mathbf{p})$$
$$\times\exp[i\hbar^{-1}(\mathbf{p}\cdot\mathbf{x}+cpt)]. \quad (3)$$

The relation between the expansion coefficients $a_+(\mathbf{p})$, $a_-^*(\mathbf{p})$ and the quantities $K_\pm(\mathbf{p})$ of reference 1 is

$$a_+(\mathbf{p})=p^{-\frac{1}{2}}K_+(\mathbf{p}),\quad a_-^*(\mathbf{p})=p^{-\frac{1}{2}}K_-(\mathbf{p}). \quad (4)$$

To quantize the theory, one assigns a_+ and a_-^* to be the destruction and creation operators fulfilling the commutation or anticommutation rules[6]

$$\begin{aligned}&[a_\pm(\mathbf{p}),a_\pm^*(\mathbf{q})]=\delta(\mathbf{p}-\mathbf{q}),\\ &[a_\pm(\mathbf{p}),a_\mp^*(\mathbf{q})]=0,\\ &[a_\pm(\mathbf{p}),a_\pm(\mathbf{q})]=0,\\ &[a_\pm(\mathbf{p}),a_\mp(\mathbf{q})]=0,\end{aligned} \quad (5)$$

so that $\phi(x)$ is the wave function operator for the field.

The covariance of the wave equation, auxiliary condition, and commutation rules with respect to continuous Lorentz transformations,

$$x_\alpha'=a_{\alpha\beta}x_\beta, \quad (6)$$

$$\psi'(x')=\exp(i\boldsymbol{\beta}\cdot\mathbf{s})\psi(x), \quad (7)$$

requires the transformation rule

$$p'^{\frac{1}{2}}a_\pm'(\mathbf{p}')=\exp[\pm i\eta_\pm(\mathbf{p})]p^{\frac{1}{2}}a_\pm(\mathbf{p}), \quad (8)$$

where $p_\alpha'=a_{\alpha\beta}p_\beta$ and $\eta_\pm(\mathbf{p})$ is some real function. To see that this is the correct transformation rule one first considers space rotations. Since

$$\exp(-i\boldsymbol{\beta}\cdot\mathbf{s})s_i\exp(i\boldsymbol{\beta}\cdot\mathbf{s})=a_{ij}s_j, \quad (9)$$

for space rotations, it is seen that

$$(c/s)\mathbf{s}\cdot\mathbf{p}'\exp(i\boldsymbol{\beta}\cdot\mathbf{s})u_\pm(\mathbf{p})=\pm cp'\exp(i\boldsymbol{\beta}\cdot\mathbf{s})u_\pm(\mathbf{p}).$$

The uniqueness of the eigenvectors except for a phase factor then implies that

$$\exp(i\boldsymbol{\beta}\cdot\mathbf{s})u_\pm(\mathbf{p})=\exp[i\eta_\pm(\mathbf{p})]u_\pm(\mathbf{p}').$$

The generalization of this result to continuous Lorentz transformations is

$$\exp(i\boldsymbol{\beta}\cdot\mathbf{s})p^s u_\pm(\mathbf{p})=\exp[i\eta_\pm(\mathbf{p})]p'^s u_\pm(\mathbf{p}'). \quad (10)$$

This equation is a consequence of the fact that a general Lorentz transformation can be written as products of space rotations and pure Lorentz transformations about the 3-axis, for which Eq. (10) holds with $\eta_\pm$ equal to zero.[1] The covariance of the wave equation and auxiliary condition then follows from Eqs. (8) and (10) by a proof similar to the one leading to Eq. (28) in reference 1. Also the transformation rule of Eq. (8), together with the property of the delta function

$$p'\delta(\mathbf{p}'-\mathbf{q}')=p\delta(\mathbf{p}-\mathbf{q}), \quad (11)$$

assures the covariance of the commutation rules, Eq. (5). With respect to the space and time reflections

$$x_i'=-x_i,\quad x_4'=x_4, \quad (12a)$$

$$x_i'=x_i,\quad x_4'=-x_4, \quad (12b)$$

the covariance of the theory is assured if the spinor components transform according to

$$\psi'(x')=[C\psi(x)]^*, \quad (13a)$$

$$\psi'(x')=[C\psi(x)]^C, \quad (13b)$$

and the operator relation is[7]

$$a_\pm'(\mathbf{p})=a_\mp(\mathbf{p}), \quad (14a)$$

$$a_\pm'(\mathbf{p})=a_\pm^C(-\mathbf{p}). \quad (14b)$$

In summary, the wave equation, auxiliary conditions, and commutation rules are covariant with respect to the full Lorentz group uniformly for all spins and both statistics.

