Possibility of Formulation of a Theory of Strongly Interacting Fermions

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A fermion field is investigated with the interaction Lagrangian density equal to $g(\bar{\psi}O_j\psi)\cdot(\bar{\psi}O_j\psi)$. This point interaction is considered as a limit of an extended one, where it is supposed that the interaction vanishes if the momentum of a particle exceeds Λ and/or if the momentum transferred in a collision of two particles exceeds λ. The relation between Λ and λ is such as to make the quantity $\lambda^2/\ln(\Lambda/\lambda)$ arbitrarily small as $\lambda\rightarrow\infty$ and $\Lambda\rightarrow\infty$. This choice of the limiting procedure considerably simplifies the investigation of the theory. It is shown that in the limit $\lambda\rightarrow\infty$, $\Lambda\rightarrow\infty$, the physical interaction between particles vanishes in all types of four-fermion interactions. The case of two interacting fields ψ and χ with different isotopic spin is also considered. Going over to the local theory, the physical interaction vanishes in this case as well.

This result shows that in the cases considered no strongly interacting fermion theory can be constructed. In the case of the weak interaction, although no logically consistent theory can be built up, there does exist the perturbation theory, as in electrodynamics, which is valid for sufficiently small energies.

1. INTRODUCTION

THE transition to the limit of a point interaction in electrodynamics[1] or meson theories[2] entails a difficulty which is connected with vanishing of the renormalized charge and the disappearance of physical interactions between particles. It would seem to be of interest to ascertain whether this difficulty can be overcome by replacing the Yukawa-type interaction $g_0(\bar{\psi}O_j\psi)\varphi_j$ between fermions and bosons by other types of interactions, such as that between bosons alone or that between fermions alone. The difficulty created by the vanishing of the renormalized charge remains in the case of a system of bosons whose interaction is determined by the operator

$$\frac{g_0}{4!}\int\varphi^4(x)dv.$$

In this case the dependence of g_c on g_0 and Λ (Λ is the cutoff momentum) has the form[3]

$$g_c=\frac{g_0}{1+\frac{3}{2}g_0\ln(\Lambda^2/\mu^2)},\tag{1}$$

and g_c vanishes for $\Lambda\rightarrow\infty$ and any arbitrary dependence of g_0 on Λ, providing that g_0 is positive. Negative values of g_0 (for which, in the limit $\Lambda\rightarrow\infty$, g_c may not vanish) are in general inadmissible because no stationary states of a boson system exist for $g_0<0$. Indeed, for boson fields a classical limiting case exists in which each state may contain many particles. For $g_0<0$ the energy of the classical field φ,

$$\tfrac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x_\mu}\right)^2+\mu^2\varphi^2\right]+\frac{g_0}{4!}\varphi^4(x),$$

is not positive definite and can decrease indefinitely with increase of the field amplitude φ. Physically this means that it should be energetically possible for an infinite number of particles to be created from vacuum. Thus the vacuum cannot exist for $g_0<0$.

The constant g_0 for the Fermi interaction

$$V=2\pi^2g_0\int(\bar{\psi}O_j\psi)(\bar{\psi}O_j\psi)dv\tag{2}$$

(where the O_j are the ordinary spin and isotopic spin operators for fermions) can have any sign, since the occupation numbers cannot exceed unity and stationary states exist for any sign of g_0. In the given case, the turning on of the interaction simply leads to a redistribution of the levels of negative and positive energy. The new stationary state with minimal energy which arises after the interaction is "turned on" is the one of a physical vacuum.

Hence if the relation between g_c and g_0 in this case were also determined by a formula analogous to (1), the renormalized charge g_c would not vanish for $g_0<0$. Meson theories could be based on interaction (2) and the mesons from the very start would be similar (in the sense of the Fermi-Yang concept[4]) to nonlocal formations of fermions.

In the following we shall consider the possibility of setting up a theory of this type.[5]

2. EQUATION FOR THE VERTEX OPERATOR

If the interaction has the same form as (2), the matrix elements will contain quadratic and logarithmic divergences. We shall cut off the diverging integrals by assuming that interaction (2) is somewhat "smeared out": it will be assumed that the interaction vanishes in the momentum representation if momenta P and

[1] L. D. Landau and I. Ya. Pomeranchuk, Doklady Akad. Nauk S.S.S.R. **102**, 489 (1955).
[2] I. Ya. Pomeranchuk, Doklady Akad. Nauk S.S.S.R. **103**, 1005 (1955); **104**, 51 (1955); **105**, 461 (1955).
[3] Pomeranchuk, Sudakov, and Ter-Martirosyan, Phys. Rev. **103**, 784 (1956). This formula was derived for $g_0<1$; if, however, two cutoff momenta are introduced it can be shown that it will be valid for an *arbitrary* positive value of g_0.
[4] E. Fermi and C. Yang, Phys. Rev. **76**, 1739 (1949).
[5] An interaction of the same type as (2) has been discussed in a number of papers. See for example, B. Jouvet, Nuovo cimento **5**, 1 (1957).

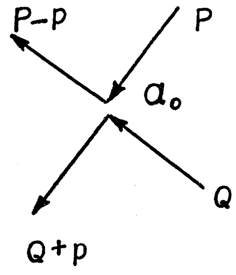

FIG. 1. Simplest four-fermion vertex.

Q, directed along the fermion lines according to Fig. 1,[6] exceed some limiting momentum Λ or if the momentum p transferred from one line to another exceeds λ, where $\lambda \ll \Lambda$.[7] In order to go over to the limit of a point interaction one should put $\Lambda \to \infty$, $\lambda \to \infty$, the relation between Λ and λ being arbitrary.

If the theory is internally consistent, one may expect the result of the limiting process to be independent of the nature of the transition. The latter, however, should not violate the general conditions required by any physically reasonable theory, such as the general theorems regarding the behavior of Green's functions,[8] gauge invariance, etc. We shall restrict ourselves to the case of a limiting transition in which λ/Λ remains arbitrarily small, since in this case the analysis is considerably simplified.

