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Analysis of 40-Mev p-p Scattering*

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Analysis of the 40-Mev p-p scattering data of Johnston and Swenson shows that either the singlet proton-proton interaction gives an anomalously large D wave and an anomalously small S wave at 40 Mev, or that angular momentum states in addition to S, P, and D are present. The anomalously small S-wave fits to the data are always accompanied by large polarizations unless F waves are present as well. Hence, recent experimental measurements at Harvard, which show that the p-p polarization at 40 Mev is small, rule out the possibility of fitting the data with only S, P, and D waves.

I. INTRODUCTION

As is well known, low-energy proton-proton scattering can be used to determine an effective range and scattering length for the singlet even S state, but gives no information about the radial variation of the interaction. However, knowledge of the singlet S phase at 40 Mev, if accurate enough, can yield a limited amount of information about this, as has been stressed, for example, by Raphael.[1] Since the 32-Mev p-p angular distribution indicates the presence of strong noncentral forces,[2] which have been shown to give large polarizations at higher energies, it is anticipated that an accurate phase-shift analysis will be required in order to disentangle this piece of information. Although the wide-angle scattering at 40 Mev is nearly isotropic, S-wave scattering alone could not give the sharp interference minimum that is observed, and since central-force scattering would give a minimum at 90° c.m. rather than isotropy (see arguments in reference 1), it is clear from the start that the analysis must be made in terms of different phase shifts in each of the ^3P states.

Starting with the simplest assumption that only S and P waves are present, which has been extensively investigated by several authors,[3,4] it is impossible to obtain a reasonable fit to the data. If the method of Clementel and Villi[4] is slightly extended to include the $^1D_2$ state, it is only possible to fit the data for a D phase of 3.1° and an S phase of less than 33°. As is argued in detail in Sec. III, both of these values are completely at variance with presently accepted ideas about the singlet interaction. Also, recent measurements at 40 Mev by the Harvard group,[5] which show that the polarization is very small, rule out even the solutions for S less than 33°, since these solutions give large polarizations. Hence this analysis indicates that higher angular momentum states are already present at 40 Mev. If the possibility of coupling to the $^3F_2$ state is included in the analysis, a multiplicity of fits to the data is discovered, and can be followed as a continuous function of the assumed singlet S phase, and even a polarization measurement at a single energy probably does not lead to a unique phase shift analysis.

II. LEAST-SQUARES SOLUTIONS

Clementel and Villi[4] have made use of the fact that if only $^3P_{0,1,2}$ are present in addition to the singlet scattering, the p-p cross section may be written as

$$k^2\sigma(\theta)=k^2\sigma_{\text{sing}}(\theta)+k^2[^3\sigma_{\text{Mott}}(\theta)] +z_1A(\theta)+z_2B(\theta)+z_3P_2(\cos\theta), \quad (1)$$

where

$$\begin{aligned} z_1&=\sum_{J=0}^{2}(2J+1)\sin^2(^3P_J),\\ z_2&=\sum_{J=0}^{2}(2J+1)\sin(^3P_J)\cos(^3P_J),\\ z_3&=\tfrac{3}{2}\sin^2(^3P_1)+\tfrac{7}{2}\sin^2(^3P_2)\\ &\quad+4\sin(^3P_0)\sin(^3P_2)\cos(^3P_0-{}^3P_2)\\ &\quad+9\sin(^3P_1)\sin(^3P_2)\cos(^3P_1-{}^3P_2), \end{aligned} \quad (2)$$

with k=c.m. wave number, $\sigma_{\text{sing}}(\theta)$=singlet spin state scattering, θ=c.m. scattering angle, $^3\sigma_{\text{Mott}}(\theta)$=triplet spin state Mott (Coulomb) scattering, $P_2(\cos\theta)$ =Legendre polynomial, ^3P_J=nuclear triplet-P-wave phase shift for total angular momentum J, and

$$\begin{aligned} A(\theta)&=1+\frac{\eta}{2}\cos\theta\left[\frac{\sin[\eta\ln(s^2)+2\phi_1]}{s^2}-\frac{\sin[\eta\ln(c^2)+2\phi_1]}{c^2}\right],\\ B(\theta)&=-\frac{\eta}{2}\cos\theta\left[\frac{\cos[\eta\ln(s^2)+2\phi_1]}{s^2}-\frac{\cos[\eta\ln(c^2)+2\phi_1]}{c^2}\right], \end{aligned} \quad (3)$$

