Dispersion Theory Treatment of Pion Production in Electron-Nucleon Collisions*

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The production of mesons in electron-nucleon collisions is treated by using the method of dispersion relations. The result is analogous to that obtained by Chew, Low, Goldberger, and Nambu for photoproduction using the same method. Apart from the appearance of longitudinal and scalar components of the virtual photon polarization, the important difference is that charge and magnetic moments have to be multiplied by the appropriate nucleon electromagnetic form factors. Two formulas are given for the scattering amplitude: one is a static approximation which is valid if the electron invariant momentum transfer λ is less than 500 Mev; the other takes account of the recoil effects, and may be trusted within 30% even up to $\lambda\sim 1$ Bev. It is suggested that the measurement of the energy and angular distribution of the electron can offer an alternative method for the determination of the nucleon form factors.

1. INTRODUCTION

EXPERIMENTS on pion production in high-energy electron-proton collisions have recently been performed at Stanford.[1] This phenomenon is quite interesting from a theoretical point of view because it offers a further tool for the investigation of the electromagnetic structure of the nucleon.

The T-matrix element for the electro-pion production is given by

$$T=-ie\frac{\langle p_2q|j_\mu|p_1\rangle\bar{u}(s_2)\gamma_\mu u(s_1)}{(s_1-s_2)^2}, \tag{1}$$

where p_1 and s_1 are the four-momenta of the initial nucleon and electron, and p_2, s_2, and q those of the final nucleon, electron, and meson, respectively; j_μ is the four-dimensional nucleon current in the Heisenberg representation; and the u's are the electron spinors.[2] [For the derivation of Eq. (1), see Appendix I.]

Equation (1) shows that the evaluation of the cross section is reduced to that of the matrix element

$$H_\mu=i\langle p_2q|j_\mu|p_1\rangle. \tag{2}$$

The right-hand side of Eq. (2) looks formally identical with the matrix element for photomeson production. The only differences are that in the present case the "photon" momentum

$$k=s_1-s_2=p_2+q-p_1$$

has a modulus $\lambda\equiv(k^2)^{\frac{1}{2}}$ different from zero, and that the polarization vector ϵ_μ, which is proportional to $i\bar{u}(s_2)\gamma_\mu u(s_1)$, can have transversal, longitudinal, and scalar components.

In Appendix II we shall also show that, as in the case of the photomeson production, the matrix element H_μ has the phase of the final pion-nucleon scattering state as a consequence of the time-reversal invariance.[3]

The first experiment performed by Panofsky *et al.*[1] looked for the energy distribution of pions at a fixed angle. To be compared with this experiment, the differential cross section obtained from Eq. (1) must be integrated over the final electron states. In this process the important contributions come from the matrix elements corresponding to $\lambda\sim 0$ which are not much different from those for the photomeson production. Indeed, computations based on the static model[4,5] fit very well with the experimental data.

New experiments are being performed[6] in which the final electron is detected, and which involve virtual photons with λ of the order of several meson masses. In this case the matrix elements contributing to the cross section are essentially different from those of the photoproduction. One will first note that the appearance of large momentum transfer makes the static approximation much less reliable than in the case of photoproduction. Indeed, due to the imaginary mass of the virtual photon, its momentum $|\mathbf{k}|$ in the "center-of-mass" system can be large even if the final nucleon is produced in the resonance region. Quantitatively,

$$\begin{aligned}|\mathbf{k}|&=(\lambda^2+k_0^2)^{\frac{1}{2}}\geqslant\lambda\simeq[2\epsilon_1\epsilon_2(1-\cos\theta)]^{\frac{1}{2}},\\ k_0&=(W^2-M^2-\lambda^2)/2W,\end{aligned} \tag{3}$$

where ϵ_1 and ϵ_2 are the initial and final energies of the electron in the laboratory system and θ its scattering

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[1] Panofsky, Woodward, and Yodh, Phys. Rev. **102**, 1392 (1956).

[2] For simplicity, the spin and the isotopic-spin states are not explicitly indicated.

[3] K. Watson, Phys. Rev. **95**, 228 (1954); K. Aizu, *Proceedings of the International Conference on Theoretical Physics, Kyoto and Tokyo*, September, 1953 (Science Council of Japan, Tokyo, 1954); E. Fermi, Suppl. Nuovo cimento **2**, 58 (1955).

[4] R. H. Dalitz and D. R. Yennie, Phys. Rev. **105**, 1598 (1957); quoted in the following as DY.

[5] R. B. Curtis, Phys. Rev. **104**, 211 (1956).

[6] W. Panofsky (private communication).

angle, M is the nucleon mass, and $W=[-(p_2+q)^2]^{\frac{1}{2}}=[-(p_1+k)^2]^{\frac{1}{2}}$ is the total energy of the meson and nucleon in their center-of-mass system. Thus for $\epsilon_1 \sim 500$ Mev and at large angles, $|\mathbf{k}|$ can easily be comparable with M, resulting in large nucleon recoil.

In this paper we shall make use of the method of dispersion relations.[7] This will enable us to take into account correctly the effects of large λ and to evaluate the most important recoil contributions.

The dispersion relations in our case are similar to the ones given in B for photoproduction. A new feature is the appearance in the inhomogeneous terms of nucleon form factors, which depend on the invariant momentum transfer λ^2. The dispersion theory, which is believed to be valid under very general assumptions, shows that these form factors are exactly the same as the ones obtained by Hofstadter[8] in elastic electron-nucleon scattering. Since the electro-meson production process involves both the proton and neutron form-factor effects, it can offer an independent method, different from the electron-deuteron scattering, for determining the neutron form factors.

The task of obtaining a scattering amplitude which satisfies both the dispersion relation and the final-state theorem is not easy unless nucleon recoils are neglected. We shall therefore obtain the first approximation by solving the dispersion relations in the static limit. This solution takes into account the important effects due to the form factors, which are mainly caused by the meson cloud around a nucleon, but neglects the appreciable nucleon recoil contributions. For $\lambda \gtrsim 500$ Mev, the solution cannot be trusted.

To obtain an improved solution of the relativistic dispersion relation we shall proceed as follows:

First, we shall prove that the imaginary part of the static solution multiplied by a λ-dependent factor is still reliable even for large λ.

