It is therefore necessary that such contaminations be less than $0.1\delta^2$ (which is 0.1% for $\delta\approx 0.1$) for a 30% accuracy in the determination of δ^2.

Since different isotopes capture μ mesons with approximately the same rate, contamination of the stopping substance with its isotopes can be shown to produce relatively small curvature in the logarithmic decay curve, and consequently can be well tolerated. An isotopic contamination of a few percent will be quite harmless.

(c) A curvature measurement, however, does not allow for a determination of the sign of δ, even if one knows the population of the two hyperfine states. A measurement of the change of curvature, or a study of the time dependence of the angular asymmetry of the decay electrons if the μ meson is not completely depolarized in slowing down, is necessary to determine the sign of δ. Both of these seem to be very difficult.

(d) For a nucleus with an even number of protons the difference $\lambda_+ - \lambda_-$, if it exists, should be very small. Also, if $I=0$, there should be only one lifetime. These obvious conclusions offer convenient "controls" in any experimental setup to detect $\lambda_+ - \lambda_-$.

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Conservation Laws in General Relativity

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The conservation laws are examined from the transformation properties of the Lagrangian. The energy-momentum complex obtained has mixed indices, $T_\mu{}^\nu$, whereas a symmetric quantity $\mathcal{T}^{\mu\nu}$ is required for the definition of angular momentum. Such a symmetric quantity has been constructed by Landau and Lifshitz. In the course of examining the relationship between these quantities, two hierarchies of complexes $T_{(n)\mu}{}^\nu$ and $\mathcal{T}_{(n)}{}^{\mu\nu}$ are constructed. Under linear coordinate transformations the former are tensor densities of weight $(n+1)$ and the latter of weight $(n+2)$. For $n=0$ these reduce to the canonical $T_\mu{}^\nu$ and the Landau-Lifshitz $\mathcal{T}^{\mu\nu}$, respectively.

By requiring the energy-momentum complex to generate the coordinate transformations, and the total energy and momentum to form a free vector, one can identify the canonical complex $T_\mu{}^\nu$ as the appropriate quantity to describe the energy and momentum of the field plus matter. Similarly, by requiring the total angular momentum to behave as a free antisymmetric tensor, one can construct, in the usual manner, an appropriate quantity from $\mathcal{T}_{(-1)}{}^{\mu\nu}$. The angular momentum complex so defined differs from that proposed by Landau and Lifshitz as well as from an independent construction by Bergmann and Thomson.

1. INTRODUCTION

CONSERVATION laws in general relativity were first formulated by Einstein in 1916.[1] By examining the behavior of the Lagrangian of the theory of gravitation under infinitesimal translations of the coordinate system, he was led to the canonical energy-momentum pseudotensor of the gravitational field. Because of the nontensor character of the pseudotensor, the local energy density of the field does not have a covariant significance. Indeed, Schrödinger[2] criticized this formulation of the conservation laws because he found a coordinate system in which all components of the pseudotensor vanished for the Schwarzschild metric. This criticism was answered only when Einstein[3] showed that total energy and momentum, the only physically meaningful quantities, are constants of the motion and transform as a free-vector[4] under linear coordinate transformations.

Except for a further examination of the relationship between conservation laws and transformation properties,[5] little has been added to the analysis by Einstein. However, in order to discuss angular momentum, a symmetric quantity for energy-momentum is desirable,[6] although not necessary.[7] The canonical pseudotensor has mixed indices, and raising one with the metric tensor does not yield a symmetric quantity. Recently

[1] A. Einstein, Berlin Ber. **42**, 1111 (1916).
[2] E. Schrödinger, Physik Z. **19**, 4 (1918).
[3] A. Einstein, Berlin Ber. 448 (1918); W. Pauli, *Relativitatstheorie* (B. G. Teubner, Leipzig, 1922), Enzyklopädie der Mathematische Wissenschaften, Vol. 2, p. 740.
[4] A free-vector is a set of quantities which are not defined at a particular point in space, yet which transform together as a vector under linear coordinate transformations.
[5] P. G. Bergmann, Phys. Rev. **75**, 680 (1949); P. G. Bergmann and R. Schiller, Phys. Rev. **89**, 4 (1953).
[6] W. Pauli, Revs. Modern Phys. **13**, 203 (1941).
[7] P. G. Bergmann and R. Thomson, Phys. Rev. **89**, 400 (1953).

Landau and Lifshitz[8] have succeeded in constructing a symmetric pseudotensor. However, following Einstein's analysis,[3] one can show that the total energy and momentum defined by L-L (Landau and Lifshitz) transforms as a vector density rather than a vector as is the case for particles.

