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Dirac-Like Wave Equations for Particles of Zero Rest Mass, and Their Quantization*

J. S. LOMONT
International Business Machines Research Center, Ossining, New York
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The basic algebraic structure of the Maxwell equations (in a particular form) is first abstracted. This structure is then used as a model for wave equations for other massless particles. Gauge-independent wave equations of Dirac type (more precisely, of Pauli type) are thus found for every half-integral positive spin. Multiple-spin equations of Dirac types are also found. The single-spin equations are then quantized. The infinite-dimensional equations are not considered.

1. INTRODUCTION

MANY papers have been written on wave equations for massless particles of arbitrary spin.[1] Most of these approaches utilize spinor analysis with the result that the equations so derived are complicated in appearance. Also, although spinor notation facilitates quick derivation of wave equations, it frequently obscures the underlying structure of a theory.

The present theory is an algebraic (nonspinor) one which takes the photon as a model for all massless particles. Most important, however, is the fact that the wave equations yielded by this theory have an extremely simple and usable form, namely the Dirac form

$$\alpha_\mu \nabla^\mu \psi = 0, \quad (\mu = 0, 1, 2, 3) \tag{1.1}$$

where the α's are square Hermitian matrices satisfying

$$[\alpha_i, \alpha_j]_+ = 2\delta_{ij}, \quad (i, j = 1, 2, 3). \tag{1.2}$$

Besides being simple in form, these equations have the following desirable properties:

(1) For every positive half-integral spin there is an equation.

(2) The dimension of the matrices of the spin s equation is only $4s$.

(3) The equations are gauge-independent; i.e., only transverse particles (or states) enter the theory.

(4) The spin $\frac{1}{2}$ and spin 1 equations are the (2-component) neutrino and photon equations, respectively.

(5) Massless particles with more than one spin state can also be described by these equations.

2. MAXWELL EQUATIONS

It has been shown by Moses[2] that the Maxwell equations can be written in a gauge-independent, Dirac-like form. If[3]

$$\psi = \begin{bmatrix} H_x - iE_x \\ H_y - iE_y \\ H_z - iE_z \\ 0 \end{bmatrix}, \tag{2.1}$$

$$\alpha_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\alpha_2 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{2.2}$$

$$\nabla^\mu = \partial/\partial x_\mu, \quad \mu = (0,1,2,3), \tag{2.3}$$

$$x_0 = -t, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z, \tag{2.4}$$

then the free-field Maxwell equations can be written

$$\alpha_\mu \nabla^\mu \psi = 0. \tag{2.5}$$

The α's are Hermitian and satisfy the Dirac anticommutation relations[4]

$$[\alpha_i, \alpha_j]_+ = 2\delta_{ij}, \quad (i, j = 1, 2, 3) \tag{2.6}$$

and the relations

$$\alpha_1 \alpha_2 = i\alpha_3, \quad (\text{cycl. } 1, 2, 3). \tag{2.7}$$

Equations (2.6) and (2.7) imply

$$[\alpha_1, \alpha_2]_- = 2i\alpha_3 \ (\text{cycl.}), \tag{2.8}$$

and conversely (2.7) and (2.8) together with the fact that $\alpha_i^2 = I$ imply (2.6).

* A preliminary report of this work was made at the 1958 New York Meeting of the American Physical Society [J. S. Lomont, Bull. Am. Phys. Soc. Ser. II, 3, 36 (1958)].

[1] See the reference list in the book by E. M. Corson; *Introduction to Tensors, Spinors, and Relativistic Wave Equations* (Hafner Publishing Company, Inc., New York, 1953), p. 177.

[2] H. E. Moses, Suppl. Nuovo cimento **7**, 1 (1958).

[3] We take $c = \hbar = 1$, $-g_{00} = g_{11} = g_{22} = g_{33} = 1$.

[4] Greek indices run from 0 to 3; Latin indices from 1 to 3.

If under the infinitesimal Lorentz transformation

$$\delta x^\lambda = \omega^{\lambda\mu} g_{\mu\nu} x^\nu, \tag{2.9}$$

$$\omega^{\lambda\mu} = -\omega^{\mu\lambda}, \tag{2.10}$$

ψ transforms according to the equation

$$\delta\psi = -\tfrac{1}{2}\omega^{\mu\nu} K_{\mu\nu}\psi, \tag{2.11}$$

$$K_{\mu\nu} = -K_{\nu\mu}, \tag{2.12}$$

where the six independent $K_{\mu\nu}$'s are 4×4 matrices, then it follows that the $K_{\mu\nu}$'s and α_λ's satisfy the commutation relations

$$[K_{\mu\nu},\alpha_\lambda]_- = -\alpha_\mu(g_{\nu\lambda}+g_{\nu 0}\alpha_\lambda)+\alpha_\nu(g_{\mu\lambda}+g_{\mu 0}\alpha_\lambda). \tag{2.13}$$

