Integral Representation of a Double Commutator

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An integral representation is found for the vacuum expectation value of a double commutator. The problem of incorporating the Jacobi identity into such a representation remains unsolved.

I. STATEMENT OF RESULTS

LET x_1, x_2, x_3 be three space-time points, and $A(x)$, $B(x)$, $C(x)$ three scalar fields which may be the same or different. The vacuum expectation value of a double commutator will be denoted by

$$D(y,z)=D_{CBA}(y,z)=\langle[C(x_3),\,[B(x_2),\,A(x_1)]]\rangle_0, \quad (1)$$

$$y=x_1-x_2, \quad z=x_2-x_3. \quad (2)$$

It is assumed that the theory is invariant under proper inhomogeneous Lorentz transformations, but not necessarily under space or time reflection.

The Fourier transform of $D(y,z)$ is

$$F(p,q)=\int\int D(y,z)\exp[ip\cdot y+iq\cdot z]d_4y d_4z. \quad (3)$$

From the assumption that fields commute at points separated by a space-like interval, we deduce

$$D(y,z)=0 \text{ if either } y^2<0 \text{ or both } z^2<0,(y+z)^2<0. \quad (4)$$

From the assumption that there exists a complete set of states which are eigenstates of the total energy with non-negative eigenvalues, we deduce

$$F(p,q)=0 \text{ if either } q^2<0 \text{ or both } p^2<0,(p-q)^2<0. \quad (5)$$

It will be shown that the conditions (4), (5) imply the existence of an integral representation

$$D(y,z)=\int_0^\infty ds\int_0^\infty dt\int_0^1 d\lambda\,\psi(s,t,\lambda)\Delta_s(y)\Delta_t(z+\lambda y), \quad (6)$$

in which $\psi(s,t,\lambda)$ is a function of the real parameters s, t, λ in the indicated ranges, while $\Delta_s(y)$ is the usual invariant commutator function for a free field with mass $s^{\frac{1}{2}}$. There is a complete symmetry between position and momentum-space, since the Fourier transform of Eq. (6) has the form

$$F(p,q)=\int_0^\infty ds\int_0^\infty dt\int_0^1 d\lambda\,\phi(s,t,\lambda)\Delta_s(q)\Delta_t(p-\lambda q), \quad (7)$$

with a weight-function $\phi(s,t,\lambda)$ which is expressible in terms of ψ. We shall prove also that the representation (6) or (7) is unique, so that the weight-function ψ or ϕ is determined when D or F is given.

The representation is preliminary to an attempted analysis of the analytic structure of the double commutator, analogous to Lehmann's original investigation[1] of the single commutator. In order to analyze the double commutator completely, one would have to find a representation which includes the information contained in the Jacobi identity

$$D_{CBA}(y,z)+D_{BAC}(-y-z,\,y)+D_{ACB}(z,\,-y-z)\equiv 0. \quad (8)$$

The problem of satisfying Eq. (8) seems quite difficult.[2] It seemed worthwhile to place the representation (6) on record, since it is simple and may be of some practical use, although it fails to include this basic symmetry property of the double commutator.

II. PROOFS

We first prove the uniqueness of the representation (6), if it exists. For this purpose we introduce the mixed position-momentum function

$$G(y,q)=\int D(y,z)\exp[iq\cdot z]d_4z. \quad (9)$$

Equation (6) is then equivalent to

$$G(y,q)=\epsilon(y\cdot q)\int_0^\infty ds\int_0^\infty dt\int_0^1 d\lambda$$
$$\times\omega(s,t,\lambda)\delta(y^2-s)\delta(q^2-t)\exp[-i\lambda y\cdot q], \quad (10)$$

with $\omega(s,t,\lambda)$ determined in terms of $\psi(s,t,\lambda)$. We have to prove that $\omega(s,t,\lambda)\equiv 0$ if $G(y,q)\equiv 0$. The integral on the right of Eq. (10) is a function of three real variables, $(y^2,q^2,y\cdot q)$. If $G\equiv 0$ this integral vanishes in the physical range where $(y\cdot q)^2\geqslant y^2q^2$. But because the range of variation of λ is bounded, the integral is an entire function of $(y\cdot q)$ for fixed y^2, q^2. If the integral vanishes for physical values of $(y\cdot q)$, it must vanish also in the nonphysical range. But then the Fourier integral with respect to the variable $(y\cdot q)$ can be inverted, giving the result $\omega(s,t,\lambda)=0$.

The proof of existence of the representation (6) is a

[1] H. Lehmann, Nuovo cimento **11**, 342 (1954).

[2] It is known that the Wightman function $W(y,z)=\langle C(x_3)B(x_2)A(x_1)\rangle_0$ can be expressed in terms of the double commutators $D(y,z)$ if and only if Eq. (8) is satisfied. Therefore an integral representation of $D(y,z)$ satisfying Eq. (8) would automatically include the deep results of Wightman and Källèn concerning the domain of regularity of $W(y,z)$. See G. Källèn, *Proceedings of Seventh Annual Rochester Conference on High-Energy Nuclear Physics, 1957* (Interscience Publishers, Inc., New York, 1957), Session IV, p. 17.