Therefore, it is of some interest to examine the relationship between the L-L and the canonical pseudotensors. In the course of this examination a whole family of mixed and symmetric pseudotensors will be constructed. The various pseudotensors have different weights. All of the mixed tensors have the same physical content (total energy and momentum) whereas the symmetric ones are all different in their physical content. Of the symmetric quantities, only the L-L pseudotensor has the same total energy and momentum as the canonical one. However, it has the wrong transformation properties.

In the next section an analysis of the relationship between covariance and conservation laws is presented. This material is essentially the same as that of Bergmann and Schiller[5] and is presented only for completeness as the results are needed. The translation of these results into the general theory of relativity is carried out in Sec. 3. Here, too, the hierarchy of pseudotensors, both symmetric and mixed, is constructed. Finally, in Sec. 4 we discuss angular momentum.

2. COVARIANCE AND THE CONSERVATION LAWS

Consider field equations which may be derived from a variational principle. The Lagrangian L may contain up to second derivatives of the field variables y_A. In order to assure covariant field equations, L should transform as a scalar density under coordinate transformations. Therefore, with respect to an infinitesimal coordinate transformation,

$$\bar{x}^\mu = x^\mu + \xi^\mu, \tag{2.1}$$

we have

$$\bar{\delta} L + (L\xi^\mu)_{,\mu} = 0. \tag{2.2}$$

Let the field variables obey a linear transformation law[9]:

$$\bar{\delta} y_A = u_{\mu A}{}^\nu \xi^\mu{}_{,\nu} - y_{A,\mu}\xi^\mu, \quad u_{\mu A}{}^\nu = F_{A\mu}{}^{B\nu} y_B, \tag{2.3}$$

where the $F_{A\mu}{}^{B\nu}$ are certain constants.[5] Expanding Eq. (2.2) with the help of (2.3) and equating to zero the coefficients of the various differential orders of ξ^μ, we obtain the following set of relations[10]:

$$t_\mu{}^\nu{}_{,\nu} + L^A y_{A,\mu} = 0, \tag{2.4a}$$

$$t_\mu{}^\nu - u_{\mu A}{}^\nu L^A = U_\mu{}^{[\nu\sigma]}{}_{,\sigma}, \tag{2.4b}$$

$$L^{\nu\sigma} + L^{\sigma\nu} = 0, \tag{2.4c}$$

$$u_{\mu A}{}^\nu \partial^{A\rho\sigma} L + u_{\mu A}{}^\rho \partial^{A\sigma\nu} L + u_{\mu A}{}^\sigma \partial^{A\nu\rho} L = 0, \tag{2.4d}$$

where the following abbreviations have been used:

$$L^A = \partial^A L - (\partial^{A\rho} L)_{,\rho} + (\partial^{A\rho\sigma} L)_{,\rho\sigma}, \tag{2.5a}$$

$$t_\mu{}^\nu = -\delta_\mu{}^\nu [L - (\partial^{A\rho\sigma} L y_{A,\sigma})_{,\rho}] + [\partial^{A\nu} L - (\partial^{A\nu\rho} L)_{,\rho}] y_{A,\mu} - (\partial^{A\nu\rho} L)_{,\mu} y_{A,\rho}, \tag{2.5b}$$

$$U_\mu{}^{[\nu\rho]} = L_\mu{}^{[\nu\rho]} - \tfrac{2}{3}[u_{\mu A}{}^\nu \partial^{A\rho\sigma} L - u_{\mu A}{}^\rho \partial^{A\nu\sigma} L]_{,\sigma} + K_\mu{}^{[\nu\rho]}, \tag{2.5c}$$

$$L_\mu{}^{[\nu\rho]} = u_{\mu A}{}^\nu \partial^{A\rho} L + 2u_{\mu A}{}^\nu{}_{,\sigma} \partial^{A\rho\sigma} L - \partial^{A\rho\nu} L y_{A,\mu}, \tag{2.5d}$$

$$K_\mu{}^{[\nu\rho]} = [\delta_\mu{}^\nu \partial^{A\rho\sigma} L - \delta_\mu{}^\rho \partial^{A\nu\sigma} L] y_{A,\sigma}. \tag{2.5e}$$

In (2.4a) use has been made of the following identity

$$\partial^{A\nu\sigma} L y_{A,\sigma\mu} = -K_\mu{}^{[\nu\sigma]}{}_{,\sigma} - (\partial^{A\nu\sigma} L)_{,\mu} y_{A,\sigma} + \delta_\mu{}^\nu (\partial^{A\rho\sigma} L y_{A,\sigma})_{,\rho}; \tag{2.6a}$$

in (2.4b) we have used the relations (2.4d) and the identity

$$[u_{\mu A}{}^\nu \partial^{A\rho\sigma} L + \tfrac{1}{3}(u_{\mu A}{}^\sigma \partial^{A\nu\rho} L - u_{\mu A}{}^\rho \partial^{A\nu\sigma} L)]_{,\rho\sigma} = [u_{\mu A}{}^\nu \partial^{A\rho\sigma} L]_{,\rho\sigma}. \tag{2.6b}$$