Then we shall insert this imaginary part in the right-hand side of the dispersion relations to generate a new real part. For large λ the new real part thus obtained will account for the more important recoil effects. It will be rather different from the real part obtained directly in the framework of the static approximation.

2. RELATIVISTIC DISPERSION RELATIONS

In this and in the next section we shall follow closely the method outlined in A and B, to which paper the reader is referred for all the details about the philosophy and the mathematical procedure. Unless explicitly stated otherwise, the notations used will be the same. Let us consider the product $H\cdot\epsilon \equiv H_\mu \epsilon_\mu$, where ϵ_μ is an arbitrary four-vector which at the end of the calculation will be identified with $ei\bar{u}\gamma_\mu u/k^2$. (We use the usual convention that repeated indices are summed.) Since $k^2 \neq 0$, and ϵ_μ has longitudinal and scalar components, the number of the fundamental relativistic and gauge-invariant forms will be six. These are taken to be

$$\begin{aligned} M_A &= \tfrac{1}{2} i\gamma_5\{\gamma,\gamma\}, && (+)\\ M_B &= 2i\gamma_5\{P,q\}, && (+)\\ M_C &= \gamma_5\{\gamma,q\}, && (-)\\ M_D &= 2\gamma_5(\{\gamma,P\}-\tfrac{1}{2}iM\{\gamma,\gamma\}), && (+)\\ M_E &= i\gamma_5\{k,q\}, && (-)\\ M_F &= \gamma_5\{k,\gamma\}, && (-) \end{aligned} \qquad (4)$$

where $P=\frac{1}{2}(p_1+p_2)$ and an abbreviation is used for the gauge-invariant combination

$$\{a,b\} \equiv a\cdot\epsilon b\cdot k - a\cdot k b\cdot\epsilon. \qquad (4')$$

The last two invariants were absent in the photoproduction case. The signs in parentheses indicate the behavior under crossing of the nucleon lines.

In terms of these invariant forms, $H\cdot\epsilon$ is written as

$$H\cdot\epsilon = M_A A + M_B B + \cdots + M_F F. \qquad (5)$$

Each of the invariant coefficients can further be decomposed, as was done in B, according to the isotopic dependence:

$$A_\alpha = \tfrac{1}{2}\{\tau_\alpha,\tau_3\}A^+ + \tfrac{1}{2}[\tau_\alpha,\tau_3]A^- + \tau_\alpha A^0, \quad \text{etc.} \qquad (5')$$

The 18 coefficients A^+, A^-, $\cdots F^0$ are now functions of the scalars

$$\nu = -P\cdot k/M, \quad \nu_B = q\cdot k/2M, \quad \text{and} \quad \lambda^2 = k^2. \qquad (6)$$

These coefficients satisfy dispersion relations of the form

$$\mathrm{Re}H_i(\nu,\nu_B,\lambda^2) = C_i(\nu,\nu_B,\lambda^2) + R_i(\lambda^2)\left(\frac{1}{\nu_B-\nu} \pm \frac{1}{\nu+\nu_B}\right) + \frac{1}{\pi}\int_{\nu_0}^{\infty} d\nu' \, \mathrm{Im}H_i(\nu',\nu_B,\lambda^2)\left(\frac{1}{\nu'-\nu} \pm \frac{1}{\nu'+\nu}\right), \qquad (7)$$

where $\nu_0 = \nu_B + 1 + 1/2M$,[9] the $\pm$ sign depending on the crossing symmetry. The residues R_i of the poles turn out to be

$$\begin{aligned} R[A^{V,S}] &= -fe^{V,S}(\lambda^2)/2,\\ R[B^{V,S}] &= fe^{V,S}(\lambda^2)/4M\nu_B,\\ R[C^{V,S}] &= R[D^{V,S}] = f\mu^{V,S}(\lambda^2)/2,\\ R[E] &= R[F] = 0, \end{aligned} \qquad (8)$$

where

$$\left.\begin{aligned} e^{V,S}(\lambda^2) &= e[F_1{}^p(\lambda^2) \pm F_1{}^n(\lambda^2)],\\ \mu'^{V,S}(\lambda^2) &= \mu_p' F_2{}^p(\lambda^2) \pm \mu_n F_2{}^n(\lambda^2), \end{aligned}\right\} \begin{aligned} &- \text{ for } V\\ &+ \text{ for } S \end{aligned} \qquad (8')$$

$$A^V \equiv A^+ \text{ or } A^-, \quad A^S \equiv A^0; \quad \text{etc.} \quad f^2/4\pi \sim 0.08.$$

F_1 and F_2 are the form factors for charge and magnetic moment distributions, respectively, as were introduced

[7] Chew, Low, Goldberger, and Nambu, Phys. Rev. 106, 1337 and 1345 (1957). These papers will be quoted as A and B, respectively.

[8] R. Hofstadter, Revs. Modern Phys. 28, 214 (1956). Other papers are quoted there.

[9] The pion mass is put equal to 1 throughout this paper.

by Hofstadter,[8] and μ_p' is the anomalous part of the magnetic moment of the proton.

It is interesting that the coefficients in Eq. (8) are identical with those for photoproduction except for the appearance of the form factors. The new invariants M_E and M_F do not have poles at $\nu=\pm\nu_B$. The terms C_i in Eq. (7) are in general polynomials in the variable ν, which our dispersion relations leave arbitrary. In order to determine these terms we shall use perturbation theory as a guide. It is seen that H also contains a meson current contribution which is independent of ν, and which does not have the nucleon form factor.[10] If we assume that the same result as the perturbation series can be obtained by iteration starting from the inhomogeneous terms of the dispersion relation, these terms must then be of the form

$$-g\frac{\gamma_5 i\gamma\cdot q}{2p_2\cdot q-1}\tau_\alpha\left(\frac{e^S+\tau_3 e^V}{2}\gamma_\mu-\frac{\mu'^S+\tau_3\mu'^V}{2}\sigma_{\mu\nu}k_\nu\right)$$
$$-g\left(\frac{e^S+\tau_3 e^V}{2}\gamma_\mu-\frac{\mu'^S+\tau_3\mu'^V}{2}\sigma_{\mu\nu}k_\nu\right)\tau_\alpha\frac{i\gamma\cdot q\gamma_5}{2p_1\cdot q+1}$$
$$+\frac{[\tau_\alpha,\tau_3]}{2}ig\gamma_5\left(\frac{e(2q-k)\cdot\epsilon}{(2q-k)\cdot k}+\frac{e^V-e}{\lambda^2}k\cdot\epsilon\right). \quad (9)$$

The first two terms are the nucleon current, and the third the meson current contributions. The last term has been formally inserted in order to preserve the over-all gauge invariance of the expression (9). Because of the continuity equation $k_\mu\bar{u}\gamma_\mu u=0$ for the electron, however, this term does not have any effect on the electron-meson production matrix element.