It will be shown that if g_0 is assumed in general to depend on the cutoff limits, one finds that for any form of this dependence, the exact solution yields in the limit of a point interaction the result that no physical interaction exists between fermions.

Of those graphs which define the vertex operator $\Gamma(P,Q,p)$ (at which the momentum arrangement corresponds to that drawn in Fig. 1), the largest contribution comes from those in which integration over virtual momenta is performed along closed loops (that is, up to Λ^2) and in which the degree of divergence of the integrals is maximal. This is the loop chain shown in Fig. 2.

We shall now find the total contribution (Γ_1) of all of these graphs to the vertex operator and show, by using the value of the vertex operator thus obtained, that the contribution of the remaining graphs (Fig. 4) to Γ is, for sufficiently small λ/Λ, arbitrarily small, and that the Green's function of a fermion is identical with the function $G_0 = (-i\boldsymbol{p} - m)^{-1} \cong i/\boldsymbol{p}$ for a free fermion.[9]

The infinite sum of quantities corresponding to the graphs in Fig. 2 satisfies the equation (see Fig. 3)

$$\Gamma_1(P,Q,p) = a_0 - i\int^{\Lambda^2} a_0 G(l-p) G(l) \Gamma_1(l,Q,p) d^4l, \quad (3)$$

where

$$a_0 = g_0(O_j \times O_j) \quad (3a)$$

is a quantity which corresponds[10] to the simplest graph in Fig. 1; $d^4l = (2\pi)^{-2} dl_1 dl_2 dl_3 dl_4$. The spinor indices are arranged in the integral term in accord with the closed loop in Fig. 3; that is, the integral contains a trace. It is easy to verify that a consequence of this is that no interference occurs between the various interactions (3a) in (3).

3. SCALAR INTERACTION

Equation (3) possesses a solution which depends only on p; in the simplest case of scalar ($O_j = 1$) or pseudoscalar ($O_j = i\gamma_5$) theory the solution is $\Gamma_1 = \alpha(p) \times (O_j \times O_j)$, where

$$\alpha(p) = g_0 \left\{ 1 + g_0 i \int^{\Lambda^2} \mathrm{Sp}[O_j G(l-p) O_j G(l) d^4l] \right\}^{-1}.$$

Inserting $G(l) = G_0(l) = i/\boldsymbol{l}$, we evaluate the quadratically divergent integral involved:

$$i\int^{\Lambda^2} \mathrm{Sp}[O_j G(l-p) O_j' G(l)] d^4l = \tfrac{1}{4}\,\mathrm{Sp}(O_j \gamma_\nu O_j' \gamma_\mu)\left(\tfrac{1}{2} J_0 \delta_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} J_1\right), \quad (4)$$

where[11]

$$J_0 = \frac{8}{3i}\int^{\Lambda^2} \frac{k^2 - (pk)^2/p^2}{(k-\frac{1}{2}p)^2 (k+\frac{1}{2}p)^2} d^4k = \frac{8}{3\pi}\int^{\Lambda^2} (k^2)^2 dk^2 \int_0^1 \frac{(1-x^2)^{\frac{3}{2}} dx}{(k^2 + \frac{1}{4}p^2)^2 - p^2 k^2 x^2} = \tfrac{1}{2}\Lambda^2 - \tfrac{1}{6} p^2 \ln(\Lambda^2/\zeta_0^2 p^2),$$

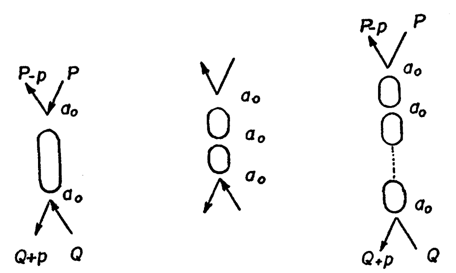

FIG. 2. Chains of closed loops, corresponding to the largest contribution to the vertex operator.

[6] In the figures, line discontinuities signify that a large momentum (exceeding λ but smaller than Λ) cannot be transferred at the point of discontinuity.

[7] This corresponds to writing (2) with a form factor $F_{\Lambda,\lambda}$:
$$\int_{-\infty}^{+\infty} V(t)dt = 2\pi^2 g_0 \int (\bar{\psi}(x) O_j \psi(x'))(\bar{\psi}(y) O_j \psi(y')) \times F_{\Lambda,\lambda}(x-x', y-y', x-y) dx dx' dy dy',$$
in which the width of the distributions with respect to $x-x'$ and $y-y'$ are identical and equal $1/\Lambda$, whereas the width of the distribution with respect to $x-y$ is $1/\lambda$.

[8] H. Lehmann, Nuovo cimento **11**, 342 (1954).

[9] Or is equal to βG_0 where β is a constant, the value of which can be determined from the equation for G (see below). We have employed the following notation here: $\boldsymbol{p} = \gamma_\mu p_\mu$; $\gamma_\mu^\dagger = \gamma_\mu$.

[10] Apart from a factor $(2\pi)^2 i$ which is neglected everywhere.