Equation (2.4a) results from the translational invariance of the theory alone. Therefore, $t_\mu{}^\nu$ may be identified with the energy-momentum pseudotensor.[1] When the field equations are satisfied, $L^A = 0$, the "weak" conservation law for energy and momentum results:

$$t_\mu{}^\nu{}_{,\nu} = 0. \tag{2.7}$$

If the Lagrangian is linear in the second derivatives and $\partial^{A\rho\sigma} L$ depends only on the y_A one can define an equivalent function

$$\mathfrak{L} = L - (\partial^{A\rho\sigma} L y_{A,\sigma})_{,\rho}. \tag{2.8}$$

Second derivatives of the field variables no longer occur in $\mathfrak{L}$. In general $\mathfrak{L}$ will not be a scalar density. However, the field equations which result from a variation of the action defined by $\mathfrak{L}$ will be the same as those obtained through use of L. The canonical pseudotensor then takes the familiar form

$$t_\mu{}^\nu = -\delta_\mu{}^\nu \mathfrak{L} + \partial^{A\nu} \mathfrak{L} y_{A,\mu}. \tag{2.9}$$

The remaining relations in Eqs. (2.4) follow from covariance with respect to arbitrary coordinate trans-

[8] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Press, Cambridge, 1951), p. 316, English translation.

[9] The $\bar{\delta}$ transformation compares the field variables at world points with the same coordinate value rather than at the same world point. That is, $\bar{\delta} y_A = \bar{y}_A(x) - y_A(x) = \delta y_A - y_{A,\mu}\xi^\mu$. The advantage of the $\bar{\delta}$ transformation is that it commutes with ordinary differentiation.

[10] The following abbreviated notation is used: $\partial^A L = \partial L/\partial y_A$, $\partial^{A\rho} L = \partial L/\partial y_{A,\rho}$, $\partial^{A\rho\sigma} L = \partial L/\partial y_{A,\rho\sigma}$. Square brackets around two indices indicate antisymmetry in the two indices: $L^{[\nu\sigma]} = -L^{[\sigma\nu]}$.

formations. Defining the quantity $T_\mu{}^\nu$ through

$$T_\mu{}^\nu = t_\mu{}^\nu - u_{\mu A}{}^\nu L^A, \tag{2.10}$$

we obtain from Eq. (2.4b) a strong conservation law

$$T_\mu{}^\nu{}_{,\nu} \equiv 0. \tag{2.11}$$

In the presence of matter the field equations become

$$L^A = P^A, \tag{2.12}$$

where P^A describes the distribution of matter. Thus,

$$T_\mu{}^\nu = t_\mu{}^\nu - u_{\mu A}{}^\nu P^A, \tag{2.13}$$

and we may say that $T_\mu{}^\nu$ is the energy-momentum complex[11] for both field and matter.

Since $t_\mu{}^\nu$ does not have tensor transformation properties, the distribution of energy between matter and field is dependent upon the choice of coordinate system. This result was first obtained by Einstein,[3] who also showed that under certain conditions (essentially, no radiation) the total energy and momentum in a closed domain of space is independent of the choice of coordinates within the domain. This latter result can be shown very easily by use of what Pauli[12] has called the "flux theorem." The total energy and momentum is given by the spatial volume integral

$$J_\mu = \int T_\mu{}^4 d^3x. \tag{2.14}$$

By Eq. (2.4b) this may be rewritten as a surface integral

$$J_\mu = \oint U_\mu{}^{[4s]} n_s dS. \tag{2.15}$$

Thus the total energy and momentum is determined by the flux of some quantity through a closed surface. It follows that an arbitrary coordinate transformation which reduces to the identity on the surface sufficiently rapidly will not affect the total energy and momentum.