It is possible to understand Eq. (9) from a different point of view. As will be shown in Appendix I, a different representation may be given for H, leading to a new type of dispersion relations in which the relevant variable is ν_B. The bound-state contribution in these relations is just the meson current term in Eq. (9).

The last auxiliary term in Eq. (9) is independent both of ν and ν_B, so that neither of these representations can fix it. We think this is more than a coincidence.

Concluding this discussion, the assumption (9) allows us to write for the C's of Eq. (7)

$$\begin{aligned}&C_A=C_B=C=C_D=C_F=0,\\&C_E^+=C_E^0=0,\\&C_E^-=\frac{2Mf}{\lambda^2}\left(\frac{-2e}{2q\cdot k-\lambda^2}+\frac{e^V(\lambda^2)}{q\cdot k}\right).\end{aligned} \quad (10)$$

One can easily see that in the limit $\lambda^2\to 0$ the equations for photoproduction are reproduced.

[10] It should actually have other correction factors depending on ν_1 and λ, but this will be neglected here.

3. STATIC APPROXIMATION

In order to give an explicit solution to Eqs. (7), it is necessary to use the connection with pion-nucleon scattering discussed in Appendix II. This can easily be done only in the "center-of-mass" system in which $k+p_1=q+p_2=0$.

As already pointed out in DY it is sufficient to evaluate the matrix elements H for the case where the polarization vector ϵ has only space components; the matrix element of the time component of the four-current can be obtained by using the continuity equation. In order to take advantage of the analogy with photoproduction, it is convenient to split ϵ into a longitudinal and a transverse part:

$$\boldsymbol{\epsilon}_L=(\boldsymbol{\epsilon}\cdot\mathbf{k})\mathbf{k}/\mathbf{k}^2,\quad \boldsymbol{\epsilon}_T=\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_L. \quad (11)$$

The transverse part of H can be expanded in terms of magnetic and electric multipoles in the same way as for photoproduction. The longitudinal part gives rise to new multipoles (longitudinal multipoles) which under space and time inversion behave like the corresponding electric amplitudes. Among the new terms, only the longitudinal quadrupole of the form

$$i(3\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\mathbf{k}-\mathbf{k}^2\boldsymbol{\sigma}\cdot\mathbf{q})\mathbf{k}\cdot\boldsymbol{\epsilon}/\mathbf{k}^2 \quad (12)$$

can lead to a final $P_{\frac{3}{2}}$ pion-nucleon resonant state.

In order to solve the equations connecting the different amplitudes, we shall use the same procedure as in B. First we consider the equations obtained by neglecting all nucleon recoils. This static approximation allows the different multipoles to be essentially decoupled (except the mixing of the $l\pm\frac{1}{2}$ states of the same multipole).[11]

For the electric and magnetic amplitudes the results of B can be largely translated to our case by (1) replacing ϵ by ϵ_T; (2) inserting the appropriate form factors in the terms generated by the nucleon current; and (3) using for F_Q and F_M the expressions

$$F_Q=\frac{3}{2}\left(\frac{1}{2\omega^2+\lambda^2}\right)\left(\frac{1}{v^2}\right)\left\{\frac{2M}{E+M}\left(1+\frac{1-v^2}{2v}\ln\frac{1-v}{1+v}\right)+\frac{|\mathbf{q}|}{|\mathbf{k}|}\left(\frac{4}{3}v-\frac{2}{v}-\frac{1-v^2}{v^2}\ln\frac{1-v}{1+v}\right)\right\}, \quad (13)$$
$$F_M=\left(\frac{M}{E+M}\right)\left(\frac{3}{2\omega^2+\lambda^2}\right)\left(\frac{1}{v^2}\right)\left(1+\frac{1-v^2}{2v}\ln\frac{1-v}{1+v}\right),$$

[11] The form of the dispersion relations depends on the transformation properties of the different multipoles under crossing. In our case (unlike pion-nucleon scattering) one can only consider a crossing of the nucleon lines, which involves a charge conjugation. The simplest way of obtaining the static limit of this operation has been suggested by G. Feldman and P. F. Matthews [Phys. Rev. **102**, 1421 (1956)]. One first applies a CPT transformation to the matrix elements, and then one goes to the static limit. The result is that the matrix element corresponding to $-\omega$ is obtained from the one corresponding to $+\omega$ by transforming $\boldsymbol{\sigma}\to-\boldsymbol{\sigma}$, $\boldsymbol{\tau}\to-\boldsymbol{\tau}$, $\mathbf{k}\to-\mathbf{k}$, $\mathbf{q}\to-\mathbf{q}$.

where

$$v=2|\mathbf{q}||\mathbf{k}|/(2\omega^2+\lambda^2),\quad \omega=(1+\mathbf{q}^2)^{\frac{1}{2}},\quad E=(M^2+\mathbf{k}^2)^{\frac{1}{2}}.$$

The only longitudinal amplitude for which corrections to the Born approximation are appreciable is the longitudinal quadrupole. The equation for this amplitude differs from the equation for the electric quadrupole only in the inhomogeneous term

$$F_L=\frac{2-|\mathbf{k}|v/|\mathbf{q}|}{6(2\omega^2+\lambda^2)v^2}\left\{\frac{-2M}{E+M}\left(4+\frac{2}{v}\ln\frac{1-v}{1+v}\right)+\frac{|\mathbf{q}|}{|\mathbf{k}|v}\left(6-\frac{v^2-3}{v}\ln\frac{1-v}{1+v}\right)\right\}.\quad (13')$$

The procedure for solving the equation will, therefore, be the same as for the latter.