* This work was performed under the auspices of the U. S. Atomic Energy Commission.
[1] R. B. Raphael, Institute for Advanced Study, Princeton, New Jersey, March 13, 1957 (to be published).
[2] R. S. Christian and H. P. Noyes, Phys. Rev. **79**, 85 (1950).
[3] R. M. Thaler and J. Bengston, Phys. Rev. **94**, 679 (1954); Thaler, Bengston, and Breit, Phys. Rev. **94**, 683 (1954); A. Garren, Phys. Rev. **101**, 419 (1956); C. A. Klein, Nuovo cimento **1**, 581 (1955) and **2**, 38 (1955).
[4] E. Clementel and C. Villi, Nuovo cimento **2**, 352 and 1165 (1955); Clementel, Villi, and Jess, Nuovo cimento **5**, 907 (1957).
[5] R. Wilson (private communication).

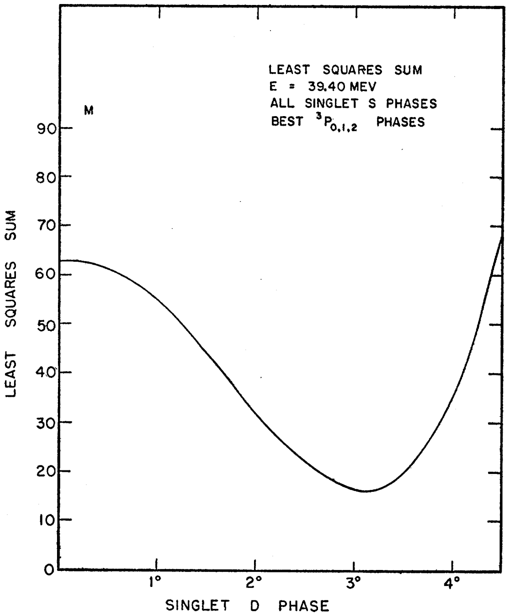

FIG. 1. Variation of the least-squares sum with the singlet D phase, assuming only S, P, and D waves present.

where $\eta = e^2/\hbar v_{\text{lab}}$, v_{lab} = velocity of incident proton in the laboratory system, $s = \sin(\theta/2)$, $c = \cos(\theta/2)$, and $\phi_1 = \tan^{-1}\eta$.

Since this expression is linear in the z's, the least-squares sum

$$M = \sum_{i=1}^{N} \left[(k^2\sigma_{\exp}(\theta_i) - k^2\sigma(\theta_i))/k^2\Delta\sigma_{\exp}(\theta_i) \right]^2, \quad (4)$$

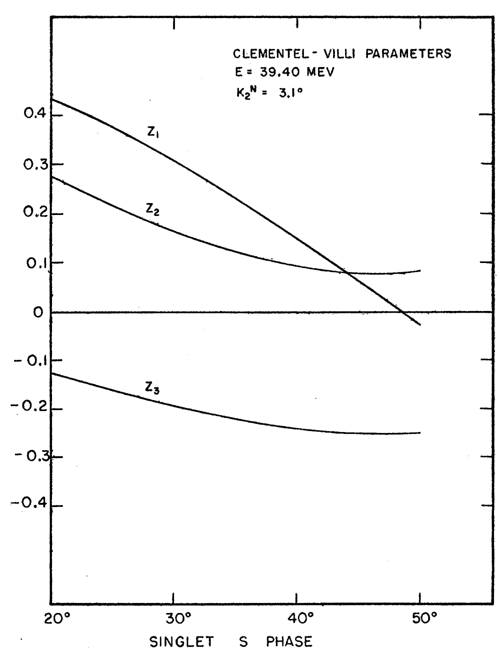

FIG. 2. Variation of the Clementel-Villi parameters with the singlet S phase for the singlet D phase which minimizes the least-squares sum.