These relations are also satisfied by the α's and K's of the Dirac electron theory. Since the K's are the infinitesimal transformations of the Lorentz group they must also satisfy the commutation relations

$$[K_{\kappa\lambda},K_{\mu\nu}]_- = -g_{\lambda\mu}K_{\kappa\nu}+g_{\lambda\nu}K_{\kappa\mu} + g_{\kappa\mu}K_{\lambda\nu} - g_{\kappa\nu}K_{\lambda\mu}. \tag{2.14}$$

3. GENERALIZATION

The basic algebraic structure of the Maxwell equation of the preceding section will now be extracted to provide a basis for describing a large class of Dirac-like wave equations for zero-rest-mass particles.

The wave equations will be assumed to be of the form

$$\alpha_\mu \nabla^\mu \psi = 0, \tag{3.1}$$

where ψ is a column matrix, the α's are Hermitian matrices of the same dimension,[5]

$$\alpha_0 = -I, \tag{3.2}$$

$$\alpha_i^2 = I, \tag{3.3}$$

and

$$\alpha_1\alpha_2 = i\alpha_3 \text{ (cycl.)}. \tag{3.4}$$

From (3.4) together with the Hermiticity of the α's it follows that

$$[\alpha_1,\alpha_2]_- = 2i\alpha_3 \text{ (cycl.)}. \tag{3.5}$$

Furthermore, if under the infinitesimal Lorentz transformation

$$\delta x^\lambda = \omega^{\lambda\mu} g_{\mu\nu} x^\nu, \tag{3.6}$$

$$\omega^{\lambda\mu} = -\omega^{\mu\lambda}, \tag{3.7}$$

the wave function ψ transforms according

$$\delta\psi = -\tfrac{1}{2}\omega^{\mu\nu} K_{\mu\nu}\psi, \tag{3.8}$$

$$K_{\mu\nu} = -K_{\nu\mu}, \tag{3.9}$$

then the K's must satisfy the commutation relations

$$[K_{\mu\nu},\alpha_\lambda]_- = -\alpha_\mu(g_{\nu\lambda}+g_{\nu 0}\alpha_\lambda)+\alpha_\mu(g_{\mu\lambda}+g_{\mu 0}\alpha_\lambda), \tag{3.10}$$

$$[K_{\kappa\lambda},K_{\mu\nu}]_- = -g_{\lambda\mu}K_{\kappa\nu}+g_{\lambda\nu}K_{\kappa\mu}+g_{\kappa\mu}K_{\lambda\nu}-g_{\kappa\nu}K_{\lambda\mu}. \tag{3.11}$$

These equations describe the structure of the theory to be investigated.

Equations (3.3), (3.4), and (3.5) of course imply the Dirac anticommutation relations

$$[\alpha_i,\alpha_j]_+ = 2\delta_{ij}. \tag{3.12}$$

The condition (3.4) on the α's differentiates these α's from the Dirac (or nonzero rest mass) α's because it can easily be shown[6] that there is no Hermitian matrix β (of the same dimension as the α's) such that

$$\beta^2 = I, \tag{3.13}$$

$$[\beta,\alpha_i]_+ = 0. \tag{3.14}$$

The α matrices used here might be called Pauli α matrices because of the analogy to the Pauli spin matrices.

The wave Eq. (3.1) can be written in Hamiltonian form as

$$H\psi = i(\partial\psi/\partial t), \tag{3.15}$$

where

$$H = -i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}. \tag{3.16}$$

Since

$$H^2 = -\nabla^2 \tag{3.17}$$

it follows that ψ satisfies the zero-rest-mass Klein-Gordon equation

$$\Box\psi = 0, \tag{3.18}$$

where

$$\Box = \nabla^2 - \nabla_0{}^2 = \nabla^\mu\nabla_\mu. \tag{3.19}$$

Finally, the covariance of the wave Eq. (3.1) under the proper, orthochronous, homogeneous Lorentz group can be demonstrated.

$$\begin{aligned}\delta(\alpha_\lambda\nabla^\lambda\psi) &= \alpha_\lambda(\delta\nabla^\lambda)\psi+\alpha_\lambda\nabla^\lambda\delta\psi \\ &= \alpha_\lambda\omega^{\lambda\mu}g_{\mu\nu}\nabla^\nu\psi - \tfrac{1}{2}\alpha_\lambda\nabla^\lambda\omega^{\mu\nu}K_{\mu\nu}\psi \\ &= \{\omega^{\lambda\mu}g_{\mu\nu}\alpha_\lambda\nabla^\nu - \tfrac{1}{2}\omega^{\mu\nu}K_{\mu\nu}\alpha_\lambda\nabla^\lambda \\ &\qquad -\tfrac{1}{2}\omega^{\mu\nu}\alpha_\mu(g_{\nu\lambda}+g_{\nu 0}\alpha_\lambda)\nabla^\lambda \\ &\qquad +\tfrac{1}{2}\omega^{\mu\nu}\alpha_\nu(g_{\mu\lambda}+g_{\mu 0}\alpha_\lambda)\nabla^\lambda\}\psi \\ &= 0.\end{aligned} \tag{3.20}$$