The recoil corrections to the static solution will be computed to the first order in $1/M$ and only for the leading terms, i.e., the magnetic dipole and the Born approximation terms. This enables us to use, also in this connection, the results of B.

Our results can be summarized in the following expression for the amplitude H.

We define $H^{\pm}$ and H^0 as

$$\langle p_2q|H\cdot\epsilon|p_1\rangle=(2\pi)^{-\frac{3}{2}}(2\omega)^{-\frac{1}{2}}\langle 2|\left[\tfrac{1}{2}\{\tau_\alpha,\tau_3\}H^+ +\tfrac{1}{2}[\tau_\alpha,\tau_3]H^- +\tau_\alpha H^0\right]|1\rangle.$$

Then, with $\mu(\lambda^2)=\mu'(\lambda^2)+e(\lambda^2)/2M$,

$$\begin{aligned}H^+/f=&\tfrac{2}{3}\{2\mathbf{q}\cdot\mathbf{k}\times\boldsymbol{\varepsilon}+i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\mathbf{q}\cdot\mathbf{k}-i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}\}\\&\times\{\tfrac{1}{2}[\mu^V(\lambda^2)/f^2q^3]+\tfrac{1}{3}ieF_M\}e^{i\delta_{33}}\sin\delta_{33}\\&-\tfrac{2}{3}\{i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_T\mathbf{q}\cdot\mathbf{k}+i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}_T\}eF_Qie^{i\delta_{33}}\sin\delta_{33}\\&-\tfrac{4}{3}i\{3\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\mathbf{k}-\mathbf{k}^2\boldsymbol{\sigma}\cdot\mathbf{q}\}(\mathbf{k}\cdot\epsilon/\mathbf{k}^2)eF_Lie^{i\delta_{33}}\sin\delta_{33}\\&+ie^V(\lambda^2)(\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{q}\cdot\boldsymbol{\varepsilon}/2M\omega),\end{aligned}\quad (14)$$

$$\begin{aligned}H^-/f=&\frac{1}{1+\omega/M}\left\{e^V(\lambda^2)i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}+e\frac{i\boldsymbol{\sigma}\cdot(\mathbf{k}-\mathbf{q})(2\mathbf{q}-\mathbf{k})\cdot\boldsymbol{\varepsilon}}{(k-q)^2+1}\right\}\\&-\tfrac{1}{3}\{2\mathbf{q}\cdot\mathbf{k}\times\boldsymbol{\varepsilon}+i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\mathbf{q}\cdot\mathbf{k}-i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}\}\\&\times\{\tfrac{1}{2}[\mu^V(\lambda^2)/f^2q^3]+\tfrac{1}{3}ieF_M\}e^{i\delta_{33}}\sin\delta_{33}\\&+\tfrac{1}{3}\{i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_T\mathbf{q}\cdot\mathbf{k}+i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}_T\}eF_Qie^{i\delta_{33}}\sin\delta_{33}\\&+\tfrac{2}{3}i\{3\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\mathbf{k}-\mathbf{k}^2\boldsymbol{\sigma}\cdot\mathbf{q}\}(\mathbf{k}\cdot\boldsymbol{\varepsilon}/\mathbf{k}^2)eF_Lie^{i\delta_{33}}\sin\delta_{33}\\&-ie^V(\lambda^2)\frac{\boldsymbol{\sigma}\cdot\mathbf{q}(\mathbf{q}+\mathbf{k})\cdot\boldsymbol{\varepsilon}}{2M\omega}-\frac{e^V(\lambda^2)-e}{\lambda^2}\mathbf{k}\cdot\boldsymbol{\varepsilon}i\boldsymbol{\sigma}\cdot(\mathbf{k}-\mathbf{q}),\end{aligned}$$

$$H^0/f=-i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\mu^S(\lambda^2)\frac{\mathbf{k}^2}{\omega}-\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\times(\mathbf{k}\times\boldsymbol{\varepsilon})}{\omega}\mu^S(\lambda^2)+i\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{q}\cdot\boldsymbol{\varepsilon}\frac{e^S(\lambda^2)}{2M\omega}.\quad (14')$$

Equation (14) is closely analogous to Eqs. (22.5)–(22.7) of B. As in that case, the resonant terms coming from the nucleon moments are much larger than the resonant terms coming from the meson current. Indeed the coefficients F_M, F_Q, and F_L decrease with increasing λ even more rapidly than the nucleon form factors.[12] This means that here also the important terms are the electric dipole, the resonant magnetic dipole, and the Born approximation part of the meson current. Thus a simplified version of Eq. (14) is

$$\begin{aligned}\frac{H^+}{f}=&\tfrac{2}{3}\{2\mathbf{q}\cdot\mathbf{k}\times\boldsymbol{\varepsilon}+i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\mathbf{q}\cdot\mathbf{k}-i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}\}\times\frac{\mu^V(\lambda^2)}{2f^2q^3}e^{i\delta_{33}}\sin\delta_{33},\\ \frac{H^-}{f}=&e^V(\lambda^2)i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}+e\frac{i\boldsymbol{\sigma}\cdot(\mathbf{k}-\mathbf{q})(2\mathbf{q}-\mathbf{k})\cdot\boldsymbol{\varepsilon}}{\lambda^2-2k\cdot q}\\&-\tfrac{1}{3}\{2\mathbf{q}\cdot\mathbf{k}\times\boldsymbol{\varepsilon}+i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\mathbf{q}\cdot\mathbf{k}-i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}\}\times\frac{\mu^V(\lambda^2)}{2f^2q^3}e^{i\delta_{33}}\sin\delta_{33},\\ H^0\sim&\,0.\end{aligned}\quad (15)$$

The procedure followed in deriving Eq. (14) suffers from the same kind of limitations as in A and B. A Legendre polynomial expansion for unphysical values of the nucleon momentum transfer $(p_1-p_2)^2=(q-k)^2$ has been carried out without a complete justification. Recoil effects are computed by assuming the convergence of a $1/M$ expansion. As already remarked in the Introduction, the k/M effects can become much more serious than in A and B. This situation can be illustrated by considering the difference between the "Breit system" (p_1+p_2 is at rest), which is the natural system for the dispersion relations, and the "center-of-mass" system ($p_1+k=p_2+q$ is at rest), which is the proper system for the phase-shift relations.[13] As λ increases, this difference becomes more pronounced. The relative velocity of the two systems, which is a measure for this difference, can be obtained from the formula

$$\frac{1}{(1-\beta^2)^{\frac{1}{2}}}=\frac{W^2+M^2+\frac{1}{2}(\lambda^2-1)}{2W[M^2+\frac{1}{4}(\lambda^2-1-2M\nu_B)]^{\frac{1}{2}}},\quad (16)$$

where ν_B is the parameter defined in Eq. (6). β ranges approximately between $(\beta_1\pm\beta_2)/2$, depending on the meson production angle, where β_1 and β_2 are the initial and final nucleon recoil velocities in the center-of-mass system. For large λ, which means large β_1, β will also be large.