From the transformation properties of the Lagrangian, we have been able to construct a quantity with mixed indices which satisfies a conservation relationship. However, in order to define an angular momentum complex, a contravariant symmetric quantity is required. Such a quantity cannot be found from the transformation properties alone because the coefficients describing coordinate transformations always involve mixed indices [see Eq. (2.3)]. If one has, however, a symmetric quantity $\Gamma^{\mu\nu}$, one can always construct a quantity $\mathcal{T}^{\mu\nu}$ which is symmetric and which satisfies

$$\mathcal{T}^{\mu\nu}{}_{,\nu} \equiv 0; \tag{2.16}$$

namely[13]

$$\mathcal{T}^{\mu\nu} = \{\Gamma^{\mu\nu}\Gamma^{\rho\sigma} - \Gamma^{\mu\sigma}\Gamma^{\nu\rho}\}_{,\rho\sigma}. \tag{2.17}$$

Quantities which are suitable for the $\Gamma^{\mu\nu}$ are easy to find; for example,

$$(\partial^{A\rho\sigma}L)y_A, \quad (\partial^{A\rho}\partial^{B\sigma}L)y_Ay_B.$$

Note, however, that with respect to linear transformations $t_\mu{}^\nu$, and hence $T_\mu{}^\nu$, transforms like a tensor density of weight $+1$. On the other hand, with the above examples for $\Gamma^{\mu\nu}$, $\mathcal{T}^{\mu\nu}$ will transform like a density of weight $+2$. Therefore, in the same manner one proves that the total energy and momentum J_μ, Eq. (2.14), transforms as a free vector,[3] one can show that

$$\mathcal{J}^\mu = \int \mathcal{T}^{\mu 4} d^3x \tag{2.18}$$

transforms as a vector density. Similarly, the angular momentum defined from Eq. (2.17) will transform like a tensor density rather than a tensor. We shall return to this question in the next section.

In order to satisfy (2.16), $\Gamma^{\mu\nu}$ need only be symmetric and need not be either of the above-mentioned quantities. For example, consider simply the Lorentz-covariant scalar meson theory. It is easy to show that in this case the energy-momentum tensor defined through Eq. (2.9) is symmetric if the lowered index is raised. Using this symmetric quantity for $\Gamma^{\mu\nu}$, a short calculation shows that $\mathcal{T}^{\mu\nu}$ does not vanish identically. Although $\mathcal{T}^{\mu\nu}$ has a vanishing divergence one would not identify it with the energy and momentum in the field. Therefore, before claiming any physical significance for the quantity defined through Eq. (2.17), one must examine its relationship to the $T_\mu{}^\nu$ defined by the transformation properties. Unfortunately, this comparison cannot be carried out in the generalized notation of this section.

3. THEORY OF GRAVITATION

The Lagrangian density in the theory of gravitation is just the curvature scalar density

$$\mathfrak{R} = (-g)^{\frac{1}{2}}R = \mathfrak{g}^{\mu\nu}R_{\mu\nu}. \tag{3.1}$$

[11] H. A. Lorentz, *Collected Papers* (Martinus Nijhoff, The Hague, 1937), Vol. 5, p. 246 ff. Lorentz used the term *complex* to denote a quantity which does not have tensor transformation properties. However, he used a different definition for energy and momentum of the gravitational field. He obtained Eq. (2.4a) but did not wish to identify the pseudotensor with the gravitational energy and momentum. In our notation, he then observed the validity of Eq. (2.11) and since $t_\mu{}^\nu{}_{,\nu} = (u_{\mu A}{}^\nu L^A)_{,\nu}$ he identified $u_{\mu A}{}^\nu L^A$ as the energy-momentum tensor of the gravitational field. It follows from Eq. (2.12) that the energy density for matter and field always vanishes (W. Pauli, reference 3). Therefore, this definition is not as satisfactory as the usual one discussed in the body of this paper. On the other hand, $t_\mu{}^\nu$ and hence $T_\mu{}^\nu$ are affine tensors; that is, they have tensor transformation properties with respect to linear coordinate transformations.

[12] W. Pauli (privately circulated letter). However, the "theorem" has been known without a name for some time. The first explicit use of the theorem was by H. Zatzkis, Phys. Rev. **81**, 1023 (1951), based on the work of P. von Freud, Ann. Math. **40**, 417 (1939). Further discussion of the theorem may be found in Bergmann and Schiller, reference 5, and J. N. Goldberg, Phys. Rev. **89**, 263 (1953); **99**, 1873 (1955).

[13] R. Sachs (private communication).

For the purpose of computing the explicit expressions in Eqs. (2.4) and (2.5), it is convenient to use the following expanded form:

$$\mathfrak{R}=\tfrac{1}{2}(\mathfrak{g}^{\iota\kappa}\mathfrak{g}_{\rho\sigma}\mathfrak{g}^{\rho\sigma}{}_{,\iota}+2\mathfrak{g}^{\iota\kappa}{}_{,\iota})_{,\kappa}+\mathfrak{L}, \tag{3.2}$$

$$\mathfrak{L}=\tfrac{1}{8}\{2\mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\lambda\iota}\mathfrak{g}_{\kappa\tau}-\mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\iota\kappa}\mathfrak{g}_{\lambda\tau}-4\delta_\kappa{}^\sigma\delta_\lambda{}^\rho\mathfrak{g}_{\iota\tau}\}\mathfrak{g}^{\iota\kappa}{}_{,\rho}\mathfrak{g}^{\lambda\tau}{}_{,\sigma}. \tag{3.3}$$