[11] The variable l has been replaced by $k = l + \frac{1}{2}p$ and the transition from pseudo-Euclidian to Euclidian metrics has been carried out: $(4/i)d^4k = (2/\pi)k^2 dk^2 (1-x^2)^{\frac{1}{2}} dx$, where $x^2 = (pk)^2/p^2k^2$. Terms which remain finite for $\Lambda \to \infty$ have been neglected in the calculations of J_0 and J_1.

$$J_1=\frac{4}{i}\int^{\Lambda^2}\frac{\frac{4}{3}\{[(pk)^2/p^2]-\frac{1}{4}k^2\}-\frac{1}{4}p^2}{(k-\frac{1}{2}p)^2(k+\frac{1}{2}p)^2}d^4k$$

$$=\frac{4}{\pi}\int^{\Lambda^2}k^2dk^2\int_0^1\frac{\frac{4}{3}k^2(x^2-\frac{1}{4})-\frac{1}{4}p^2}{(k^2+\frac{1}{4}p^2)^2-p^2k^2x^2}(1-x^2)^{\frac{1}{2}}dx$$

$$=-\tfrac{1}{6}p^2\ln(\Lambda^2/\zeta_1^2p^2),$$

and ζ_0 and ζ_1 are constants whose exact values are of no significance for the following. Substituting these values of J_0 and J_1 in (4), we get

$$\alpha(p)=g_0\{1+g_0[\Lambda^2-\tfrac{1}{2}p^2\ln(\Lambda^2/\zeta p^2)]\}^{-1}, \tag{5}$$

where $\zeta=(\zeta_0^2\zeta_1)^{\frac{1}{3}}$. The quadratic divergence in the denominator cancels out if $g_0=-1/\Lambda^2$ (or if $1+g_0\Lambda^2$ is a quantity which decreases with increase of Λ^2); if, however, $g_0\neq-1/\Lambda^2$, $\alpha(p)$ will be practically independent of p^2 (it may be recalled that $p^2\leqslant\lambda^2\ll\Lambda^2$) and for $\Lambda\rightarrow\infty$ it vanishes as $1/\Lambda^2$. Postponing the analysis of this case, we shall first assume that

$$1+g_0\Lambda^2=\mu^2/\Lambda^2,$$

where μ is a quantity of the order of the lower cutoff limit. Neglecting μ^2/Λ^2 compared with unity, we get from (5)

$$\alpha(p)=-1/(p^2L+\mu^2), \tag{5a}$$

where $L=\ln(\Lambda/\lambda)$ [ζp^2 has been replaced by λ^2 as a result of which $\alpha(p)$ simply increases]. With the aid of this value of $\alpha(p)$, we shall estimate the magnitude of the contribution from the graph in Fig. 4(a) which determines the difference $G-G_0$, where $G\equiv G_0\beta$; that is, the difference between β and unity.

The spinor indices can be neglected in estimating the integral corresponding to Fig. 4(a), since it is only the order of magnitude that matters. The integral for Fig. 4(a) has the following form (α being a function of f^2 in the given case):

$$g_0\int^{\Lambda^2}\frac{\alpha(f^2)d^4f}{p-f}\int^{\Lambda^2}\frac{d^4l}{l(l+f)}. \tag{6}$$

Integration over l yields Λ^2 (inasmuch as $f^2\leqslant\lambda^2\ll\Lambda^2$ and the integral over l diverges and therefore $l+f\approx l$). For integration over f, the factor $(p-f)^{-1}$ should be expanded into a series in p:

$$(p-f)^{-1}=-\frac{1}{f}-\frac{1}{f}p\frac{1}{f}.$$

Integration of the first term yields zero; if (5a) is taken into account, the second term yields $(1/L)\times\ln(\lambda^2/p^2)$ after integration. Taking into account that $g_0=-1/\Lambda^2$, we get

$$G^{-1}-G_0^{-1}=-ip\frac{\ln(\lambda^2/p^2)}{L},$$

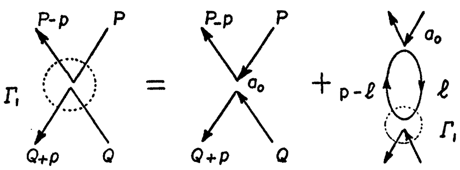

FIG. 3. Graphical representation of Eq. (3). (*Note.*—Following the plus sign, "$p-l$" should read "$l-p$.")

or, for $L\gg\ln(\lambda^2/m^2)$,

$$\beta-1=-\frac{\ln(\lambda^2/p^2)}{L}.$$

Hence for sufficiently large values of L [and for $L\gg\ln(\lambda^2/m^2)$], the difference between β and unity is arbitrarily small.

In a similar manner, we consider the ratio of the integrals corresponding to Figs. 4(b), (c), (d), and (e) [not included in Eq. (3)] to $\alpha(p)$. For Fig. 4(b) this ratio is

$$\frac{1}{\alpha(p)}\int^{\Lambda^2}\frac{\alpha(p)\alpha(f)}{(P-f)(Q+f)}d^4f.$$

Depending on the magnitude of the momenta P and Q, *viz.*, $\lambda\ll P\sim Q\ll\Lambda$, $Q\ll\lambda\ll P\ll\Lambda$, or $p\sim P\sim Q\ll\lambda\ll\Lambda$, we obtain, respectively, taking into account (5a),

$$\frac{\lambda^2}{PQL},\quad \frac{P}{Q}\frac{\ln(\lambda^2/p^2)}{L},\quad\text{or}\quad \frac{\ln(\lambda^2/p^2)}{L}.$$

In any of the cases considered the ratio can be made arbitrarily small.

For the diagrams in Fig. 4(c), we get (see reference 6)

$$\frac{1}{\alpha(p)}\int^{\Lambda^2}d^4l\int^{\Lambda^2}d^4l'\int^{\lambda^2}d^4k\frac{1}{l(l+p)(l+k)}$$

$$\times\frac{1}{l'(l'+k)(l'-p)}\alpha^2(p)\alpha(k)\alpha(p-k).$$

To calculate the integral over l (and l') it should be expanded in k, and p should be set equal to zero. The result is

$$\alpha(p)\int^{\lambda^2}k^2\alpha(k)\alpha(p-k)[\ln(\Lambda^2/k^2)]^2d^4k$$

$$\cong\frac{1}{p^2L}\int^{\lambda^2}\frac{d^4k}{k^2}\cong\frac{\lambda^2}{p^2L}.$$

Figure 4(d) leads to the following integral:

$$\frac{1}{\alpha(p)}\int^{\Lambda^2}d^4l\int^{\lambda^2}d^4k\frac{1}{l(l+p)(l+k)(l+p+k)}\alpha^2(p)\alpha(k).$$