4. LIE ALGEBRA $\mathfrak{A}$

From (3.5), (3.10), and (3.11) one sees immediately that the 9 matrices α_1, α_2, α_3, K_{23}, K_{31}, K_{12}, K_{01}, K_{02}, K_{03}, form the basis of a Lie algebra $\mathfrak{A}$. All possible matrix forms of the α's and K's can therefore be found by finding all representations of this algebra. Since the algebra $\mathfrak{A}$ is semi-simple [as is shown by (4.8)] the representations of $\mathfrak{A}$ are all direct sums of irreducible representations. Therefore, only the irreducible representations need to be found. It must be born in mind that only those representations satisfying the condition (3.4) are usable.

[5] I is the unit matrix of appropriate dimension.

[6] I am indebted to H. E. Moses for pointing this out in a private discussion.

If we let

$$\mathbf{K}=(K_{23},K_{31},K_{12}), \tag{4.1}$$

$$\mathfrak{K}=(K_{01},K_{02},K_{03}), \tag{4.2}$$

$$\mathbf{M}=\mathbf{K}+\tfrac{1}{2}i\boldsymbol{\alpha}, \tag{4.3}$$

$$\mathfrak{M}=\mathfrak{K}+\tfrac{1}{2}\boldsymbol{\alpha}, \tag{4.4}$$

$$(M_{23},M_{31},M_{12})=\mathbf{M}, \tag{4.5}$$

$$(M_{01},M_{02},M_{03})=\mathfrak{M}, \tag{4.6}$$

then the $M_{\mu\nu}$'s satisfy the commutation relations of the Lie algebra $\mathfrak{L}$ of the Lorentz group

$$[M_{\kappa\lambda},M_{\mu\nu}]_-=-g_{\lambda\mu}M_{\kappa\nu}+g_{\lambda\nu}M_{\kappa\mu}+g_{\kappa\mu}M_{\lambda\nu}-g_{\kappa\nu}M_{\lambda\mu}. \tag{4.7}$$

Furthermore, the $M_{\mu\nu}$'s commute with the α's, and the α's form a basis of the Lie algebra $\mathfrak{R}$ of the three-dimensional rotation group. Hence, the Lie algebra $\mathfrak{A}$ of the α's and K's is the direct sum of the two Lie algebras $\mathfrak{L}$ and $\mathfrak{R}$,

$$\mathfrak{A}=\mathfrak{L}\dotplus\mathfrak{R}. \tag{4.8}$$

This relation essentially reduces the study of $\mathfrak{A}$ to the study of the well-known algebras $\mathfrak{L}$ and $\mathfrak{R}$.

5. IRREDUCIBLE REPRESENTATIONS OF $\mathfrak{A}$

In this paper attention will be restricted to the finite-dimensional irreducible representations of $\mathfrak{A}$. These can be constructed from the finite-dimensional, irreducible representations of $\mathfrak{L}$ and $\mathfrak{R}$ very easily.

The irreducible representations of $\mathfrak{L}$ can be labeled by two discrete indices m, $n=0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$. The irreducible representation $\Delta_{m,n}$ has dimension $(2m+1)\times(2n+1)$. The irreducible representations of $\mathfrak{R}$ can be labeled by a single index $j=0, \frac{1}{2}, 1, \frac{3}{2}\cdots$. The irreducible representation Δ_j has dimension $(2j+1)$. If

$$M_{\mu\nu}\to D^{(m,n)}(M_{\mu\nu}) \quad \text{in} \quad \Delta_{m,n}, \tag{5.1}$$

$$\alpha_i\to D^{(j)}(\alpha_i) \quad \text{in} \quad \Delta_j, \tag{5.2}$$

then

$$M_{\mu\nu}\to D^{(m,n)}(M_{\mu\nu})\otimes I^{(j)}, \tag{5.3}$$

$$\alpha_i\to I^{(m,n)}\otimes D^{(j)}(\alpha_i), \tag{5.4}$$

where $I^{(j)}$ and $I^{(m,n)}$ are the $(2j+1)$- and $(2m+1)(2n+1)$-dimensional unit matrices, respectively, in an irreducible representation of $\mathfrak{A}$. Furthermore, every irreducible representation of $\mathfrak{A}$ is of this form (for some m, n, j).