The commutation rules for the wave function $\phi(x)$ are determined from Eqs. (3) and (5) to be

$$[\phi_m(\mathbf{x},t),\phi_n(\mathbf{x}',t)]_\pm=0, \quad (15)$$

$$[\phi_m(\mathbf{x},t),\phi_n^*(\mathbf{x}',t)]_\pm$$
$$=(2\pi\hbar)^{-3}\int d\mathbf{p}[(u_+)_m(u_+)_n{}^C$$
$$\pm(u_-)_m(u_-)_n{}^C]\exp[i\hbar^{-1}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')], \quad (16)$$

[4] M. Fierz, Helv. Phys. Acta **12**, 3 (1939).
[5] W. Pauli, Phys. Rev. **58**, 716 (1940).
[6] The superscript asterisk indicates the Hermitian conjugation of the creation and destruction operators. The symbols $[\ ]_-$, $[\ ]_+$, $[\ ]$ denote the commutator, the anticommutator, and either the commutator or anticommutator, respectively.

[7] It is convenient to choose a representation in which $a_\pm$ is real except perhaps for a phase factor as introduced by Eq. (8). This means that $a_\pm^C$ differs from $a_\pm$ at most by a phase factor.

where the plus signs apply for fermions and the minus for bosons. By operating as indicated by Eq. (2), one finds for bosons/fermions the commutation/anti-commutation rules for the spinor components $\psi(x)$ to be

$$[\psi_m(\mathbf{x},t),\psi_n(\mathbf{x}',t)]=0, \quad (17)$$

$$[\psi_m(\mathbf{x},t),\psi_n^*(\mathbf{x}',t)]$$
$$=\sum_l\{[H(\mathbf{x})/c]^{2s-1}\}_{ml}(2\pi\hbar)^{-3}\int d\mathbf{p}$$
$$\times[(u_+)_l(u_+)_n{}^C\pm(u_-)_l(u_-)_n{}^C]$$
$$\times\exp[i\hbar^{-1}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')], \quad (18)$$

where the plus sign applies for fermions with half-integral spin or bosons with integral spin, and the minus sign applies for fermions with integral spin or bosons with half-integral spin. These commutators can easily be found from the explicit formulas for $(u_\pm)_m$ given in reference 1.

In the special case of spin $\frac{1}{2}$ fermions, the $u_\pm$ form a complete set and Eq. (18) reduces to

$$[\psi_m(\mathbf{x},t),\psi_n^*(\mathbf{x}',t)]_+=\delta_{mn}\delta(\mathbf{x}-\mathbf{x}'). \quad (19)$$

In the special case of spin 1 bosons the factor of H permits the zero-eigenvalue function to be added in, so there is again a complete set and the equation reduces to

$$[\psi_m(\mathbf{x},t),\psi_n^*(\mathbf{x}',t)]_-=[H(\mathbf{x})/c]_{mn}\delta(\mathbf{x}-\mathbf{x}'). \quad (20)$$

One can see that this reduces to the usual commutators for the Maxwell field by specializing to the representation[8] in which $(s_i)_{jk}=-i\epsilon_{ijk}$ and by expressing ψ_j in terms of Hermitian operators E_j, B_j according to

$$\psi_j=(8\pi c)^{-\frac{1}{2}}(E_j+iB_j), \quad (21)$$

$$E_j=\tfrac{1}{2}(8\pi c)^{\frac{1}{2}}(\psi_j+\psi_j^*), \quad (22)$$

$$B_j=-\tfrac{1}{2}i(8\pi c)^{\frac{1}{2}}(\psi_j-\psi_j^*). \quad (23)$$

Then it is easily verified that E_j, B_j fulfil Maxwell's equations and so are to be identified with the operators of the electromagnetic field. From Eqs. (17) and (20) their commutation rules are found to be

$$[E_i(\mathbf{x},t),E_j(\mathbf{x}',t)]_-=[B_i(\mathbf{x},t),B_j(\mathbf{x}',t)]_-=0,$$
$$[E_i(\mathbf{x},t),B_j(\mathbf{x}',t)]_-=4\pi ic\hbar\epsilon_{ijk}(\partial/\partial x_k')\delta(\mathbf{x}-\mathbf{x}'),$$

in agreement with the usual treatment.[9]