It is clear from Eq. (3.2) that $\mathfrak{R}$ satisfies the necessary condition for the existence of a suitable first-order Lagrangian which is defined in (3.3). A somewhat tedious, but not difficult, calculation gives

$$U_\mu{}^{[\nu\rho]}=\mathfrak{g}_{\mu\lambda}H^{[\lambda\sigma][\nu\rho]}{}_{,\sigma}, \tag{3.4a}$$

$$H^{[\lambda\sigma][\nu\rho]}=\mathfrak{g}^{\lambda\nu}\mathfrak{g}^{\rho\sigma}-\mathfrak{g}^{\lambda\rho}\mathfrak{g}^{\nu\sigma}, \tag{3.4b}$$

$$\begin{aligned}t_\mu{}^\nu=&-\tfrac{1}{8}\delta_\mu{}^\nu[2\mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\lambda\iota}\mathfrak{g}_{\kappa\tau}-\mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\iota\kappa}\mathfrak{g}_{\lambda\tau}\\&-4\delta_\kappa{}^\sigma\delta_\lambda{}^\rho\mathfrak{g}_{\iota\tau}]\mathfrak{g}^{\iota\kappa}{}_{,\rho}\mathfrak{g}^{\lambda\tau}{}_{,\sigma}+\tfrac{1}{4}[2\mathfrak{g}^{\nu\sigma}\mathfrak{g}_{\lambda\iota}\mathfrak{g}_{\kappa\tau}\\&-\mathfrak{g}^{\nu\sigma}\mathfrak{g}_{\iota\kappa}\mathfrak{g}_{\lambda\tau}-4\delta_\lambda{}^\nu\delta_\kappa{}^\sigma\mathfrak{g}_{\iota\tau}]\mathfrak{g}^{\lambda\tau}{}_{,\sigma}\mathfrak{g}^{\iota\kappa}{}_{,\mu},\end{aligned} \tag{3.5}$$

$$u_{\mu A}{}^\nu L^A=2(-g)^{\frac{1}{2}}G_\mu{}^\nu=2\mathfrak{G}_\mu{}^\nu. \tag{3.6}$$

Therefore,

$$T_\mu{}^\nu=t_\mu{}^\nu-2\mathfrak{G}_\mu{}^\nu. \tag{3.7}$$

At first sight the minus sign in the above equation is disconcerting. However, in the presence of matter we have[14]

$$G^{\mu\nu}=-8\pi\kappa P^{\mu\nu}, \tag{3.8}$$

where $P^{\mu\nu}$ is the matter tensor. Therefore, the total energy will be positive.

The quantity defined in Eq. (3.4b) permits the construction of a symmetric energy-momentum complex in the manner of (2.17)[15]:

$$\mathcal{T}^{\mu\nu}=H^{[\mu\sigma][\nu\rho]}{}_{,\rho\sigma}. \tag{3.9}$$

This quantity is just that proposed by Landau and Lifshitz.[8] In order to see the relationship between this quantity and the mixed quantity defined by Eq. (3.7) we use (3.7) and (2.4b).

$$-2\mathfrak{G}_\mu{}^\nu=U_\mu{}^{[\nu\sigma]}{}_{,\sigma}-t_\mu{}^\nu. \tag{3.10}$$

Raising the covariant index with the contravariant metric tensor density, we obtain

$$-2\mathfrak{g}^{\mu\lambda}\mathfrak{G}_\lambda{}^\nu=H^{[\mu\rho][\nu\sigma]}{}_{,\rho\sigma}-\tau^{\mu\nu}, \tag{3.11a}$$

$$\tau^{\mu\nu}=\mathfrak{g}^{\mu\lambda}{}_{,\sigma}U_\lambda{}^{[\nu\sigma]}+\mathfrak{g}^{\mu\lambda}t_\lambda{}^\nu. \tag{3.11b}$$

In view of the symmetry of $\mathfrak{G}^{\mu\nu}$ and $\mathcal{T}^{\mu\nu}$, it follows that the quantity $\tau^{\mu\nu}$ is also symmetric and is just the L-L pseudotensor. It is clear that the L-L pseudotensor is homogeneous quadratic in the first derivatives of the $\mathfrak{g}^{\mu\nu}$, just as is $t_\mu{}^\nu$.

[14] P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1947), Chap. 12.

[15] Indeed, we have $(\partial^{A\rho\sigma}L)y_A=3\mathfrak{g}^{\rho\sigma}$ and $(\partial^{A\rho}\partial^{B\sigma}L)y_Ay_B=-3\mathfrak{g}^{\rho\sigma}$.