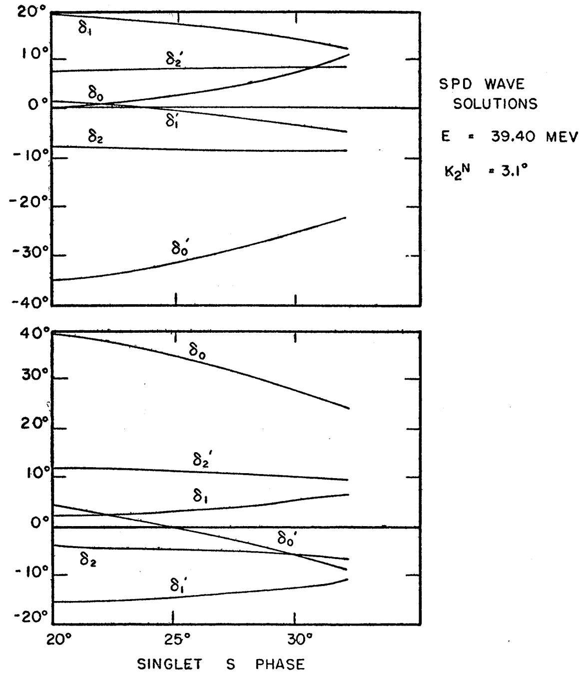

FIG. 3. The four sets of P phases which minimize the least-squares sum under the assumption that only S, P, and D phases are present.

where N = number of experimental points, $\sigma_{\exp}(\theta_i)$ = experimental cross section at c.m. angle θ_i, $\sigma(\theta_i)$ = calculated cross section at θ_i, and $\Delta\sigma_{\exp}(\theta_i)$ = uncertainty in the experimental cross section measurement, has a single minimum as a function of the z's. If it is assumed that the only singlet states present are $^1S_0$ and $^1D_2$, one can calculate M at this minimum as a function of the two corresponding singlet phase shifts K_0^N and K_2^N. When this is done for the 40-Mev p-p scattering data of Johnston and Swenson,[6] it is found that M is independ-

TABLE I. Phase shift sets assuming S, P, and D waves, with an S phase of 40°.

$^3P_0$	$^3P_1$	$^3P_2$	$^1D_2$	M	$P(40°)$	No. angles used[a]
−8.51	−7.04	7.76	2.12	29.1	0.0622	15
−8.66	−7.06	7.67	2.08	13.1	0.0661	13
−11.73	−5.72	7.55	2.19	27.8	0.0725	15
−18.09	−1.56	6.38	1.86	35.3	0.0750	15
10.11	9.31	−5.68	2.11	29.5	−0.0658	15
9.91	9.27	−5.73	2.05	13.4	−0.0656	13
16.53	6.36	−5.14	2.14	28.4	−0.0778	15
21.79	2.67	−3.38	1.68	40.0	−0.0691	15
13.03	−8.55	5.82	0.47	61.3	0.0401	15
12.88	−8.62	5.74	0.42	25.9	0.0395	13
20.78	−5.69	3.06	0.23	62.9	0.0234	15

[a] 15 experimental points were used in the least-squares fit. The rows marked "13 angles used" correspond to the "15-angle" rows directly above them and were obtained by eliminating the two smallest angles (4° and 5° in the lab system) from the least-squares fit.

[6] L. H. Johnston and D. Swenson, Linear Accelerator Laboratory Report, University of Minnesota, March, 1957 (unpublished), p. 9; Bull. Am. Phys. Soc. Ser. II, 2, 180 (1957); Phys. Rev. 111, 212 (1958), preceding paper.

ent of the value of K_0^N and varies with K_2^N according to the curve given in Fig. 1. It is seen from this curve that the best fit to the data is for $K_2^N=3.1°$. Presumably this unique value for the D phase arises because when Eq. (1) is fitted to the data, the fit requires an appreciable $(\cos\theta)^4$ term, and under the above assumptions, the coefficient of this term is a function of K_2^N alone.