The restriction (3.4) means that the usable matrices $D^{(j)}(\alpha_i)$ must also form an irreducible representation of the Pauli ring.[1] Since the only irreducible representation of the Pauli ring is two-dimensional, only the irreducible representation

$$\alpha_i\to D^{(\frac{1}{2})}(\alpha_i) \tag{5.5}$$

of $\mathfrak{R}$ is usable.

The usable irreducible representations of $\mathfrak{A}$ are therefore

$$M_{\mu\nu}\to D^{(m,n)}(M_{\mu\nu})\otimes I^{(\frac{1}{2})}, \tag{5.6}$$

$$\alpha_i\to I^{(m,n)}\otimes D^{(\frac{1}{2})}(\alpha_i). \tag{5.7}$$

For $D^{(\frac{1}{2})}(\alpha_i)$ one can take the Pauli spin matrices

$$D^{(\frac{1}{2})}(\alpha_1)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad D^{(\frac{1}{2})}(\alpha_2)=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad D^{(\frac{1}{2})}(\alpha_3)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{5.8}$$

The α matrices (5.7) are now seen to be Hermitian. Let us call the usable irreducible representation of $\mathfrak{A}$, defined by (5.6) and (5.7), $\Gamma_{m,n}$.

6. SUBALGEBRA $\mathfrak{L}'$

The elements $K_{\mu\nu}$ of $\mathfrak{A}$ generate a subalgebra which is isomorphic to the Lie algebra of the Lorentz group, so it will be called $\mathfrak{L}'$ (to distinguish it from the subalgebra $\mathfrak{L}$ generated by $M_{\mu\nu}$). When the irreducible representation $\Gamma_{m,n}$ of $\mathfrak{A}$ is restricted to the subalgebra $\mathfrak{L}'$ it becomes a representation of $\mathfrak{L}'$. Let us call this "subduced" representation $\Gamma_{m,n}^{(s)}$. The question to be answered in this section is how $\Gamma_{m,n}^{(s)}$ decomposes into irreducible representations of $\mathfrak{L}'$.

To answer this question we shall evaluate the eigenvalues of the two Casimir operators

$$C_1=\mathbf{K}^2-\mathfrak{K}^2, \tag{6.1}$$

$$C_2=\mathbf{K}\cdot\mathfrak{K}. \tag{6.2}$$

These satisfy the relations

$$[C_1,K_{\mu\nu}]_-=0, \tag{6.3}$$

$$[C_2,K_{\mu\nu}]_-=0, \tag{6.4}$$

so that in an irreducible representation of $\mathfrak{L}'$ they are represented by scalar matrices. Let us furthermore define

$$D_1=\mathbf{M}^2-\mathfrak{M}^2, \tag{6.5}$$

$$D_2=\mathbf{M}\cdot\mathfrak{M}. \tag{6.6}$$

Then

$$\begin{aligned}C_1&=(\mathbf{M}-\tfrac{1}{2}i\boldsymbol{\alpha})^2-(\mathfrak{M}-\tfrac{1}{2}\boldsymbol{\alpha})^2\\&=\mathbf{M}^2-i\mathbf{M}\cdot\boldsymbol{\alpha}-\tfrac{1}{4}\boldsymbol{\alpha}^2-\mathfrak{M}^2+\mathfrak{M}\cdot\boldsymbol{\alpha}-\tfrac{1}{4}\boldsymbol{\alpha}^2\\&=\mathbf{M}^2-\mathfrak{M}^2-\tfrac{3}{2}-i(\mathbf{M}+i\mathfrak{M})\cdot\boldsymbol{\alpha},\end{aligned}$$

or

$$C_1-D_1+\tfrac{3}{2}=-i(\mathbf{M}+i\mathfrak{M})\cdot\boldsymbol{\alpha}. \tag{6.7}$$

Similarly

$$C_2-D_2-\tfrac{3}{4}i=-\tfrac{1}{2}(\mathbf{M}+i\mathfrak{M})\cdot\boldsymbol{\alpha}. \tag{6.8}$$

Using the relations

$$[(\mathbf{M}\cdot\boldsymbol{\alpha}),(\mathfrak{M}\cdot\boldsymbol{\alpha})]_+=2D_2+2i\mathfrak{M}\cdot\boldsymbol{\alpha}, \tag{6.9}$$

$$(\mathbf{M}\cdot\boldsymbol{\alpha})^2=\mathbf{M}^2+i\mathbf{M}\cdot\boldsymbol{\alpha}, \tag{6.10}$$

$$(\mathfrak{M}\cdot\boldsymbol{\alpha})^2=\mathfrak{M}^2-i\mathfrak{M}\cdot\boldsymbol{\alpha}, \tag{6.11}$$

one easily finds

$$\{(\mathbf{M}+i\mathfrak{M})\cdot\boldsymbol{\alpha}\}^2=D_1+2iD_2+2i(\mathbf{M}+i\mathfrak{M})\cdot\boldsymbol{\alpha}. \tag{6.12}$$