As was noted at the end of the previous section, under linear coordinate transformations, $T_\mu{}^\nu$ and $t_\mu{}^\nu$ transform like tensor densities of weight one, whereas $\mathcal{T}^{\mu\nu}$ and $\tau^{\mu\nu}$ transform like tensor densities of weight two. It follows that the total energy and momentum transform like free-vectors and free-vector densities, respectively. The question naturally arises whether one can construct a symmetric quantity with a weight one. Indeed, one can construct a symmetric quantity of arbitrary weight. Multiplying Eq. (3.11a) by $(-g)^{n/2}$, we have

$$-2(-g)^{(n+1)/2}\mathfrak{G}^{\mu\nu}=H_{(n)}{}^{[\mu\rho][\nu\sigma]}{}_{,\rho\sigma}-\tau_{(n)}{}^{\mu\nu}, \tag{3.12}$$

with

$$H_{(n)}{}^{[\mu\rho][\nu\sigma]}=(-g)^{n/2}H^{[\mu\rho][\nu\sigma]}, \tag{3.13a}$$

and

$$\begin{aligned}\tau_{(n)}{}^{\mu\nu}=(-g)^{n/2}\{&\tau^{\mu\nu}+\tfrac{1}{2}n[(\ln|g|)_{,\sigma\tau}\\&+\tfrac{1}{2}n(\ln|g|)_{,\sigma}(\ln|g|)_{,\tau}]H^{[\mu\tau][\nu\sigma]}\\&+\tfrac{1}{2}n(\ln|g|)_{,\sigma}[H^{[\mu\tau][\nu\sigma]}+H^{[\nu\tau][\mu\sigma]}]_{,\tau}\}.\end{aligned} \tag{3.13b}$$

Note that except for the L-L quantity ($n=0$) the above defined pseudotensors depend on the second derivatives of the field variables. One can now define a total energy-momentum complex by

$$\mathcal{T}_{(n)}{}^{\mu\nu}=H_{(n)}{}^{[\mu\rho][\nu\sigma]}{}_{,\rho\sigma}. \tag{3.14}$$

Smilarly one can define quantities of arbitrary weight with the mixed quantities:

$$U_{(n)\mu}{}^{[\nu\sigma]}=(-g)^{n/2}U_\mu{}^{[\nu\sigma]}, \tag{3.15a}$$

$$t_{(n)\mu}{}^\nu=(-g)^{n/2}[t_\mu{}^\nu+\tfrac{1}{2}nU_\mu{}^{[\nu\sigma]}(\ln|g|)_{,\sigma}], \tag{3.15b}$$

$$T_{(n)\mu}{}^\nu=U_{(n)\mu}{}^{[\nu\sigma]}{}_{,\sigma}. \tag{3.15c}$$

Note that for all n the pseudotensor thus defined contains only first derivatives.

Which of the infinite number of quantities defined through Eqs. (2.14) and (2.15c) are meaningful? Certainly the local distribution of energy and momentum has no physical significance in general relativity. The quantities which can have a meaning are the total energy and momentum defined through

$$\begin{aligned}J_{(n)\mu}&=\frac{1}{16\pi\kappa}\int T_{(n)\mu}{}^4d^3x\\&=\frac{1}{16\pi\kappa}\oint U_{(n)\mu}{}^{[4s]}n_sdS,\end{aligned} \tag{3.16}$$

or

$$\begin{aligned}\mathcal{J}_{(n)}{}^\mu&=\frac{1}{16\pi\kappa}\int \mathcal{T}_{(n)}{}^{\mu4}d^3x\\&=\frac{1}{16\pi\kappa}\oint H_{(n)}{}^{[\mu\tau][4s]}{}_{,\tau}n_sdS.\end{aligned} \tag{3.17}$$

In order to make these integrals meaningful, the surface of integration must be taken at infinity. We assume that the metric approaches the Minkowski metric at least as $1/r$. Therefore, at infinity we may set

$$\mathfrak{g}^{\mu\nu}=\eta^{\mu\nu}(-1,-1,-1,+1), \quad -g=1, \tag{3.18}$$

but the derivatives of the metric must not be neglected. Under these conditions, it follows that for all n, the $J_{(n)\mu}$ are equal:

$$J_{(n)\mu}=J_\mu. \tag{3.19}$$

On the other hand, from Eq. (2.13a),

$$\mathcal{J}_{(n)}{}^\mu=\mathcal{J}_{(0)}{}^\mu+\frac{1}{16\pi\kappa}\frac{n}{2}\oint(-g)_{,\tau}[\eta^{4\mu}\eta^{\tau s}-\eta^{\mu s}\eta^{4\tau}]n_s dS. \tag{3.20}$$