The values of z_1, z_2, and z_3 for $K_2^N=3.1°$ are given in Fig. 2 as a function of K_0^N. When the trigonometric equations (2) are inverted to obtain the four[4] sets of P phases, it is found that the required value of z_3 is too negative to be compatible with the equations for z_1 and

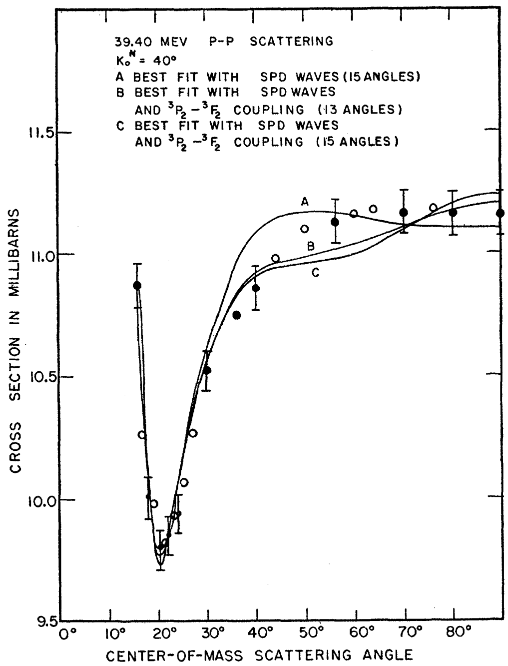

FIG. 4. Comparison of the best 5- and 6-phase shift fits to the data for an assumed S phase of 40°. The experimental points that have been used in computing the least-squares sum are shown with errors. The open circles are experimental points with the same errors. Points at 8°, 10°, 12°, and 14° were used in addition to those shown in the figure for the 15-angle fit. The 13-angle fit excluded the points at 8° and 10°.

z_2 if the S phase is greater than 33°. The P phases are plotted in Fig. 3 from an S phase of 20° up to this limiting value. It is possible to find least-squares solutions for larger S and smaller D phases, as is indicated in Table I, but these give statistically poorer fits to the data as is shown in Fig. 4. Since, as is discussed below, an S phase as small as 33° and a D phase as large as 3.1° are very unlikely, we now investigate whether the presence of higher partial waves will improve the fit to the data for more reasonable values of the singlet parameters.

A tensor interaction in the triplet odd states will couple $^3F_2$ to $^3P_2$. Since the coupling parameter varies

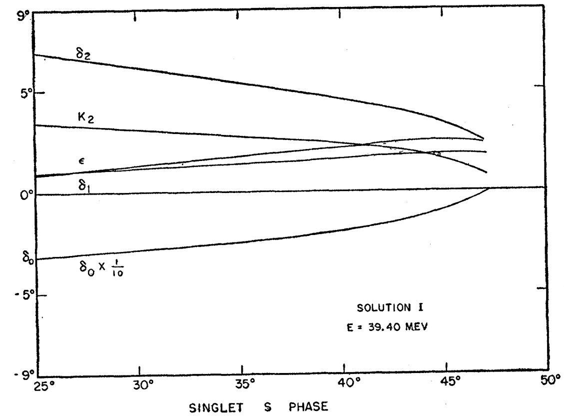

FIG. 5. Phase shift set which minimizes the least-squares sum; note that the coupling parameter ϵ is that defined by Stapp.[8] δ_0, δ_1, and δ_2 are the ^3P phase shifts. K_2 is the $^1D_2$ phase shift. Solution I.

as k^4 while the $^3F_2$ phase itself varies as k^5, one could have the coupling parameter present at 40 Mev without an appreciable $^3F_2$ phase; this is in fact the case for the model proposed by Gammel and Thaler.[7] Alternatively, a $\boldsymbol{\sigma}\cdot\mathbf{L}$ force of sufficiently long range could lead to a $^3F_2$ phase without coupling. It is found that either assumption, or both together, lead to indistinguishable fits to the data. Examples of phase shift sets obtained by a straightforward least-squares search are given in Figs. 5–8 as a function of the assumed singlet S phase, and an example of the multiplicity of fits obtainable for an S phase of 40° is given in Table II. The fit to the data, in comparison with an S, P, D fit using the same S phase, is given in Fig. 4. Note that the coupling parameter is that defined by Stapp,[8] *not* the Blatt-Biedenharn coupling parameter.