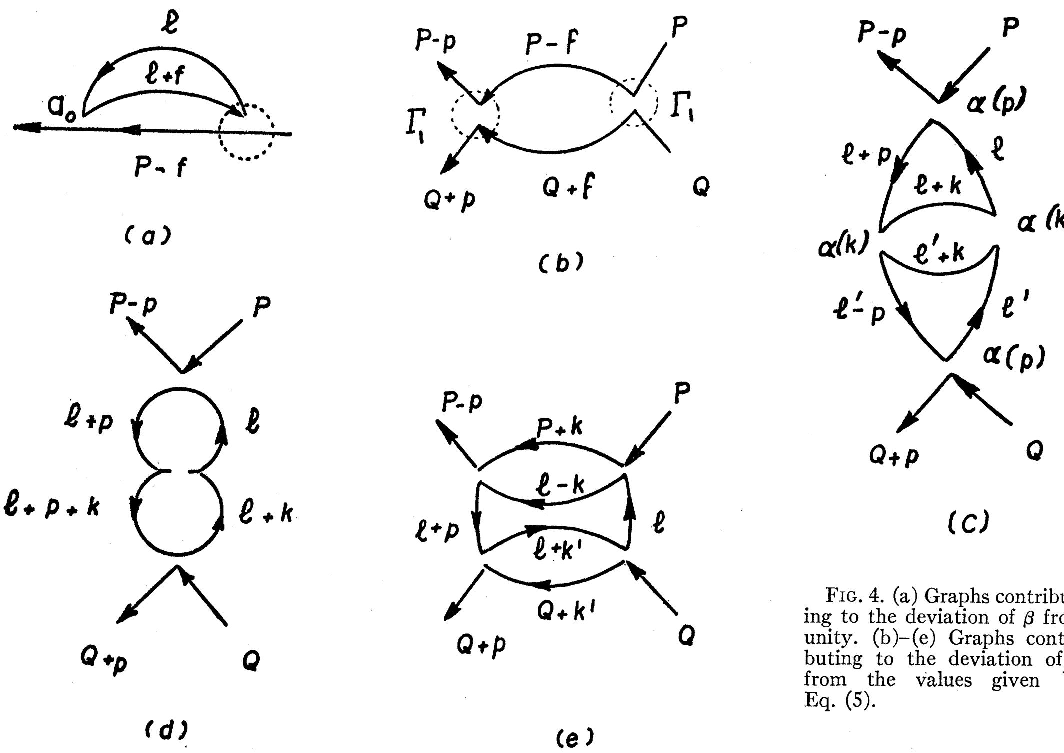

FIG. 4. (a) Graphs contributing to the deviation of β from unity. (b)–(e) Graphs contributing to the deviation of α from the values given by Eq. (5).

Integration over l yields $\ln(\Lambda^2/k^2)$, and we obtain

$$\frac{1}{p^2L}\int^{\lambda^2}\alpha(k)\ln\left(\frac{\Lambda^2}{k^2}\right)d^4k\cong\frac{\lambda^2}{p^2L}.$$

Finally, we shall consider Fig. 4(e). The integration up to a small limit performed in Fig. 4(b) is repeated twice in this diagram. One may therefore expect that it will be smaller than Fig. 4(b). The corresponding ratio is

$$\frac{1}{\alpha(p)}\int^{\Lambda^2}d^4l\int^{\lambda^2}d^4k\int^{\lambda^2}d^4k'$$
$$\times\frac{\alpha(k)\alpha(k')\alpha(k-p)\alpha(k'-p)}{l(l+p)(l-k)(l+k')(P+k)(Q+k')}.$$

Depending on the magnitudes of P and Q, namely, $\lambda\ll P\sim Q\ll\Lambda$, $p\sim Q\ll\lambda\ll P$, or $p\sim P\sim Q\ll\lambda$, we correspondingly get

$$\frac{p^2}{PQ}\left(\frac{\ln\lambda^2/p^2}{L}\right)^2,\quad \frac{p}{P}\frac{\ln\lambda^2/p^2}{L^2},\quad\text{or}\quad\frac{1}{L^2}.$$

These quantities can be made arbitrarily small if the limiting procedure is carried out in such a manner that λ^2/m^2L remains sufficiently small for $\Lambda\rightarrow\infty$. More complex graphs of the vertex operator not indicated in Fig. 4 are proportional to higher powers of the same ratio λ^2/m^2L (or of a smaller quantity) and hence are certainly small.

The expression (5a) obtained for $\alpha(p)$ is thus an exact solution.

The physical interaction between two fermions is determined by the product $\alpha\beta^2$. The results obtained above indicate that $\alpha\beta^2$ vanishes for $\Lambda\rightarrow\infty$; that is, physical interaction between point fermions is absent.

The case when $1+g_0\Lambda^2$ is negative should be excluded. Indeed, if $1+g_0\lambda^2$ is negative and equal to -1, by order of magnitude, the logarithmic term in the expression for Γ can be neglected. Evaluating the additional term in the Green's function G with a negative Γ, we see that β would exceed unity, and this is inconsistent with the general theorems of field theory.[8]

If the absolute value of $1+g_0\Lambda^2$ is less than $g_0\lambda^2$, a pole in the space values of p^2 will arise in the formula for Γ. Since, in treating the interaction between fermions as an interaction between bosons, Γ has the meaning of a boson Green's function, it is evident that this pole corresponds to an imaginary boson mass. Such bosons would yield a term $-|m|^2\varphi^2$ in the Hamiltonian and this would point to nonstability of the vacuum in this case.

Returning to the case $g_0\neq-1/\Lambda^2$, we note that Eq. (3) should be solved simultaneously with the equation for the Green's function $G(p)$,

$$\left[-ip-\frac{g_0}{2}\int a_0G(p-f)\Gamma_1(f)G(l)G(l+f)d^4ld^4f\right]G(p)=1,$$

[the order of the spinor indices corresponds to Fig. 4(a); that is, the integral term contains an integral over l of the same type as (4)].