Another trouble peculiar to the virtual photon arises when its time component k_0 vanishes.[4] According to

[12] With a root mean square radius of 0.8×10^{-13} cm, the corresponding form factor will be reduced to $\frac{1}{2}$ for $\lambda\sim600$ Mev.

[13] Of course the two systems coincide in the static limit.

Eq. (3) and the fact that $W \geqslant M+1$, this can happen for $\lambda^2 \geqslant 2M+1$ or $\lambda \gtrsim 500$ Mev. In such a case the longitudinal component of the nucleon current must vanish according to the continuity equation, but it is not satisfied by our static results where k_0 and $q_0=\omega$ are not distinguishable. As k_0 becomes small, the time component becomes predominant over the longitudinal current in the ratio $|\mathbf{k}|/k_0$ so that it will not be advisable to compute the former from the latter by means of the continuity equation. Rather we should have calculated the charge component independently, and derive the longitudinal part from it. However, since the recoil correction is also large when this is necessary, it seems reasonable to limit the validity of our results to such regions where λ is less than 500 Mev.

The procedure outlined in the next section will enable us to drop this limitation, which is a very unpleasant one since the form factors do depend on just this quantity λ.

4. EVALUATION OF THE RECOIL EFFECTS FOR LARGE MOMENTUM TRANSFERS

The previous discussion has shown that Eqs. (14) and (15) are not reliable for $\lambda \gtrsim 500$ Mev because the high momentum of the initial "photon"-nucleon state produces considerable higher multipoles. However, even in this case the final pion-nucleon state is still at low energy (in its center-of-mass system), and therefore is completely dominated by the 33 resonance. This means that the only amplitudes having an appreciable imaginary part are still the three multipoles leading to the resonant state. Thus if one introduces a multipole expansion in the dispersion relations, one will obtain three equations connecting the three amplitudes with themselves; the remaining equations give the real amplitudes of the other multipoles in terms of the imaginary parts of the resonant ones.[14]

Our procedure now will be as follows: first we solve the equations for the resonant amplitudes, then insert the imaginary part of the solution thus obtained into the right-hand side of the dispersion relation to create the real part of the whole scattering amplitude. This procedure is successful because the recoil is more effective in producing new multipoles than in any large change of the resonant amplitudes, which would only be of the order of 30%.

Let us fix our attention on the equations for the resonant amplitudes. The effects of the recoil appear only in the crossed terms of the form

$$\frac{R_i}{\nu_B+\nu}, \quad \int \frac{\mathrm{Im} H_i(\nu')}{\nu'+\nu} d\nu'.$$

The contribution of the second term is very small in the static case because of the large denominator (compared to that for the uncrossed terms) and the factor $\frac{1}{9}$ coming from the crossing matrix for the spin and isotopic spin operators. It is easy to verify that the situation does not change appreciably for large recoils. It follows then that each 33 amplitude is mainly coupled with itself and thus the only important sources of recoil are the terms which constitute the whole Born contribution to the 33 state.

We will neglect here the resonant terms created by the meson current; they were evaluated in the last section, and their effect as compared to the nucleon magnetic dipole term was found to be even smaller for large λ. The electric and longitudinal quadrupole terms generated by the nucleon current are purely recoil effects. A direct calculation shows that they are negligible as compared to the magnetic dipole.

In solving the equation for the nucleon magnetic dipole term, the static approximation for the spinor matrix elements of A, B, C, and D does not introduce an appreciable error. The main source of deviation from the static approximation comes from the denominator

$$\frac{1}{\nu_B+\nu} \sim \frac{1}{2E\omega(1+\alpha\cos\theta)}, \quad \alpha = |\mathbf{k}|\,|\mathbf{q}|/E\omega,$$

the static limit of which was $1/2M\omega$. It is easy to take care of the ratio M/E since (to within 5%) one has $E=(M^2+\mathbf{k}^2)^{\frac{1}{2}} \sim (M^2+\lambda^2)^{\frac{1}{2}}$, which is a pure parameter. The effect of the $\cos\theta$ term is more difficult to treat, since the coefficient β depends on q/ω which can vary between 0 and 1. We shall here neglect this effect. This will introduce an error of the order $\frac{3}{5}\alpha^2$, ranging from 10% for $\lambda=500$ Mev to 25% for $\lambda=1$ Bev. In this approximation the solution of the magnetic dipole amplitude will differ from the static limit only by a factor $M/(M^2+\lambda^2)^{\frac{1}{2}}$.

The imaginary part of the scattering amplitude in the c.m. system will be[15]

$$(\delta_{\alpha 3}-\tfrac{1}{3}\tau_\alpha\tau_3)(3\mathbf{q}\cdot\mathbf{k}\times\boldsymbol{\varepsilon}-\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{k}\times\boldsymbol{\varepsilon}) \times M(M^2+\lambda^2)^{-\frac{1}{2}}\mu^V(\lambda^2)h(w), \quad (17)$$

where

$$w=(W^2-M^2)/2M\sim\omega, \quad h(w)=\sin^2\delta_{33}(\omega)/q^3;$$

ω and q are known functions of W or w.

In order to insert this expression in the right-hand side of Eq. (7), we express the spin dependence in the center-of-mass system in Eq. (17) in terms of the six relativistic invariants. We then get

[14] The main simplification in the case of the static approximation was that only states of the same orbital angular momentum are coupled. Here this is not the case because of the large difference between the Breit and center-of-mass systems.