Since 15 points were used for the fitting, the expected value of M for a six-parameter curve is 9, whereas no

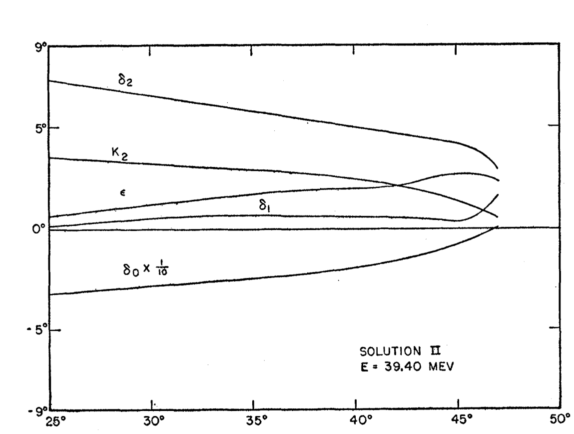

FIG. 6. Solution II. See caption for Fig. 5.

[7] Gammel, Christian, and Thaler, Phys. Rev. **105**, 311 (1957); J. L. Gammel and R. M. Thaler, Phys. Rev. **107**, 291 (1957).
[8] Stapp, Ypsilantis, and Metropolis, Phys. Rev. **105**, 302 (1957).

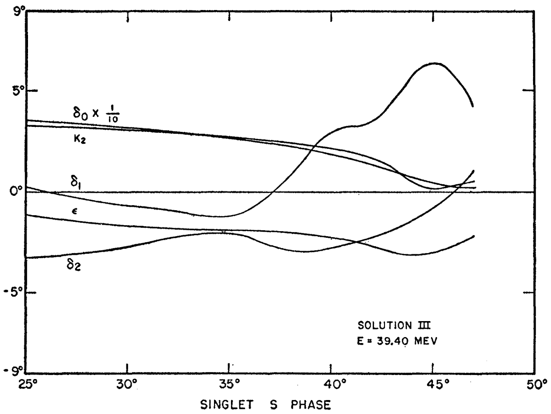

FIG. 7. See caption for Fig. 5. Solution III.

value of M less than 16.8 has been found. However, if the one point at 4° laboratory scattering angle is dropped, the M value is reduced to about the expected value. To insure that the attempt to fit the 4° point was not dominating the results, we made least-squares fits to 15-point curves and to 13-point curves (the latter having the two smallest angles, at 4° and 5° lab, removed), using the same starting point for the search. As Table II shows, the phase shift solutions were about the same for both sets of data. With the 13-point search, the M values are just as expected, showing that the solutions are consistent with the quoted errors. The calculated and experimental cross sections at the 4° lab angle differed by 2.5 standard deviations. Comparison between the 15-point and 13-point solutions is given in Fig. 4.

The solutions given in Figs. 5–8 and Table II make no pretense of being exhaustive. The plotted curves were obtained by systematically following an originally arbitrarily located set as a function of the S phase, but random searches often led to additional solutions which did not fall on any of these curves. Also, including the $^3F_2$ phase either with or without the coupling parameter led to still different solutions.

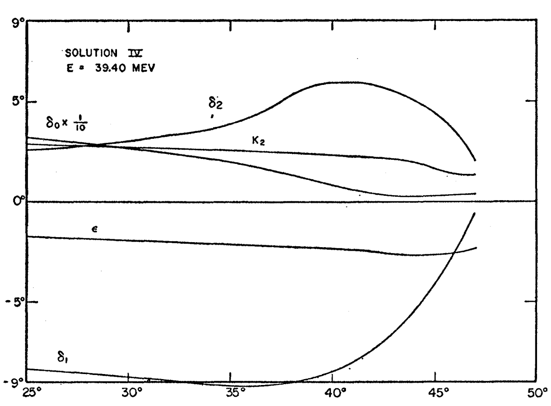

FIG. 8. Solution IV. See caption for Fig. 5.

After the work was completed, we received preliminary data from the Harvard group[5] which gave the maximum polarization at 46 Mev as $1.2 \pm 1.3\%$. This small value of the polarization not only eliminates the possibility of obtaining a satisfactory fit to the data with S, P, and D waves, but it also eliminates all of the solutions in Table II except one (this is not to imply that the one remaining is in any sense unique).