simple deduction from the work of Jost and Lehmann[3] on the single commutator. Suppose that the vector y has the special form $(2a,0,0,0)$ with $a>0$. Write

$$D(y,z)=d(a,w), \quad w=z+\tfrac{1}{2}y, \tag{11}$$

and consider $d(a,w)$ as a function of the vector $w=(w_0,\mathbf{w})$ for fixed a. By Eq. (5), the Fourier transform of $d(a,w)$ with respect to w is zero outside the light-cone. By Eq. (4), the function $d(a,w)$ itself vanishes in the region

$$|\mathbf{w}|>|w_0|+a. \tag{12}$$

According to Theorem 2 of Jost and Lehmann,[3] these conditions are sufficient to ensure that $d(a,w)$ possesses a representation

$$d(a,w)=\int d_3u\int_0^\infty dt[\Phi_1(\mathbf{u},t)+\Phi_2(\mathbf{u},t)(\partial/\partial w_0)]\Delta_t(w-\mathbf{u}), \tag{13}$$

in which the integration over the three-vector $\mathbf{u}$ extends over the sphere $|\mathbf{u}|\leqslant a$. Since $d(a,w)$ has three-dimensional rotational symmetry, the weight-functions Φ_1 and Φ_2 will be functions of $\mathbf{u}^2$ and t only.

The problem now is to convert the "horizontal" representation (13), in which the auxiliary variable is the space-like vector $\mathbf{u}$, into a "vertical" representation in which the auxiliary variable is a purely time-like vector. Let $\tilde{v}$ denote the time-like vector $(v,0,0,0)$. The conversion is made by means of the identity

$$\int d_3u\, \delta(\mathbf{u}^2-b^2)\Delta_t(w-\mathbf{u})=\pi\int_{-b}^{b} dv\, I_0([t(b^2-v^2)]^{\frac{1}{2}})\Delta_t(w-\tilde{v}), \tag{14}$$

in which I_0 is the Bessel function with imaginary argument. We defer the verification of Eq. (14) to the appendix. Substituting Eq. (14) into Eq. (13), we find

$$d(a,w)=\int_0^\infty dt\int_{-a}^{a} dv\, \mu(a,t,v)\Delta_t(w-\tilde{v}), \tag{15}$$

with

$$\mu(a,t,v)=2\pi\int_{|v|}^{a} bdb\, \Phi_1(b^2,t)I_0([t(b^2-v^2)]^{\frac{1}{2}})+2\pi(\partial/\partial v)\int_{|v|}^{a} bdb\, \Phi_2(b^2,t)I_0([t(b^2-v^2)]^{\frac{1}{2}}). \tag{16}$$

Writing $\tilde{v}=(\tfrac{1}{2}-\lambda)y$ and using Eq. (11), we deduce from Eq. (15)

$$D(y,z)=\int_0^\infty dt\int_0^1 d\lambda \times\nu(a,t,\lambda)\Delta_t(z+\lambda y), \quad y=(2a,0,0,0). \tag{17}$$

We have proved Eq. (17) only for positive a. Writing $a^2=y^2/4$, and using the invariance of $D(y,z)$ under proper Lorentz transformations, we may conclude from Eq. (17) that Eq. (6) holds for all y in the future light-cone. For space-like y, both sides of Eq. (6) are zero. It remains to prove that Eq. (6) holds for y in the past light-cone with the same weight-function ψ. For this purpose we invoke the TCP invariance of the theory, which according to Jost[4] is a consequence of the local commutativity which has already been assumed. The *TCP* invariance implies

$$D(y,z)=D(-y,\,-z). \tag{18}$$

Since the right side of Eq. (6) is even in y and z jointly, Eq. (18) is sufficient to complete the proof.

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APPENDIX. VERIFICATION OF EQ. (14)

We wish to verify that Eq. (14) holds for every 4-vector w. Take the Fourier transform of Eq. (14) with respect to w; all components vanish on both sides except on the mass shell with mass $t^{\frac{1}{2}}$. Thus Eq. (14) reduces to

$$2\frac{\sin[b(r^2-t)^{\frac{1}{2}}]}{(r^2-t)^{\frac{1}{2}}}=\epsilon(b)\int_{-|b|}^{|b|} dv\, e^{ivr}I_0([t(b^2-v^2)]^{\frac{1}{2}}). \tag{19}$$

Both sides of Eq. (19) are entire functions of t. It is therefore sufficient to verify Eq. (19) for real negative t. Setting $t=-m^2$ and taking the Fourier transform with respect to r, Eq. (19) becomes

$$\epsilon(b)\theta(b^2-v^2)J_0[m(b^2-v^2)^{\frac{1}{2}}]=\frac{1}{\pi i}\int\int_{-\infty}^{\infty} dxdr\, \epsilon(x)\delta[x^2-r^2-m^2]\times\exp[i(bx-vr)]. \tag{20}$$

This formula is the two-dimensional analog to the standard expression for the commutator-function of a free field in four dimensions. To prove Eq. (20), observe that both sides are invariant under Lorentz transformations of the 2-vectors (b,v) and (x,r). The right side, being an odd invariant function, must vanish when (b,v) is space-like. It remains only to verify Eq. (20) for time-like (b,v); in this case, because of the Lorentz invariance, we may assume without loss of generality that $v=0$. When $v=0$, Eq. (20) reduces to

$$\epsilon(b)J_0(mb)=\frac{2}{\pi}\int_0^\infty \sin(mb\cosh\theta)d\theta, \tag{21}$$

which is a standard formula for the Bessel function.

[3] R. Jost and H. Lehmann, Nuovo cimento **5**, 1598 (1957).

[4] R. Jost, Helv. Phys. Acta **30**, 409 (1957).