Combining (6.12) with (6.7) and with (6.8), one finds

$$(C_1-D_1+\tfrac{3}{2})^2-2(C_1-D_1+\tfrac{3}{2})+D_1+2iD_2=0, \quad (6.13)$$

$$4(C_2-D_2-\tfrac{3}{4}i)^2+4i(C_2-D_2-\tfrac{3}{4}i)-D_1-2iD_2=0. \quad (6.14)$$

We are now in a position to compute the eigenvalues of C_1 and C_2 in the representation $\Gamma_{m,n}$. In the irreducible representation $\Delta_{m,n}$ of $\mathcal{L}$ the operators D_1 and D_2 are scalar operators (since they commute with $M_{\mu\nu}$), and are given by the well-known expressions

$$D_1=-2\{m(m+1)+n(n+1)\}I, \quad (6.15)$$

$$D_2=i\{m(m+1)-n(n+1)\}I, \quad (6.16)$$

where I is the $(2m+1)(2n+1)$-dimensional unit matrix. From (5.6) we see that in the irreducible representation $\Gamma_{m,n}$ of $\mathcal{A}$ the operators D_1 and D_2 are given by (6.15) and (6.16) if I is taken to be the $2(2m+1)(2n+1)$-dimensional unit matrix. If C_1 and C_2 are in diagonal form the Eqs. (6.13) and (6.14) become [with the use of (6.15) and (6.16)] quadratic equations for the eigenvalues of C_1 and C_2. Solving these equations, one finds for the two distinct eigenvalues of C_1 and of C_2 the following:

$$C_1:\ {}^{(1)}-2\{(m+\tfrac{1}{2})(m+\tfrac{3}{2})+n(n+1)\},\quad {}^{(2)}-2\{(m-\tfrac{1}{2})(m+\tfrac{1}{2})+n(n+1)\}, \quad (6.17)$$

$$C_2:\ {}^{(1)}i\{(m+\tfrac{1}{2})(m+\tfrac{3}{2})-n(n+1)\},\quad {}^{(2)}i\{(m-\tfrac{1}{2})(m+\tfrac{1}{2})-n(n+1)\}. \quad (6.18)$$

Corresponding to each irreducible component of $\Gamma^{(s)}_{m,n}$ there is one pair of eigenvalues (γ_1,γ_2), where γ_1 and γ_2 are eigenvalues of C_1 and C_2, respectively. From (6.15), (6.16), (6.17), and (6.18) we see that there are only two possible pairs:

$$(1):\ \gamma_1=-2\{(m+\tfrac{1}{2})(m+\tfrac{3}{2})+n(n+1)\},\quad \gamma_2=i\{(m+\tfrac{1}{2})(m+\tfrac{3}{2})-n(n+1)\}, \quad (6.19)$$

$$(2):\ \gamma_1=-2\{(m-\tfrac{1}{2})(m+\tfrac{1}{2})+n(n+1)\},\quad \gamma_2=i\{(m-\tfrac{1}{2})(m+\tfrac{1}{2})-n(n+1)\}. \quad (6.20)$$

These two pairs correspond to the irreducible components $\Delta_{m+\frac{1}{2},n}$ and $\Delta_{m-\frac{1}{2},n}$. Hence

$$\Gamma^{(s)}_{m,n}=\Delta_{m+\frac{1}{2},n}\oplus\Delta_{m-\frac{1}{2},n}. \quad (6.21)$$

This is the desired result.

7. SPIN

Let us consider now the subalgebra of $\mathcal{L}'$ generated by K_{23}, K_{31}, and K_{12}. Since this algebra is isomorphic to the Lie algebra of the three-dimensional rotation group let us call it $\mathcal{R}'$ (the prime distinguishing it from the isomorphic algebra $\mathcal{R}$ generated by the α's). Under the infinitesimal coordinate rotation

$$\delta\mathbf{x}=-\boldsymbol{\omega}\times\mathbf{x}, \quad (7.1)$$

the wave function ψ transforms according to the equation

$$\delta\psi=-\boldsymbol{\omega}\cdot\mathbf{K}\psi. \quad (7.2)$$

Hence, a knowledge of the matrices representing K_{23}, K_{31}, and K_{12} will tell us how ψ transforms under a rotation, and therefore what the spin of a particle described by ψ can be. The eigenvalues of $-K^2$ are in fact the possible values of spin of a particle described by ψ.