III. DISCUSSION

Since the foregoing analysis indicates a good fit to the angular distribution data (but not to the polarization data) for an S wave of less than 33° and a D wave of about 3.1°, it will be useful to see what values might reasonably be expected for these parameters. In making our estimates, we will ignore the difference between the true p-p phase shifts and the purely nuclear phase shifts, since direct calculation at[2] 32 Mev showed that this Coulomb correction is less than a degree for the S phase and negligible for the D phase. For orientation, we first calculate the S phase from the effective range expansion of $k \cot K_0^N$ in the shape-independent approximation (corresponding to a potential shape intermediate between exponential and Gaussian) and obtain 42.7° at 40 Mev. Longer-tailed potentials would give a larger value, and short-tailed or hard-core potentials a smaller one. As the high-energy p-p scattering clearly shows a repulsive interaction at short distances in the singlet state,[9] this value is to be taken as an upper limit. Actually, even if we knew the shape parameters of the

TABLE II. Phase shift sets assuming S, P, and D waves and $^3P_2$–$^3F_2$ coupling, with an S phase of 40°.

$^1D_2$	$^3P_0$	$^3P_1$	$^3P_2$	ϵ_2	M	$P(40°)$	No. angles used[a]
2.31	−5.77	−7.36	7.51	−1.91	16.8	0.0462	15
2.26	−6.37	−7.39	7.51	−1.63	6.3	0.0485	13
1.95	−10.66	−5.06	7.51	−2.32	17.5	0.0524	15
1.15	−13.74	−2.35	7.28	−2.76	18.3	0.0518	15
0.70	−14.30	−1.17	7.18	−2.98	20.2	0.0494	15
1.15	−6.67	−5.82	8.04	2.73	18.1	0.0694	15
1.22	−7.02	−5.94	7.96	2.59	9.0	0.0703	13
2.22	−15.85	−2.46	6.53	1.98	16.8	0.0825	15
2.24	−19.54	3.46	3.58	2.16	17.8	0.0502	15
1.42	11.64	8.67	−4.55	−2.70	18.9	−0.0716	15
1.40	11.33	8.78	−4.70	−2.56	9.6	−0.0720	13
2.16	16.89	5.60	−4.48	−2.12	17.1	−0.0824	15
2.27	20.71	2.15	−3.14	−1.98	16.9	−0.0750	15
2.33	−2.41	11.23	−4.17	2.11	18.0	−0.0302	15
2.25	−2.84	11.27	−4.17	1.99	8.5	−0.0305	13
1.83	14.49	7.22	−4.71	2.31	16.9	−0.0525	15
1.45	16.10	6.50	−4.13	2.66	18.3	−0.0469	15
2.34	4.21	−8.97	6.46	−2.12	18.2	0.0298	15
2.23	3.70	−9.12	6.56	−1.89	7.5	0.0313	13
2.29	7.05	−9.00	5.95	−2.13	17.9	0.0268	15
2.16	16.90	−6.93	3.04	−2.21	17.6	0.0073	15
2.22	21.35	−3.12	0.04	−2.23	18.3	−0.0311	15

[a] 15 experimental points were used in the least-squares fit. The rows marked "13 angles used" correspond to the "15-angle" rows directly above them and were obtained by eliminating the two smallest angles (4° and 5° in the lab system) from the least-squares fit.

[9] H. P. Noyes and H. G. Camnitz, Phys. Rev. 88, 1206 (1952).

effective range theory, we would not expect this expansion to converge much above 10 Mev, as was pointed out by Chew and Goldberger.[10] But as was shown by Raphael,[11] the logarithmic derivative $k \cot(K_0{}^N + k\bar{r})$ is much less energy-dependent, since $\bar{r}$ can be chosen in such a way that the first energy-dependent term to appear in the expansion of this quantity is of order k^4 or higher. If the logarithmic derivative is taken to be independent of energy, as in the boundary condition model,[12] we find an S phase of 39.5° at 40 Mev and of $-43.1°$ at 310 Mev. Thaler[13] has argued that the physically acceptable phase shift solution at 310 Mev is the Stapp-Ypsilantis-Metropolis Solution I,[8] which has an S phase of $-20.2°$. If we add to our expansion the small k^4 term of the logarithmic derivative which gives this value to the S phase at 310 Mev, the S phase predicted at 40 Mev changes by only one degree to 40.5°. The D phase for a Yukawa potential at this energy would be 2.4° and for a square well 1.3°. Monotonic potentials would be expected to give D phases between these values, and core potentials should give D phases close to the lower value. These estimates are substantiated by the calculation of Gammel and Thaler,[7] who fitted a model to the low-energy scattering and to the Stapp S and D phases