For $n=0$, we have the satisfying result

$$J_\mu=\eta_{\mu\rho}\mathcal{J}_{(0)}{}^\rho. \tag{3.21}$$

In general, the surface integral in (3.20) will not vanish. If we wish all $\mathcal{J}_{(n)}{}^\mu$ to be equal, then we require the additional assumption that in physically acceptable coordinate systems, for large r,

$$g_{,r}\sim 1/r^3. \tag{3.22}$$

Let us consider the Schwarzschild solution in two coordinate systems which satisfy the conditions of Eq. (2.18):

$$(\mathrm{I})^{16}: \quad ds^2=\left(1-\frac{2\kappa m}{r}\right)(dx^4)^2-\left[\delta_{rs}+\left(\frac{2\kappa m}{r-2\kappa m}\right)\frac{x^r}{r}\frac{x^s}{r}\right]dx^r dx^s,$$

$$(\mathrm{II})^{17}: \quad ds^2=\left[\frac{1-\kappa m/2r}{1+\kappa m/2r}\right]^2(dx^4)^2-\left(1+\frac{\kappa m}{2r}\right)^4[(dx^1)^2+(dx^2)^2+(dx^3)^2].$$

In solution (I), $g=-1$ everywhere, so that all $\mathcal{J}_{(n)}{}^\mu$ are equal. However, in solution (II)

$$-g=\left(1-\frac{\kappa m}{2r}\right)^2\left(1+\frac{\kappa m}{2r}\right)^{10}. \tag{3.23}$$

As a result,

$$\mathcal{J}_{(n)}{}^\mu=\mathcal{J}_{(0)}{}^\mu-\tfrac{1}{2}n\delta_{\mu 4}m. \tag{3.24}$$

Since

$$\mathcal{J}_{(0)}{}^\mu=\delta_{\mu 4}m, \tag{3.25}$$

it follows that for $n=2$ the total energy vanishes, while for $n>2$ it is negative.

Clearly, the existence of a complex with a vanishing divergence is insufficient evidence for the conservation of a physically interesting quantity. In this case we have shown that in the theory of gravitation there are an infinite number of divergences which are related to the conservation of energy and momentum. From the relations of the defined complexes to the field equations, it appears that all are equally good for a qualitative description of the properties of energy and momentum. Nonetheless, it is desirable to have a unique quantity define the physically interesting properties of the field. In order to pick out a satisfactory quantity we propose two guiding principles: (1) the energy-momentum complex should generate the infinitesimal coordinate transformations, and (2) the total energy and momentum should transform as the energy and momentum of a free particle, hence, as a free-vector. These requirements are satisfied only by the quantities defined through the transformation properties of the Lagrangian[18] as in Sec. 2. Thus, we may say that $t_\mu{}^\nu$, Eq. (3.5), describes the energy and momentum of the gravitational field, while $T_\mu{}^\nu$, Eq. (3.7), describes the energy and momentum of the field plus matter. The argument presented here is hardly conclusive. It may be possible to find other arguments which will permit a different choice or, perhaps, no choice at all.

4. ANGULAR MOMENTUM

The definition of angular momentum is usually based on a symmetric energy-momentum complex[6] although a nonsymmetric form was used by Bergmann and Thomson.[7] In the previous section an infinite number of symmetric complexes were constructed. Thus, an infinite number of quantities which are suitable as angular momentum complexes may be constructed:

$$\begin{aligned}M_{(n)}{}^{[\nu\sigma]\mu}&=x^\nu\mathcal{T}_{(n)}{}^{\sigma\mu}-x^\sigma\mathcal{T}_{(n)}{}^{\nu\mu}\\&=\{x^\nu H_{(n)}{}^{[\sigma\iota][\mu\kappa]}{}_{,\iota}-x^\sigma H_{(n)}{}^{[\nu\iota][\mu\kappa]}{}_{,\iota}+H_{(n)}{}^{[\nu\sigma]\mu\kappa}\}_{,\kappa}.\end{aligned} \tag{4.1}$$

The total angular momentum corresponding to the above quantities is given by

$$\begin{aligned}\mathfrak{M}_{(n)}{}^{[\nu\sigma]}&=\int M_{(n)}{}^{[\nu\sigma]4}d^3x\\&=\oint\{x^\nu H_{(n)}{}^{[\sigma\iota][4s]}{}_{,\iota}-x^\sigma H_{(n)}{}^{[\nu\iota][4s]}{}_{,\iota}+H_{(n)}{}^{[\nu\sigma][4s]}\}n_s dS.\end{aligned} \tag{4.2}$$

Because of the differentiation occurring inside the brackets, the value of the total angular momentum will depend on n. As in the case of the total energy and momentum, the total angular momentum is unchanged by coordinate transformations which reduce to the identity transformations on the surface of integration sufficiently fast. Furthermore, with respect to linear coordinate transformations $\mathfrak{M}_{(n)}{}^{[\nu\sigma]}$ transforms as a tensor density of weight $(n+1)$.