[15] The covariant form of the $P_{\frac{3}{2}}$ projection operator in Eq. (17) is given by $\{3q'\cdot(k\times\epsilon)'-q'\cdot\gamma(k\times\epsilon)'\cdot\gamma\}(-iP'\cdot\gamma+M)/2W$, where $P'=p_1+k=p_2+q$, $P'^2=-W^2$, $q_\mu'=q_\mu-P_\mu'P'\cdot q/P'^2$, and $(k\times\epsilon)_\mu'=-i\epsilon_{\mu\nu\lambda\rho}k_\nu\epsilon_\lambda P_\rho'/M$. Making use of the Dirac equation, the above operator may be transformed into a linear combination of the six invariants. In Eq. (18) only leading terms of the expansion of S_i in $|\mathbf{q}|/M$ are kept.

$$\text{Im}H_i = S_i M \mu^V(\lambda^2) h(w)/(M^2+\lambda^2)^{\frac{1}{2}}, \quad (i=1,\cdots 16)$$

$$\begin{aligned}
&S_i^0 = 0,\\
&S_A^+ = -2S_A^- = \tfrac{2}{3}\left(w - \frac{\lambda^2}{M} + 6\nu_B\right),\\
&S_B^+ = -2S_B^- = -\left(1 - \frac{\lambda^2}{3M\nu_B}\right)\Big/ M,\\
&S_C^+ = -2S_C^- = -\tfrac{2}{3}\left(1 + \frac{3\lambda^2}{4M^2}\right),\\
&S_D^+ = -2S_D^- = \tfrac{4}{3},\\
&S_E^+ = -2S_E^- = \frac{2w}{3M\nu_B},\\
&S_F^\pm = 0.
\end{aligned} \tag{18}$$

By substituting Eqs. (17) and (18) into Eq. (7), we obtain for our scattering amplitude the final relativistic expression

$$\begin{aligned}
H_i = {}& C_i(\nu_1,\lambda^2) + R_i(\lambda^2)\left(\frac{1}{\nu_B - \nu} \pm \frac{1}{\nu_B + \nu}\right)\\
&+ \frac{M}{(M^2+\lambda^2)^{\frac{1}{2}}}\mu^V(\lambda^2)\left\{\frac{1}{\pi}P\int_{1+1/2M} h(w')S_i(w'+\nu_B,\nu_B,\lambda^2)\right.\\
&\left.\times\left(\frac{1}{w'+\nu_B-\nu} \pm \frac{1}{w'+\nu_B+\nu}\right) + ih(\nu-\nu_B)\right\}.
\end{aligned} \tag{19}$$

As usual, for practical applications, the upper limit of integration has to be taken around 4μ where the effects of the 33 resonance are no longer important.

5. DISCUSSION

Our results show that experiments on electron-meson production for large electron angles can be very important for a measurement of the nucleon form factors.

More specifically, the terms $e^V(\lambda^2)$ and $\mu^V(\lambda^2)$ are linear combinations of the proton and neutron form factors. Experimental measurement of those factors can give an independent check of the neutron form factors obtained from deuteron experiments.

An advantage of the method proposed here is the possibility of performing independent measurements of the form factors by looking at the energy spectrum of the final electron for different fixed scattering angles (or better, for different fixed λ).

In this way it is possible both to check experimentally the physical assumptions upon which our theory is built and to measure the charge and magnetic moment distributions of the neutron. For $\lambda < 500$ Mev, the simple formula (15) shows only two main differences from the result of the Chew-Low theory:

(1) The S-wave term is multiplied by a nucleon charge form factor.

(2) The form factors are functions of the relativistic momentum transfer k^2 and not of $\mathbf{k}^2$. As already pointed out, at higher λ important corrections of kinematic nature have to be taken into account. These corrections have the main effect of producing higher multipoles. The dynamics of the system, represented by the 33 resonance in the final pion-nucleon state, is still unchanged.

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APPENDIX I

We shall provide in this Appendix a formal derivation of Eq. (1). The proof will be based on the Heisenberg representation technique developed by Lehmann, Symanzik, and Zimmermann.[16]

We can extend the definition given in LSZ for a boson field to the case of a fermion field:

$$\psi_{ks}(t) = -i\int \bar{f}_{ks}\gamma_4\psi(\mathbf{x},t)d^3x, \tag{A1}$$

where $\bar{f}_{ks}(\mathbf{x},t)$ is a negative frequency solution of the Dirac equation for the electron

$$(\gamma^T\cdot\partial/\partial x - m)\bar{f}_{ks} = 0,$$

with momentum k and spin s. The same definitions apply to incoming and outgoing fields, which turn out to be time-independent.

Using the equation of motion, one can easily obtain

$$\psi_{ks}^{\text{out}} = \psi_{ks}^{\text{in}} - \int \bar{f}_{ks}(x)O(x)d^4x, \tag{A2}$$

$$\psi_{ks}(0) = \psi_{ks}^{\text{in}} - \int \bar{f}_{ks}(x)O(x)\eta(-x_0)d^4x, \tag{A2'}$$

$$\bar{\psi}_{ks}(0) = \bar{\psi}_{ks}^{\text{in}} - \int \bar{O}(x)f_{ks}(x)\eta(-x_0)d^4x, \tag{A2''}$$

where

$$O(x) = ie\gamma\cdot A(x)\psi(x),\ \bar{O}(x) = ie\bar{\psi}(x)\gamma\cdot A(x), \tag{A3}$$
(+renormalization terms).

[16] Lehmann, Symanzik, and Zimmermann, Nuovo cimento 1, 205 (1955); quoted as LSZ hereafter.