Neglecting terms of the form $p^2 \ln(\Lambda^2/p^2)$ compared with Λ^2 and taking into account that in this case the integral (4) is equal to Λ^2, we obtain the following solution:

$$\Gamma_1(p) = (O_j \times O_j)\alpha; \quad G(p) = i\beta/p,$$

where $\alpha = g_0[1+g_0\beta^2\Lambda^2]^{-1}$ and $\beta = [1+\frac{1}{8}g_0\lambda^2\Lambda^2\alpha\beta^3]^{-1}$ are quantities which do not depend on p. Inserting α in the formula for β, we obtain an equation of the fourth degree with respect to β. Its solution can easily be found in the cases when $g_0\Lambda^2 \ll 1$ or $g_0\Lambda^2 \sim 1$, $g_0\lambda^2 \ll 1$. In the first case we get $\beta = (4/g_0\lambda^2)[(1+\frac{1}{2}g_0\lambda^2)^{\frac{1}{2}}-1]$, and in the second, $\beta \cong 1$. In both cases (which include the complete range of values of g_0 which are of any interest), the quantity $\xi = \alpha\beta^2\Lambda^2$, in the limit for $\Lambda \rightarrow \infty$, remains of the order of unity. Now it is easy to demonstrate that the unaccounted graphs of the type depicted in Fig. 4 are much less than α. For example, the ratio of the quantities corresponding to the graphs in Figs. 4(b) and (d) to α is $\alpha\beta^2(\lambda^4/PQ) = \xi(\lambda^4/PQ\Lambda^2)$, and $\alpha^2\beta^4\Lambda^2\lambda^2 = \xi^2(\lambda^2/\Lambda^2)$. These quantities are arbitrarily small when $\lambda^2/\Lambda^2 \rightarrow 0$. It follows that the solutions obtained for Γ and G are exact ones, and since $\alpha\beta^2\Lambda^2 \sim 1$, the physical interaction $\alpha\beta^2$ vanishes as $1/\Lambda^2$ for $\Lambda^2 \rightarrow \infty$.

A point which was important for the foregoing analysis was that the integral J_0 contains Λ^2. The magnitude of the quadratically divergent integral, however, may significantly depend on the form factor employed in computing this quantity. Suppose, for example, that the quadratically divergent integral is made to vanish by using an oscillating form factor. Then instead of (5) we get

$$\alpha(p) = \frac{g_0}{1-\frac{1}{2}g_0 p^2 \ln(\Lambda^2/p^2)}. \tag{7}$$

If g_0 behaves as $1/m^2L$ for $\Lambda \rightarrow \infty$, the difference between β and 1 will be of the order of λ^2/m^2L. Indeed, in formula (6) the integral over l now does not contain a quadratically divergent part and hence is equal to $\sim f^2 \ln(\Lambda^2/f^2)$. The remaining integral over f yields $g_0\lambda^2 p \sim (\lambda^2/m^2L)p$, if one takes into account that $\alpha \sim 1/f^2L$ and expands in a series in p.

If g_0 is not small, the equations for α and β should be solved simultaneously. The equation for β can be satisfied in this case by assuming that β is independent of p. Then, instead of (7), we get

$$\alpha(p) = \frac{g_0}{1-\frac{1}{2}g_0\beta^2 p^2 \ln(\Lambda^2/p^2)}. \tag{7a}$$

The integral in the equation for β has the following form:

$$g_0\beta^3 \int \frac{\alpha(f)}{p-f} d^4f \int \frac{d^4l}{l(l-f)} \cong g_0\beta^3 p \int \frac{\alpha(f)}{f^2} d^4f \cdot f^2 \ln\frac{\Lambda^2}{f^2}.$$

For a sufficiently large L, when $g_0\beta^2\lambda^2 L \gg 1$, this integral equals $g_0\beta\lambda^2 p$ and the following equation is obtained for β:

$$\beta[1+\tfrac{1}{16}g_0\beta\lambda^2] = 1.$$

Hence

$$\beta = \frac{8}{g_0\lambda^2}[(1+\tfrac{1}{4}g_0^2\lambda^4)^{\frac{1}{2}}-1];$$

that is, β is independent of L. It therefore follows from (7a) that for a sufficiently large value of L,

$$\alpha\beta^2 \sim 1/p^2L$$

[if $p^2 \rightarrow 0$, formula (7a) should be refined by taking into account the finiteness of the mass; then $\alpha\beta^2 \sim 1/m^2L$].

It can readily be verified that in this case the diagrams of Fig. 4 and the more complicated ones are small compared with α in (7a).

Thus the conclusion drawn above regarding the vanishing of the interaction of point fermions does not depend on whether the quadratically divergent integral is considered equal to Λ^2 or to zero. This confirms our viewpoint that the *physical* results of the theory (in the given case, the vanishing of the interaction) should not depend on the form of the cutoff factor.

4. VECTOR AND TENSOR INTERACTION

Consider now the vector interaction theories ($O_j = i\gamma_\mu$, or $O_j = i\gamma_5\gamma_\mu$). We start with the pseudovector theory. According to (4), insertion of Γ_1 in the form $\alpha(p) \times (i\gamma_5\gamma_\mu \times i\gamma_5\gamma_\mu)$ into (3) leads to the appearance of a new spinor form, $(1/p^2)(i\gamma_5 p \times i\gamma_5 p)$, in the right-hand part of the equation. Thus the solution of (3) should be sought in the form

$$\Gamma_1 = \alpha(p)(i\gamma_5\gamma_\mu \times i\gamma_5\gamma_\mu) + \alpha_1(p)(i\gamma_5 p \times i\gamma_5 p)\frac{1}{p^2}. \tag{8}$$