From (6.21) it is seen that under a Lorentz transformation ψ is transformed by the representation $\Delta_{m+\frac{1}{2},n}\oplus\Delta_{m-\frac{1}{2},n}$. To find how ψ transforms under a rotation it is only necessary to find how these representations decompose when restricted to rotations. Let $\Delta_{m+\frac{1}{2},n}{}^{(s)}$ and $\Delta_{m-\frac{1}{2},n}{}^{(s)}$ be the representations of $\mathcal{R}'$ "subduced" by $\Delta_{m+\frac{1}{2},n}$ and $\Delta_{m-\frac{1}{2},n}$. Then, since

$$\Delta_{m+\frac{1}{2},n}{}^{(s)}=\Delta_{m+\frac{1}{2}}\otimes\Delta_n, \quad (7.3)$$

$$\Delta_{m-\frac{1}{2},n}{}^{(s)}=\Delta_{m-\frac{1}{2}}\otimes\Delta_n, \quad (7.4)$$

where Δ_j is the $(2j+1)$-dimensional irreducible representation of $\mathcal{R}'$, it follows from the Clebsch-Gordan series that

$$\Delta_{m+\frac{1}{2},n}{}^{(s)}=\sum_{j=|m+\frac{1}{2}-n|}^{m+\frac{1}{2}+n}\Delta_j, \quad (7.5)$$

$$\Delta_{m-\frac{1}{2},n}{}^{(s)}=\sum_{j=|m-\frac{1}{2}-n|}^{m-\frac{1}{2}+n}\Delta_j. \quad (7.6)$$

The wave equation will thus in general describe multiple-spin particles.

8. ZERO COMPONENTS OF ψ

From (2.1) we see that the ψ for the photon field has one component which is zero in all frames. It will now be shown that this condition can be generalized in a covariant way.

Let us consider ψ in the representation in which $\Gamma_{m,n}{}^{(s)}$ is completely reduced into $\Delta_{m+\frac{1}{2},n}$ and $\Delta_{m-\frac{1}{2},n}$, and let us require the components of ψ transforming under $\Delta_{m-\frac{1}{2},n}$ to be zero in one frame. These components will then be zero in all frames since they do not mix with the nonzero components when transformed. Thus, one can require

$$\psi=\begin{pmatrix}\chi\\0\end{pmatrix}, \quad (8.1)$$

where χ is $(2m+2)(2n+1)$-dimensional, and the zero part is zero in all frames.

It must also be shown that (8.1) is valid at all times if it is valid at one time. If the Hamiltonian (3.16) is expressed in the block form

$$H=\begin{pmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{pmatrix}, \quad (8.2)$$

then

$$H\psi=\begin{pmatrix}h_{11} & \chi\\ h_{21} & \chi\end{pmatrix}. \quad (8.3)$$

Consequently, if ψ is an eigenvector of H then

$$h_{21}\chi=0, \tag{8.4}$$

so that if ψ is a linear combination of eigenvectors of H, (8.4) is still valid. Hence, for such a ψ

$$i\frac{\partial\psi}{\partial t}=H\psi=\begin{pmatrix}h_{11}\chi\\ h_{21}\chi\end{pmatrix}=\begin{pmatrix}h_{11}\chi\\ 0\end{pmatrix}, \tag{8.5}$$

and we see that the zero components are zero at all times.

9. SINGLE-SPIN EQUATIONS

The class of equations defined by putting $n=0$, and the components of ψ transforming under $\Delta_{m-\frac{1}{2},0}$ equal to zero will now be considered in detail.[7] Furthermore, only these equations will be considered from here on. For clarity the index m will now be replaced by the index $s-\frac{1}{2}$:

$$m+\tfrac{1}{2}=s. \tag{9.1}$$

Since the nonzero components of ψ transform under a rotation according to the irreducible representation Δ_s of $\mathfrak{R}'$, the particles described by ψ have spin s.

The irreducible representation $\Delta_{s-\frac{1}{2},0}$ of $\mathfrak{L}$ is given by letting

$$\mathbf{M}\rightarrow -i\boldsymbol{\sigma}(s-\tfrac{1}{2}), \tag{9.2}$$

$$\mathfrak{M}\rightarrow -\boldsymbol{\sigma}(s-\tfrac{1}{2}), \tag{9.3}$$

where $\boldsymbol{\sigma}(s-\frac{1}{2})$ is an irreducible $2s$-dimensional triplet of matrices satisfying

$$[\sigma_1,\sigma_2]_-=i\sigma_3 \text{ (cycl.)}. \tag{9.4}$$

That is, $\boldsymbol{\sigma}(s-\frac{1}{2})$ is the spin vector matrix for spin $s-\frac{1}{2}$.