Using Eq. (A2) and recalling that any incoming operator is connected to the corresponding outgoing operator by the transformation $\psi^{\text{out}}=S\psi^{\text{in}}S^{-1}$, we can write the matrix element for electropion production of pions in the form

$$\langle p_2qs_2|T|p_1s_1\rangle=i\langle p_2qs_2|\bar{O}(0)|p_1\rangle u(s_1). \tag{A4}$$

Using Eq. (A2″) and the relation:

$$\langle p_2qs_2|\bar{O}(0)|p_1\rangle=\langle p_2q|[\psi_{s_2}{}^{\text{in}},\bar{O}(0)]_+|p_1\rangle,$$

we obtain

$$-i\langle p_2qs_2|T|p_1s_1\rangle=\int\langle p_2q|[\psi_{s_2}(\mathbf{x},0),\bar{O}(0)]|p_1\rangle d^3x$$
$$+\int f_{s_2}(x)\langle p_2q|[O(x),\bar{O}(0)]_+\eta(-x_0)|p_1\rangle d^4x. \tag{A5}$$

In the lowest order in the electromagnetic coupling constant only the equal-time commutator gives a contribution. Using Eqs. (A1) and (A3), this turns out to be

$$\langle p_2q|[\psi_{s_2}(\mathbf{x},0),\bar{O}(0)]_+|p_1\rangle$$
$$=-ie\bar{u}(s_2)\gamma_\mu\langle p_2q|A_\mu(0)|p_1\rangle. \tag{A6}$$

With the aid of the field equations, the matrix element $\langle p_2q|A_\mu(0)|p_1\rangle$ can be expressed in terms of the current j_μ generated by the nucleon and meson fields. We thus obtain

$$\langle p_2qs_2|T|p_1s_1\rangle$$
$$=-ie\langle p_2q|j_\mu|p_1\rangle\bar{u}(s_2)\gamma_\mu u(s_1)/(s_1-s_2)^2. \tag{A7}$$

The meson wave function in $\langle p_2q|j_\mu|p_1\rangle$ may again be "taken out" by using equations similar to Eqs. (A2) for the meson fields ϕ_α:

$$H_\mu=\langle p_2q|j_\mu|p_1\rangle$$
$$=iv_\alpha{}^*(q)\int\{\langle p_2|\eta(x_0)[j_\alpha(x),j_\mu(0)]|p_1\rangle e^{-iq\cdot x}d^4x$$
$$+\langle p_2|[\phi_\alpha(\mathbf{x},0),j_\mu(0)]|p_1\rangle e^{-i\mathbf{q}\cdot\mathbf{x}}d^3x\}, \tag{A8}$$

where $j_\alpha=(-\Box+\mu^2)\phi_\alpha$ and $v_\alpha(q)$ is the free-meson wave function. The first term is a causal amplitude of the familiar variety, whereas the second term, being an equal-time commutator, does not depend on the variable ν used in dispersion relations.

The second term is therefore part of the constant terms in the dispersion relations for H, and actually corresponds, apart from a renormalization constant, to the meson current contribution in the Born approximation.

To determine the nature of the constant terms in the dispersion relations, it is helpful to give for H a different representation by taking out the final nucleon instead of the meson:

$$H_\mu=i\bar{u}(p_2)\int\{\langle q|\eta(x_0)[g(x),j_\mu(0)]|p_1\rangle e^{-ip_2\cdot x}d^4x$$
$$-i\langle q|[\gamma_4\psi(\mathbf{x},0),j_\mu(0)]|p_1\rangle e^{-i\mathbf{p}_2\cdot\mathbf{x}}\}, \tag{A8′}$$

where $g(x)=(\gamma\cdot\partial/\partial x+M)\psi(x)$. The first term of Eq. (A8′) is a Fourier transform of a causal amplitude with a Fourier variable p_2+k. This will allow a dispersion relation to be written down with respect to a variable different from ν, and its unphysical (crossed) imaginary part will be related to the matrix element $\langle q|j_\mu|n\rangle$, n being any state with nucleon number zero. In particular, the contribution coming from the one-meson state gives rise to a pole which exactly corresponds to the meson current term in Eq. (7) with the correct renormalized electric charge and mesonic coupling constant.

APPENDIX II

The theorem which we are going to prove in this Appendix is a general one of which the present case is a particular application. Naturally it also contains as a special case the theorem proved independently by Watson, Fermi, and Aizu,[3] on the phases of the photoproduction matrix elements.

The proof is carried out most conveniently in the Heisenberg representation. In this representation, let $A(x)$ be an operator, and $|\alpha^{\text{in}}\rangle$ and $|\beta^{\text{in}}\rangle$ incoming states as defined, for instance, in LSZ.

We shall make the following assumptions:

(a) Hermitian character of the operator $A(x)$;
(b) unitarity of the S matrix;
(c) the usual transformation properties under time reversal of operators and state vectors in the Heisenberg representation.

Let us consider the matrix element

$$\begin{gathered}\langle\beta^{\text{in}}|A(0)|\alpha^{\text{in}}\rangle^*=\langle\alpha_t{}^{\text{out}}|A_t(0)|\beta_t{}^{\text{out}}\rangle,\\ A_t(0)=TA(0)T^{-1}.\end{gathered} \tag{A9}$$

Here $|\alpha_t{}^{\text{out}}\rangle$ and $|\beta_t{}^{\text{out}}\rangle$ are the time reversed states of $|\alpha^{\text{in}}\rangle$ and $|\beta^{\text{in}}\rangle$, respectively; T is the Wigner time-reversal operator.

Using the relation between "incoming" and "outgoing" operators,

$$S\phi^{\text{in}}S^\dagger=\phi^{\text{out}},$$

we can write Eq. (A9) as

$$\langle\alpha^{\text{in}}|A(0)|\beta^{\text{in}}\rangle^*=\langle\alpha_t{}^{\text{in}}|S^\dagger A_t(0)S|\beta_t{}^{\text{in}}\rangle. \tag{A10}$$

Equation (A10) forms the basis of our theorem.