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[16] Reference 14, p. 203.

[17] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, Oxford, 1934), p. 205.

[18] Bergmann and Schiller, reference 5 and W. Pauli, reference 3.

All of the above complexes may be equally good for a description of angular momentum. However, as in the case of energy and momentum, a unique quantity is desirable. Since a symmetric form does not arise naturally from the transformation properties of the Lagrangian, the only heuristic principle available is that the total angular momentum should be an antisymmetric free-tensor, corresponding to the transformation properties of the angular momentum of free particles. This requirement picks out $M_{(-1)}{}^{[\nu\sigma]\mu}$ as the appropriate quantity to describe angular momentum.

Landau and Lifshitz take $M_{(0)}{}^{[\nu\sigma]\mu}$ to be the angular momentum. The total angular momentum defined in this manner will transform under linear coordinate transformations as a tensor density of weight one. However, as is shown by Eqs. (3.4), (3.9), and (3.21), the L-L quantities are more simply related to the canonical energy-momentum complex than the other symmetric quantities. Moreover, of the symmetric pseudotensors constructed, only the L-L $\tau^{\mu\nu}$ does not contain second derivatives of the metric tensor. It is this property which leads to (3.21). Therefore, from an aesthetic point of view, one would prefer to work with the L-L quantities. However, if one is to make a choice based on transformation properties, the L-L quantities must be rejected. In the case of angular momentum this type of argument is even less convincing than in the case of energy and momentum. It is to be hoped that better arguments will be developed.

However, the quantity $M_{(-1)}{}^{[\nu\sigma]\mu}$ can be compared with the angular momentum complex $\Omega^{[\nu\sigma]\mu}$ obtained by Bergmann and Thomson.[7]

$$M^{[\nu\sigma]\mu}=\Omega^{[\nu\sigma]\mu}-\{[\ln(-g)^{\frac{1}{2}}]_{,\iota}[x^{\nu}H_{(-1)}{}^{[\sigma\iota][\mu\kappa]}-x^{\sigma}H_{(-1)}{}^{[\nu\iota][\mu\kappa]}]+H_{(-1)}{}^{[\sigma\nu][\mu\kappa]}\}_{,\kappa}. \quad (4.3)$$

Thus the difference between the two quantities is the divergence of an antisymmetric form. For the total angular momentum we obtain

$$\mathfrak{M}_{(-1)}{}^{[\nu\sigma]}=\int M_{(-1)}{}^{[\nu\sigma]4}d^3x =\int \Omega^{[\nu\sigma]4}d^3x+\oint Z^{[\nu\sigma][4s]}n_s dS, \quad (4.4)$$

where $Z^{[\sigma\nu][\mu\kappa]}$ is the term in the brackets on the right-hand side of Eq. (4.3). It is tempting to say that the surface integral vanishes and that $M_{(-1)}{}^{[\nu\sigma]\mu}$ describes the same physical quantity as $\Omega^{[\nu\sigma]\mu}$. However, from Eq. (4.1) it is clear that $M_{(-1)}{}^{[\nu\sigma]\mu}$ itself is the divergence of an antisymmetric form. Furthermore, we recall from Eqs. (2.15) and (3.16) that the total energy and momentum can be written as surface integrals. In general, we cannot expect that any of these surface integrals will vanish. Therefore, we cannot conclude that $M_{(-1)}{}^{[\nu\sigma]\mu}$ and $\Omega^{[\nu\sigma]\mu}$ describe the same physical quantity.

5. CONCLUSION

By considering the transformation properties of the total energy-momentum and angular momentum we have chosen certain quantities for the corresponding affine tensors. However, the proof of these transformation properties follows from the strict conservation of the total energy-momentum and angular momentum:

$$J_{\mu,4}=\mathfrak{M}^{[\nu\sigma]}{}_{,4}=0.$$

If radiation occurs these conditions and the proof of the transformation properties break down. However, in all physical situations the same quantities should define the content of our space, so that the existence of radiation should not alter what we call energy and angular momentum. Therefore, the choice made here is not as suspect in this regard as might otherwise appear. We wish to emphasize again, however, the tentative nature of the choice made here for the physically meaningful quantities. The fact that a choice can be made on the basis of transformation properties does not mean that this point of view is fruitful, or even meaningful.

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