Inserting (8) into Eq. (3) and separately equating the coefficients in each spinor form, we get

$$\begin{aligned} \alpha(p) &= g_0 - g_0(J_0+J_1), \\ \alpha_1(p) &= 2g_0J_1\alpha(p) - g_0(J_0-J_1)\alpha_1(p), \end{aligned} \tag{9}$$

or

$$\begin{aligned} \alpha(p) &= \frac{g_0}{1+g_0(J_0+J_1)}, \\ \alpha(p)+\alpha_1(p) &= \frac{g_0}{1+g_0(J_0-J_1)}. \end{aligned} \tag{10}$$

Taking into account the values of the integrals J_0 and J_1, we get

$$\begin{aligned} \alpha(p) &= \frac{g_0}{1+g_0[\frac{1}{2}\Lambda^2 - \frac{1}{3}p^2 \ln(\Lambda^2/p^2)]}, \\ \alpha(p)+\alpha_1(p) &= \frac{g_0}{1+\frac{1}{2}g_0\Lambda^2}; \end{aligned} \tag{11}$$

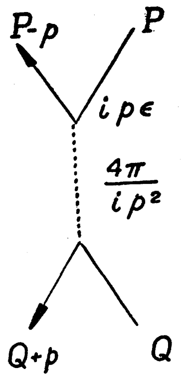

FIG. 5. Simplest graph corresponding to the second-order interaction arising from the term in $\mathfrak{L}_f^{(0)}$.

that is $\alpha(p)$ has the same form as in (5) and for decreases with increasing Λ^2 as $1/L$ for $g_0 \sim 1/\Lambda^2$, whereas $\alpha_1(p)$ does not contain $\ln(\Lambda^2/p^2)$ in the denominator and does not tend to zero for $\Lambda^2 \to \infty$.

The term containing $\alpha_1(p)$, however, does not correspond to any real interaction, as it can be excluded with help of a transformation similar to a guage transformation. Let us examine, for example, a system consisting of fermions and of scalar bosons not interacting with the fermions. The Lagrangian of the system has the form

$$\mathfrak{L} = \mathfrak{L}_b^{(0)} + \mathfrak{L}_f^{(0)} + \mathfrak{L}_f^{(1)},$$

where $\int \mathfrak{L}_f^{(1)} dv = -V$, $\mathfrak{L}_b^{(0)}$ is the Lagrangian of the boson free field, and $\mathfrak{L}_f^{(0)} = -\bar{\psi}(\gamma_\mu \partial/\partial x_\mu + m)\psi$ is the Lagrangian for free fermions.

The fermion system can be characterized by a wave field ψ_1 related to ψ by the relation

$$\psi_1 = \exp[i\gamma_5 \bar{\epsilon}\varphi(x)]\psi; \quad \bar{\psi}_1 = \bar{\psi}\exp[i\gamma_5 \bar{\epsilon}\varphi(x)], \quad (12)$$

where $\varphi(x)$ is a quantized boson field and $\bar{\epsilon}$ is an operator defined in the momentum representation by the function $\epsilon(p^2)$ [i.e., $\bar{\epsilon} = \epsilon(-\partial^2/\partial x_\mu^2)$]. Under this transformation of the field $\psi(x)$, the Lagrangian $\mathfrak{L}_f^{(1)}$ [i.e., interaction (2)] remains constant, whereas $\mathfrak{L}_f^{(0)}$ changes:

$$\mathfrak{L}_f^{(0)} = \mathfrak{L}_{f1}^{(0)} + \mathfrak{L}_1 + \mathfrak{L}_2,$$

where

$$\mathfrak{L}_1 = -\bar{\psi}_1 i\gamma_\mu\gamma_5 \bar{\epsilon}\frac{\partial\varphi}{\partial x_\mu}\psi_1,$$

and

$$\mathfrak{L}_2 = m\bar{\psi}_1\{\exp[-2i\gamma_5\bar{\epsilon}\varphi(x)] - 1\}\psi_1.$$

The Lagrangian $\mathfrak{L}_1$ corresponds to the addition to (2) of a fictitious interaction

$$V' = \int \bar{\psi}_1 i\gamma_\mu\gamma_5\bar{\epsilon}\frac{\partial\varphi}{\partial x_\mu}\psi_1 dv.$$

In second order, it leads to scattering of fermions by fermions, $(f_0/p_2)(i\gamma_5 p \times i\gamma_5 p)$ corresponding to the simplest diagram in Fig. 5. Here $f_0 = \epsilon^2(p^2)/\pi \geqslant 0$. The presence of this interaction (that is, of all the diagrams created from the simplest one in Fig. 5) can be taken into account in calculating Γ_1 if instead of (3a) we insert in (3) the expression

$$a_0' = g_0(i\gamma_5\gamma_\mu \times i\gamma_5\gamma_\mu) + (f_0/p_2)(i\gamma_5 p \times i\gamma_5 p).$$

After simple transformations, the following results can be derived from Eq. (3):

$$\alpha(p) = \frac{g_0}{1 + g_0(J_0 + J_1)};$$
$$\alpha(p) + \alpha_1(p) = \frac{g_0 + f_0}{1 + (g_0 + f_0)(J_0 - J_1)}. \quad (11a)$$

Inserting the upper equation in the lower, we note that $\alpha_1(p)$ vanishes if

$$f_0 = -2g_0^2 J_1/(1 + 2g_0 J_1). \quad (13)$$

The expression for $\alpha(p)$ is identical with (11); it yields the result that the interaction is maximal if $g_0 = -2/\Lambda^2$. From (13) we then obtain

$$f_0 = (p^2/3\Lambda^4)L.$$

Thus, in order to exclude the term $\alpha_1(p)$ in the expression for Γ_1, it is sufficient to choose $\epsilon(p^2)$ equal to $[(p^2/3\Lambda^4)L]^{\frac{1}{2}}$; it is arbitrarily small for $\Lambda \to \infty$. It was thus sufficient to consider the fictitious interaction V' in the lower approximation, the Lagrangian $\mathfrak{L}_2$ disappearing for $\Lambda \to \infty$.