III. CONNECTION BETWEEN SPIN AND STATISTICS

To discuss the integrals in Eq. (18), it is convenient to choose the 3-axis of the coordinates in the $\mathbf{x}-\mathbf{x}'$ direction and to substitute for $u_\pm(\mathbf{p})$ from Eq. (10) of reference 1 using polar coordinates. One finds

$$[\psi_m(\mathbf{x},t),\psi_n^*(\mathbf{x}',t)]$$
$$=\frac{1}{(2\pi\hbar)^3}\int\frac{d\mathbf{p}}{p}\frac{(2s)!(p_1+ip_2)^{s-m}(p_1-ip_2)^{s-n}}{2^{2s}[(s+m)!(s-m)!(s+n)!(s-n)!]^{\frac{1}{2}}}$$
$$\times[(p+p_3)^{m+n}\mp(-p+p_3)^{m+n}]$$
$$\times\exp[i\hbar^{-1}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')]$$
$$=2\pi(2\pi\hbar)^{-3}\delta_{mn}\int\int\sin\theta d\theta dp(2s)!$$
$$\times[2^{2s}(s+m)!(s-m)!]^{-1}p^{2s+1}(\sin^2\theta)^{s-m}$$
$$\times[(1+\cos\theta)^{2m}\mp(-1)^{2m}(1-\cos\theta)^{2m}]$$
$$\times\exp[i\hbar^{-1}p|\mathbf{x}-\mathbf{x}'|\cos\theta]$$
$$=\delta_{mn}\sum_k c_kI(2s+1,\,k). \quad (24)$$

Here the minus sign applies for half-integral fermions or integral bosons and the plus sign otherwise. The integrals I are defined by

$$I(2s+1,\,k)=\int\int\sin\theta d\theta dp\; p^{2s+1}\cos^k\theta$$
$$\times\exp[i\hbar^{-1}p|\mathbf{x}-\mathbf{x}'|\,\cos\theta], \quad (25)$$

and the c_k are numbers which can easily be found in any special case. It is seen that $2s+1$ is even/odd for half-integral/integral spin, and that only even/odd values of k arise for Fermi/Bose statistics. When $2s+1$ and k are both even or both odd, the integrals have the value

$$I(2s+1,\,k)=(-1)^{\frac{1}{2}(2s+1-k)}\frac{2\pi}{i^k}\frac{d^k}{d\alpha^k}\left(\frac{1}{\alpha}\frac{d^{2s-k}}{d\alpha^{2s-k}}\delta(\alpha)\right), \quad (26)$$

and otherwise they have the value

$$I(2s+1,\,k)=(-1)^{\frac{1}{2}(2s-k)}\frac{2}{i^k}\frac{d^k}{d\alpha^k}\left[\frac{1}{\alpha}\frac{d^{2s-k}}{d\alpha^{2s-k}}\left(\frac{\mathcal{P}(\alpha)}{\alpha}\right)\right], \quad (27)$$

where α is an abbreviation for $\hbar^{-1}|\mathbf{x}-\mathbf{x}'|$ and Heitler's notation[10] for the $\mathcal{P}$ function is used. One sees, therefore, that $[\psi_m(\mathbf{x},t),\psi_n^*(\mathbf{x}',t)]$ is zero when $\mathbf{x}\neq\mathbf{x}'$ for half-integral fermions and for integral bosons, and is not zero when $\mathbf{x}\neq\mathbf{x}'$ for integral fermions and for half-integral bosons. Furthermore, for half-integral fermions and integral bosons, $[\psi_m(\mathbf{x},t),\psi_n^*(\mathbf{x}',t')]$ must be zero for any two space-like events because the commutation rules were assigned covariantly. Since

$$[\psi_m(\mathbf{x},t),\psi_n^*(\mathbf{x}',t')]$$

as a function of $\mathbf{x}$, t satisfies the wave equation, it follows from Huygens' principle that the commutator

[8] R. H. Good, Jr., Phys. Rev. **105**, 1914 (1957).