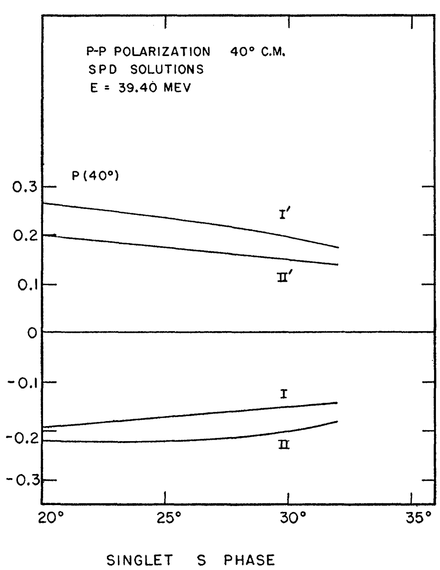

FIG. 9. Polarization at a scattering angle of 40° c.m. to be expected if the phase shift sets given in Fig. 3 are assumed; I and I′ correspond to the upper two sets in Fig. 3; II and II′ correspond to the lower two sets.

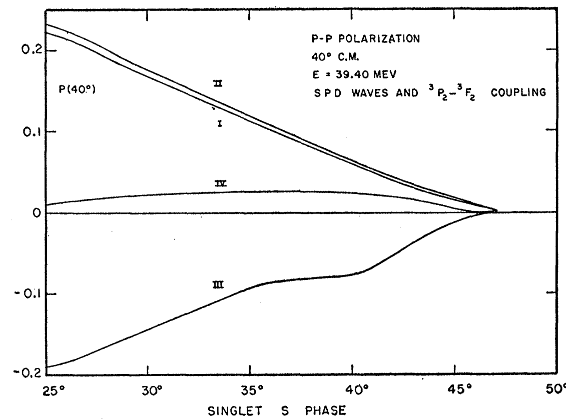

FIG. 10. Polarizations to be expected from the phase shift sets given in Figs. 5–8 at a scattering angle of 40° c.m.

at 310 Mev; they give an S phase of 39.82° and a D phase of 1.37° at 40 Mev.

From the above estimates it is clear that only a violently velocity-dependent singlet interaction could give rise to an S phase differing by more than a couple of degrees from 40°, or a D phase which did not lie between 1° and 2.5°. We have seen that it is possible to find solutions which meet these requirements, but that there are also equally good fits to the data with anomalously small S waves and anomalously large D waves. If only S, P, and D waves are present, these fits are accompanied by polarizations of about 20%, as can be seen in Fig. 9. Consequently the recent Harvard polarization measurements,[5] which give a maximum polarization of $1.2 \pm 1.3\%$ at 46 Mev, prove unambiguously the presence of F waves at this energy. Unfortunately, the small polarization that is observed does not completely rule out the possibility of anomalous singlet phases, since Solution IV gives small polarizations even when the S phases are small. (See Fig. 10.)

The tentative conclusion we reach from the above discussion is discouraging, since it shows that F waves make a substantial contribution to 40-Mev p-p scattering, and that it is unlikely that a polarization measurement would be able to give a unique phase shift solution. It is therefore believed that a minimal requirement for unique phase-shift analyses in this energy region is a set of experiments that follow both the angular distribution and the polarization as a function of energy, and even this cannot be guaranteed to lead to success.

IV. ACKNOWLEDGMENTS

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[10] G. F. Chew and M. L. Goldberger, Phys. Rev. **75**, 1637 (1949).
[11] R. B. Raphael, Phys. Rev. **102**, 905 (1956).
[12] G. Breit and W. Bouricius, Phys. Rev. **75**, 1029 (1949); H. Feshbach and E. Lomon, Phys. Rev. **102**, 891 (1956).
[13] R. M. Thaler, *Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics 1957* (Interscience Publishers, Inc., New York, 1957).