Using (9.2), (9.3), (5.6), (5.7), (5.8), (4.3), and (4.4) one finds for the representation $\Gamma_{s-\frac{1}{2},0}$ of $\mathfrak{A}$

$$K\rightarrow -i\boldsymbol{\sigma}(s-\tfrac{1}{2})\otimes I^{(\frac{1}{2})}-\tfrac{1}{2}iI^{(s-\frac{1}{2},0)}\otimes D^{(\frac{1}{2})}(\boldsymbol{\alpha}), \tag{9.5}$$

$$\mathfrak{K}\rightarrow -\boldsymbol{\sigma}(s-\tfrac{1}{2})\otimes I^{(\frac{1}{2})}-\tfrac{1}{2}I^{(s-\frac{1}{2},0)}\otimes D^{(\frac{1}{2})}(\boldsymbol{\alpha}), \tag{9.6}$$

$$\boldsymbol{\alpha}\rightarrow I^{(s-\frac{1}{2},0)}\otimes D^{(\frac{1}{2})}(\boldsymbol{\alpha}). \tag{9.7}$$

A unitary matrix U which transforms the representation $\Gamma_{s-\frac{1}{2},0}{}^{(s)}$ given by (9.5) and (9.6) into completely reduced form can now be given explicitly.[8] Let

$$u_1=-\begin{bmatrix}(2s)^{\frac{1}{2}} & 0 & 0 & \cdots & 0\\ 0 & (2s-1)^{\frac{1}{2}} & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & & & & \sqrt{1}\\ 0 & & & & 0\end{bmatrix}, \quad (2s+1)\times 2s, \tag{9.8}$$

$$u_2=\begin{bmatrix}0 & 0 & 0 & \cdots & 0\\ \sqrt{1} & 0 & 0 & \cdots & \\ 0 & \sqrt{2} & 0 & \cdots & \\ \vdots & \vdots & \vdots & & \\ 0 & & & & (2s)^{\frac{1}{2}}\end{bmatrix}, \quad (2s+1)\times 2s, \tag{9.9}$$

$$v_1=(-1)^{2s}\begin{bmatrix}0 & \sqrt{1} & 0 & 0 & \cdots & 0\\ 0 & 0 & \sqrt{2} & 0 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & & & & & (2s-1)^{\frac{1}{2}}\end{bmatrix}, \quad (2s-1)\times 2s, \tag{9.10}$$

$$v_2=(-1)^{2s}\begin{bmatrix}(2s-1)^{\frac{1}{2}} & 0 & 0 & \cdots & 0 & 0\\ 0 & (2s-2)^{\frac{1}{2}} & 0 & \cdots & & \\ \vdots & \vdots & \vdots & & & \\ 0 & & & & \sqrt{1} & 0\end{bmatrix}, \quad (2s-1)\times 2s. \tag{9.11}$$

The $4s$-dimensional unitary matrix U is defined by

$$U=\frac{1}{(2s)^{\frac{1}{2}}}\begin{pmatrix}u_1 & u_2\\ v_1 & v_2\end{pmatrix}. \tag{9.12}$$

If $K_{\mu\nu}$ is given by (9.5) and (9.6) then

$$U\mathbf{K}U^{-1}=-i\begin{pmatrix}\boldsymbol{\sigma}(s) & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}(s-1)\end{pmatrix}. \tag{9.13}$$

The transformed α's can be explicitly computed also. The result is

$$U\boldsymbol{\alpha}U^{-1}=\begin{pmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{b}^\dagger & \mathbf{d}\end{pmatrix}, \tag{9.14}$$

where

$$\mathbf{a}=(1/s)\boldsymbol{\sigma}(s), \tag{9.15}$$

$$\mathbf{d}=-(1/s)\boldsymbol{\sigma}(s-1), \tag{9.16}$$

$$b_1=\frac{(-1)^{2s-1}}{2s}\begin{bmatrix}[(2s-1)2s]^{\frac{1}{2}} & 0 & 0 & \cdots\\ 0 & [(2s-2)(2s-1)]^{\frac{1}{2}} & 0 & \cdots\\ -(1\times 2)^{\frac{1}{2}} & 0 & [(2s-3)(2s-2)]^{\frac{1}{2}} & \cdots\\ 0 & -(2\times 3)^{\frac{1}{2}} & 0 & \cdots\\ \vdots & \vdots & \vdots & \end{bmatrix}, \tag{9.17}$$

[7] An equivalent single-spin theory was found by C. L. Hammer and R. H. Good, Phys. Rev. **108**, 882 (1957). The Hammer-Good form of the Maxwell equations, $(\nabla_0+\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\psi=0$, where $\psi_j=E_j+iH_j$, was also given by L. Silberstein, Ann. Physik **22**, 579 (1907), **24**, 783 (1907), Phil. Mag. **23**, 790 (1912), and P. Weiss, Proc. Roy. Irish Acad. **A46**, 129 (1941).