[9] See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), second edition, p. 377.

[10] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1954), third edition, Sec. 8.

is zero for any two events $\mathbf{x},t$ and $\mathbf{x}',t'$ which are not on a light cone relative to each other. It is reasonable to assume that all physical theories will have the property that the commutator or anticommutator vanishes for points outside the light cone because it follows then that any local interaction which is linear in the spinor components of the interacting particles

$$H_{\text{int}}=\psi_A\psi_B\cdots+\psi_A^*\psi_B^*\cdots,$$

and which involves an even number of fermions, commutes with itself when evaluated for two spacelike events.

IV. OPERATOR ASSIGNMENTS

The assignments for the number of particles N, the energy $\mathcal{H}$, and the momentum P_j are

$$N=\int d\mathbf{p}[a_+^*a_++a_-^*a_-], \tag{28}$$

$$\mathcal{H}=\int d\mathbf{p}[cpa_+^*a_++cpa_-^*a_-], \tag{29}$$

$$P_j=\int d\mathbf{p}[p_ja_+^*a_++(-p_j)a_-^*a_-]. \tag{30}$$

One finds the q-number energy and momentum assignments by summing the number of particles operator a^*a times the eigenvalue of the c-number energy operator $(H/|H|)H$ and momentum operator $(H/|H|)p_j$. From the transformation properties of the $a_\pm$ given in Sec. II, it is seen that N is a scalar with respect to the full Lorentz group and that $(P_j,i\mathcal{H}/c)$ is a four-vector (a pseudovector under time reflection). It is interesting that the states of negative eigenvalues of H are treated the same way for both statistics although a hole theory interpretation for the bosons is impossible.

In contrast to the c-number theory, the energy operator is also the Hamiltonian for the system since

$$[\phi,\mathcal{H}]_-=i\hbar\partial\phi/\partial t, \tag{31}$$

for both statistics. Also the momentum is the space displacement operator since

$$[\phi,P_j]_-=-i\hbar\partial\phi/\partial x_j. \tag{32}$$

In terms of the wave function operator ϕ, the operators above have the form

$$N=\int d\mathbf{x}\,\phi^*\phi, \tag{33}$$

$$\mathcal{H}=\int d\mathbf{x}\,\phi^*(H/|H|)H\phi, \tag{34}$$

$$P_j=\int d\mathbf{x}\,\phi^*(H/|H|)p_j\phi, \tag{35}$$

for Bose statistics (infinite constants are disregarded and a sum on the spinor indices is understood). For Fermi statistics the anticommutation rules introduce and additional factor of $(H/|H|)$ so that

$$N=\int d\mathbf{x}\,\phi^*(H/|H|)\phi, \tag{36}$$

$$\mathcal{H}=\int d\mathbf{x}\,\phi^*H\phi, \tag{37}$$

$$P_j=\int d\mathbf{x}\,\phi^*p_j\phi. \tag{38}$$

One can also define an angular momentum operator by using the c-number results as a guide. One finds

$$\mathcal{J}_j=\int d\mathbf{x}\,\phi^*(H/|H|)(\mathbf{x}\times\mathbf{p}+\hbar\mathbf{s})_j\phi, \tag{39}$$

for Bose statistics and a similar formula without the $(H/|H|)$ factor for Fermi statistics. The quantity $\epsilon_{ijk}\mathcal{J}_k$ is the space-space part of

$$\Theta_{\mu\nu}=-2\int d^4p\,\delta(p_\rho p_\rho)p^{\frac{1}{2}}a^*(p_4/|p_4|)\times(p_\nu x_\mu-p_\mu x_\nu+\hbar T_{\mu\nu})p^{\frac{1}{2}}a, \tag{40}$$

where

$$a(\mathbf{p},p_4)=a_+(\mathbf{p}) \quad \text{when} \quad -ip_4>0,$$
$$=a_-^*(\mathbf{p}) \quad \text{when} \quad -ip_4<0.$$

This is a Lorentz tensor, regular under space reflection and a pseudotensor under time reflection.