Formula (11a) for $\alpha(p)$ is in all respects similar to the expression (5) obtained in scalar theories and therefore, as in the latter cases, it leads to the disappearance of physical interaction for $\Lambda \to \infty$.

We shall now consider the vector interaction theory. In this case the quadratically divergent integral can be considered to equal zero. Indeed, in vector theory the integral (4) can be written in the following form:

$$\int T\langle j_\nu(x) j_\mu(y)\rangle_0 e^{ip(x-y)} d^4x d^4y, \quad (14)$$

where $j_\nu(x) = \bar{\psi}(x)\gamma_\nu\psi(x)$ and[12] $\partial j_\nu(x)/\partial x_\nu = 0$. If we put $p = 0$ in (14), we formally obtain a quadratically divergent integral. Its structure is the same as that of the integral which in electrodynamics determines the photon mass (since in electrodynamics j_ν satisfies the continuity equation), and it therefore must vanish.[13]

[12] This equation is valid not only for free operators but for coupled operators as well.

[13] A formal proof, not based on the analogy with electrodynamics, can be given, for example, in the following way. Consider the integral

$$\int T\langle j_\nu(x) j_\mu(y)\rangle \frac{\partial\varphi(x)}{\partial x_\nu} d^4x d^4y = \int \frac{\partial}{\partial x_\nu} T\langle j_\nu(x) j_\mu(y)\rangle \varphi(x) d^4x d^4y - \int T\left\langle \frac{\partial j_\nu(x)}{\partial x_\nu} j_\mu(y)\right\rangle \varphi(x) d^4x d^4y.$$

The first integral on the right-hand side is zero if $j_\nu(x)$ vanishes at infinity; the second integral is zero because $\partial j_\nu(x)/\partial x_\nu = 0$. If $\partial\varphi(x)/\partial x_\nu$ is a sufficiently slowly varying function of the

In vector theory the term $\alpha_1(p)(ip\times ip/p^2)$ appears in Γ_1, and formula (10) is valid as before; however, instead of (11), we have

$$\alpha(p)=g_0[1-\tfrac{1}{3}g_0p^2\ln(\Lambda^2/p^2)]^{-1};\qquad \alpha(p)+\alpha_1(p)=g_0. \quad (11b)$$

We now take notice of the fact that for real particles the interaction $\alpha_1(p)(ip\times ip/p^2)$ is identically equal to zero. Indeed, p is the difference between the final and initial particle momenta. Since the wave functions of the initial and final states obey the free Dirac equation, the matrix element of ip will vanish.[14] As $\alpha(p)$ in (11b) is practically identical with (7), the reasoning and also the results obtained in Sec. 3 for scalar theory (in which the quadratic integral vanishes) are also valid in the given case.

It remains for us to consider tensor theory. Replacing O_j in (4) by $\sigma_{\mu\nu}=\frac{1}{2}i(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$, we see that

$$\mathrm{Sp}(\sigma_{\alpha\beta}\gamma_\nu\sigma_{\lambda\rho}\gamma_\nu)=0.$$

Thus no quadratic divergence is involved in tensor theory. The new spinor forms for Γ_1 arise, but in tensor theory $\alpha_1(p)$ contain $\ln\Lambda^2/p^2$ in the denominator. The considerations do not differ from those applied in the case of scalar theory (in which the quadratic integral is assumed to be zero).

Consequently, if only a single fermion field is considered, physical interaction will be absent in all types of four-fermion interaction.

5. INTERACTION OF SEVERAL FIELDS

The interaction between several fields will now be considered. If two types of neutral particles exist (the respective fields being designated by ψ and χ), we arrive at three types of interaction:

$$V=2\pi^2\int[g_1(\bar{\psi}O_j\psi)(\bar{\psi}O_j\psi)+g_2(\bar{\chi}O_j\chi)(\bar{\chi}O_j\chi)+2g_3(\bar{\psi}O_j\psi)(\bar{\chi}O_j\chi)]dv, \quad (15)$$

the respective constants being g_1, g_2, g_3. One may inquire whether these constants can be chosen in such a way so as to cancel out the logarithm in the denominator of the expression for $\alpha(p)$, which is of the same type as (5).

For convenience, the field of the two particles will be characterized by the column matrix

$$\phi=\begin{pmatrix}\psi\\ \chi\end{pmatrix}.$$

Then in interaction (15),

$$a_0=\sum_{i=1}^{3}g_iV_i',$$

where

$$V_1'=Q_1'\times Q_1';\quad V_2'=Q_2'\times Q_2';$$
$$V_3'=Q_1'\times Q_2'+Q_2'\times Q_1';$$
$$Q_1'=O_j\begin{pmatrix}1&0\\0&0\end{pmatrix};\quad Q_2'=O_j\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

will correspond to the simplest diagram in Fig. 1 and Eq. (3) will have a solution of the type

$$\Gamma_1=\sum_{i=1}^{3}\alpha_i(p)V_i'.$$

We shall consider scalar theory ($O_j=1$, $O_j=i\gamma_5$), similar results obtaining in the other theories.

Inserting the foregoing expressions for a_0 and Γ_1 in (3), we obtain, after equating the coefficients before V_i', the following set of equations for α_i:

$$\begin{aligned}(1+g_1I)\alpha_1+g_3\alpha_3I&=g_1,\\ (1+g_2I)\alpha_2+g_3\alpha_3I&=g_2,\\ \tfrac{1}{2}g_3\alpha_1I+\tfrac{1}{2}g_3\alpha_2I+[1+\tfrac{1}{2}(g_1+g_2)I]\alpha_3&=g_3,\end{aligned}\quad (16)$$

where $I=2J_0+J_1\cong\Lambda^2-p^2L$. The solutions of these equations are

$$\alpha_1=\frac{g_1+f^2I}{\Delta},\quad \alpha_2=\frac{g_2+f^2I}{\Delta},\quad \alpha_3=\frac{g_3}{\Delta}, \quad (17)$$