We consider the case where $|\alpha^{\text{in}}\rangle$ is a state of one nucleon, $|\beta^{\text{in}}\rangle$ a state of one nucleon and one meson, and $A(x)$ the four-current operator $j_\mu(x)$ for which

$$j_\mu{}^t(0)=\theta_\mu j_\mu(0),\quad \begin{array}{ll}\theta_\mu=1 & \text{for}\quad \mu=0\\ \theta_\mu=-1 & \text{for}\quad \mu=1,2,3.\end{array} \tag{A11}$$

Equation (A9) then becomes:

$$\langle\alpha^{\text{in}}|j_\mu(0)|\beta^{\text{in}}\rangle^* =\sum_n \theta_\mu\langle\alpha_t{}^{\text{in}}|j_\mu(0)|n^{\text{in}}\rangle\langle n^{\text{in}}|S|\beta_t{}^{\text{in}}\rangle, \tag{A12}$$

where the $|n^{\text{in}}\rangle$ are a complete set of "incoming" states. Retaining only those states $|n^{\text{in}}\rangle$ with one nucleon and one meson (which is certainly valid below many-meson thresholds), and choosing a representation in which J, I, and L are diagonal, we obtain

$$\langle j',m',i'^{\text{in}}|j_\mu(0)|j,m,i,l\rangle^* =(-1)^{m'-m}\theta_\mu\langle j',-m',i'^{\text{in}}|j_\mu(0)|j,-m,i,l\rangle \times e^{2i\delta_{jil}}. \tag{A13}$$

Here we have made use of the fact that the S matrix for pion-nucleon scattering is diagonal in this representation, and that

$$T|j,m,i\rangle=(-1)^m|j,-m,i\rangle.$$

δ_{jil} is the pion-nucleon scattering phase shift.

It will be sufficient to evaluate Eq. (A12) for $\mu=0$ and 3; the relation for the other components can be obtained by performing a rotation. In both cases the only nonvanishing matrix elements are for $m=m'$. Performing an inversion of the y axis in order to transform $-m$ into $+m$, we finally obtain

$$\langle j,m,i|j_0(0)|j,m,i,l\rangle^*=\langle j,m,i|j_0(0)|j,m,i,l\rangle e^{2i\delta_{jil}}, \tag{A14}$$

$$\langle j,m,i|j_3(0)|j,m,i,l\rangle^* =-\langle j,m,i|j_3(0)|j,m,i,l\rangle e^{2i\delta_{jil}}. \tag{A14$'$}$$

APPENDIX III

In DY it is shown that the electropion production cross sections can be obtained in terms of Φ_e, defined as

$$\Phi_e=\tfrac{1}{2}\,\mathrm{Tr}[(-ip_1\cdot\gamma+M)\bar{H}_\mu(-ip_2\cdot\gamma+M)H_\nu] \times(\delta_{\mu\nu}+S_\mu S_\nu/\lambda^2), \tag{A15}$$

where $s_\mu=(s_1+s_2)_\mu$. The evaluation of Φ_e starting from our expression for H is not an easy task because of the interference terms between the different invariants. We want to propose here a method which could simplify such a calculation.

Let us introduce the following definitions:

$$\begin{aligned}&\Delta_\mu=(p_1-p_2)_\mu/2,\quad N_\mu=\epsilon_{\mu\nu\rho\sigma}P_\nu k_\rho\Delta_\sigma,\\ &\alpha=S-NS\cdot N/N^2=a_1P+a_2\Delta+a_3k,\\ &\beta=b_1P+b_2\Delta+b_3k,\end{aligned} \tag{A16}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor and

$$\begin{aligned}&a_1=\left(\frac{S\Delta k}{P\Delta k}\right)\Big/N^2,\quad a_2=\left(\frac{PSk}{P\Delta k}\right)\Big/N^2,\\ &a_3=\left(\frac{P\Delta S}{P\Delta k}\right)\Big/N^2,\\ &b_1=\left(\frac{Sk}{\Delta k}\right)\Big/\lambda^2,\quad b_2=\left(\frac{\alpha k}{kP}\right)\Big/\lambda^2,\\ &b_3=\left(\frac{\alpha k}{P\Delta}\right)\Big/\lambda^2,\\ &N^2=\left(\frac{P\Delta k}{P\Delta k}\right).\end{aligned} \tag{A16$'$}$$

The notation $(c_1c_2c_3/d_1d_2d_3)$ means $\det|c_i\cdot d_j|$.

It is easy to verify that α, β, k, N form a complete orthogonal set of vectors and that

$$\alpha^2=\lambda^2/N^2,\quad \beta^2=-a_1b_2+a_2b_1. \tag{A17}$$

We can now introduce the six new invariants:

$$\begin{aligned}&M_1=i\gamma_5\alpha\cdot\epsilon,\quad && M_2=i\gamma_5\beta\cdot\epsilon,\\ &M_3=\gamma\cdot N\alpha\cdot\epsilon,\quad && M_4=\gamma\cdot N\beta\cdot\epsilon,\\ &M_5=iN\cdot\epsilon,\quad && M_6=\gamma_5\gamma\cdot NN\cdot\epsilon,\end{aligned} \tag{A18}$$

and consider the expansion of the matrix T in terms of those invariants:

$$T=f_1M_1+\cdots f_6M_6. \tag{A19}$$

The new coefficients f_i are related to the older ones by

$$\begin{aligned}
f_1&=\frac{\lambda^2}{\Delta^2\alpha^2}[-b_3A+2(2b_3+b_2)\Delta^2B+M^2b_1C-2b_1\Delta^2E+Mb_1F],\\
f_2&=-\frac{N^2}{\Delta^2\beta^2}[-a_3A+2(2a_3+a_2)\Delta^2B+M^2a_1C-2a_1\Delta^2E+Ma_1F],\\
f_3&=\frac{1}{\Delta^2\alpha^2}[-Ma_1A+(a_1P\cdot q+a_2\Delta\cdot q)C-2a_1\Delta^2D+(a_1P\cdot k+a_2\Delta\cdot k)F],\\
f_4&=\frac{1}{\Delta^2\beta^2}[-Mb_1A+(b_1P\cdot q+b_2\Delta\cdot q)C-2b_1\Delta^2D+(b_1P\cdot k+b_2\Delta\cdot k)F],\\
f_5&=-\frac{1}{P^2}(A+2D),\\
f_6&=\frac{1}{N^2P^2}[-MP\cdot kA+P^2q\cdot kC-2P\cdot k\Delta^2D+\lambda^2P^2F].
\end{aligned} \tag{A20}$$

Introducing (A18) into (A15), we obtain

$$\Phi_e=|\Delta^2\alpha^2(1-\alpha^2/S^2)|(f_1^2+|N^2|f_3^2) +|\Delta^2\beta^2|(f_2^2+|N^2|f_4^2)+|N^2P^2|(f_5^2+|N^2|f_6^2). \tag{A21}$$