where $\Delta=1+2gI+f^2I^2$, $g=\frac{1}{2}(g_1+g_2)$, and $f^2=g_1g_2-g_3^2$. Taking into account only terms linear in p^2L (it can easily be seen that these are the leading terms if the $g_i\Lambda^2$ are of the order of unity), we obtain

$$\Lambda^2\Delta=(1+F_0)\Lambda^2-F_1p^2L,$$

where $F_0=2\tilde{g}+\tilde{f}^2$ and $F_1=2(\tilde{g}+\tilde{f}^2)$. The dimensionless constants $\tilde{g}=\Lambda^2g$ and $\tilde{f}=\Lambda^2f$ have been introduced; in the case of interest these quantities are of the order of unity. In the expression $\Lambda^2\Delta$ [that is, in the denominator of the $\alpha_i(p)$] there will be no term proportional to Λ^2, and the quantity L will cancel out if $1+F_0=0$ and $F_1=0$. Inserting the values of F_0 and F_1, we get $\tilde{g}=-1$ and $\tilde{f}^2=1$; that is, $\tilde{g}_1+\tilde{g}_2=-2$ and $\tilde{g}_1\tilde{g}_2-\tilde{g}_3^2=1$, where $\tilde{g}_i=\Lambda^2g_i$. Hence $\tilde{g}_1=-2-\tilde{g}_2$ and $-\tilde{g}_3^2=(1+\tilde{g}_2)^2$.

Thus quadratic and logarithmic infinities can be removed from the denominator of the expressions for α_i only when the constant g_3 is imaginary (if the numerators of the expressions for α_i are multiplied by Λ^2 they will be of the order of unity, providing $g_i\sim1$).

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coordinates, it can be written before the integral; and since the vector $\partial\varphi/\partial x_\nu$ is arbitrary, we obtain

$$\int T\langle j_\nu(x)j_\mu(y)\rangle d^4xd^4y=0.$$

[14] See the similar arguments in R. Feynman's paper [Phys. Rev. 76, 769 (1949)]. At small p^2 the denominator of the expression $(ip\times ip)/p^2$ will not vanish if finiteness of the mass is taken into account. Vanishing of the interaction α_1 can also be proved by employing a transformation similar to (12).

Therefore, for Hermitian Lagrangians, physical interaction will vanish in this case also if $\Lambda^2 \to \infty$.

If the quadratic integral vanishes (vector and tensor theories), the same type of reasoning as that used in Sec. 3 for a single field should be applied. No physical interaction will arise in this case either.

We now turn to consideration of charged fields. Care should be exercised in using form factors to analyze the charged field. Thus if the form factor is introduced simply as the integration limit in momentum space, a contradiction with Ward's identity will arise when $\lambda^2 \ll \Lambda^2$ (in particular, because of the interaction, a neutral particle acquires a charge). In order to avoid this inconsistency, a form factor should be introduced which has, for example, the form

$$\left(\frac{\Lambda^2}{(p-A)^2-\Lambda^2}\right)^2,$$

where A is the electromagnetic potential.[15] Gauge invariance will apply to the relation between λ^2 and Λ^2, and hence the methods employed above may be used.

Consider now the interaction between two fields with isotopic spin $\frac{1}{2}$ (field ψ) and isotopic spin 1 (field Σ). Instead of (15) we obtain

$$V=2\pi^2\int [g_1(\bar{\psi}\psi)(\bar{\psi}\psi)+g_2(\bar{\Sigma}\Sigma)(\bar{\Sigma}\Sigma)+2g_3(\bar{\psi}\psi)(\bar{\Sigma}\Sigma) +g_4(\bar{\psi}\tau_\alpha\psi)(\bar{\psi}\tau_\alpha\psi)+g_5(\bar{\Sigma}T_\alpha\Sigma)(\bar{\Sigma}T_\alpha\Sigma) +2g_6(\bar{\psi}\tau_\alpha\psi)(\bar{\Sigma}T_\alpha\Sigma)+g_7(\bar{\Sigma}S_{\alpha\beta}\Sigma)(\bar{\Sigma}S_{\alpha\beta}\Sigma)]dv, \quad (18)$$

where the τ_α are operators of isotopic spin $\frac{1}{2}$ and T_α of spin 1, and $S_{\alpha\beta}=T_\alpha T_\beta+T_\beta T_\alpha-\frac{4}{3}\delta_{\alpha\beta}$; for the sake of brevity, the operators of ordinary spin are not explicitly included. To compute the integral (4), the trace over the isotopic spin variables should also be taken. It can readily be verified in this case that interaction (18) splits into three independent sets: (1) g_1, g_2, g_3; (2) g_4, g_5, g_6; and (3) g_7. The first two can be reduced to the case of interaction of two neutral fields, discussed above, by changing the notation. Thus, in the first case, the following substitution should be made: $2g_1=g_1'$; $3g_2=g_2'$; $6^{\frac{1}{2}}g_3=g_3'$; $2\alpha_1=\alpha_1'$; $3\alpha_2=\alpha_2'$; and $6^{\frac{1}{2}}\alpha_3=\alpha_3'$. In the second case, we obtain the previous equation (16) by performing the substitution $2g_i=g_i'$. In the third case, only one field is involved.

More complicated cases of interaction may be considered, such as the interaction between three different fields. Then, besides interactions of type (18), an interaction of the β-decay type $(\bar{\psi}\psi)(\bar{\varphi}\chi)$ can be set up. These cases may be handled by the same methods, but the problem becomes more unwieldy. There are no apparent physical reasons to expect other results to be obtained in these more complex cases, although a study of them would seem to be of interest.

Thus the expectation expressed in the introduction, that choice of the sign of the interaction constant might lead to a result differing from that in other cases of local interaction, is not confined.

In all of the simplest cases of local interaction considered up to the present, physical interaction between the fields is absent.

[15] The authors are thankful to B. L. Ioffe for pointing out that a form factor of the indicated type removes the contradiction with Ward's identity.