[8] Reference 1, p. 51.

$$b_2=\frac{-i(-1)^{2s-1}}{2s}\begin{bmatrix} [(2s-1)2s]^{\frac{1}{2}} & 0 & 0 & \cdots \\ 0 & [(2s-2)(2s-1)]^{\frac{1}{2}} & 0 & \cdots \\ (1\times 2)^{\frac{1}{2}} & 0 & [(2s-3)(2s-2)]^{\frac{1}{2}} & \cdots \\ 0 & (2\times 3)^{\frac{1}{2}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}, \tag{9.18}$$

$$b_3=\frac{(-1)^{s-1}}{s}\begin{bmatrix} 0 & 0 & 0 & \cdots \\ [1\times(2s-1)]^{\frac{1}{2}} & 0 & 0 & \cdots \\ 0 & [2\times(2s-2)]^{\frac{1}{2}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}. \tag{9.19}$$

The form of $\boldsymbol{\sigma}$ used here is (for spin s)

$$\sigma_1=-\tfrac{1}{2}\begin{bmatrix} 0 & (1\times 2s)^{\frac{1}{2}} & 0 & 0 & \cdots \\ (1\times 2s)^{\frac{1}{2}} & 0 & [2\times(2s-1)]^{\frac{1}{2}} & 0 & \cdots \\ 0 & [2\times(2s-1)]^{\frac{1}{2}} & 0 & [3\times(2s-2)]^{\frac{1}{2}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \tag{9.20}$$

$$\sigma_2=\tfrac{1}{2}i\begin{bmatrix} 0 & (1\times 2s)^{\frac{1}{2}} & 0 & 0 & \cdots \\ -(1\times 2s)^{\frac{1}{2}} & 0 & [2\times(2s-1)]^{\frac{1}{2}} & 0 & \cdots \\ 0 & -[2\times(2s-1)]^{\frac{1}{2}} & 0 & [3\times(2s-2)]^{\frac{1}{2}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}, \tag{9.21}$$

$$\sigma_3=\begin{bmatrix} s & 0 & 0 & \cdots \\ 0 & s-1 & 0 & \cdots \\ 0 & 0 & s-2 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}. \tag{9.22}$$

For $s=\frac{1}{2}$ and 1 these single-spin wave equations become the equations for the neutrino and photon fields, respectively.

Using the above explicit form of $\boldsymbol{\alpha}$, one can find the eigenvectors of H. If one considers only the nonzero components of ψ and puts

$$\psi=u(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{9.23}$$

then one finds that there are only two states belonging to a given momentum $\mathbf{k}$. These are the states with spin parallel and antiparallel to $\mathbf{k}$. The corresponding eigenvalues of H are k and $-k$, respectively.[9] For the mth component of $u_\pm(\mathbf{k})$ one finds

$$u_{\pm,m}(\mathbf{k})=[2k(k\mp k_z)]^{-s}\left[\binom{2s}{s+m}\right]^{\frac{1}{2}} \times(\mp l^*)^{s+m}(k\mp k_z)^{s-m}, \tag{9.24}$$

where

$$l=k_x+ik_y. \tag{9.25}$$

It should be noted that

$$\mathbf{k}\cdot\boldsymbol{\sigma}u_\pm(\mathbf{k})=\pm sku_\pm(\mathbf{k}) \tag{9.26}$$

and

$$u_\epsilon^\dagger(\mathbf{k})u_{\epsilon'}(\mathbf{k})=\delta_{\epsilon\epsilon'}. \tag{9.27}$$

[9] Denoting the parallel and antiparallel states by $+$ and $-$, respectively.

10. QUANTIZATION

Let[10] $\psi(\mathbf{x},t)$ be the nonzero part of ψ and let us expand it in the form

$$\psi(\mathbf{x},t)=C\int d^3k\, k^{s-1}b_+(\mathbf{k})u_+(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-kt)} +C\int d^3k\, k^{s-1}b_-^\dagger(\mathbf{k})u_-(k)e^{i(\mathbf{k}\cdot\mathbf{x}+kt)}, \tag{10.1}$$

where C is a constant. Then the following commutation (or anticommutation) rules will be shown to be covariant.

$$[b_\epsilon(\mathbf{k}),b_\rho(\mathbf{l})]=[b_\epsilon^\dagger(\mathbf{k}),b_\rho^\dagger(\mathbf{l})]=0, \tag{10.2}$$

$$[b_\epsilon(\mathbf{k}),b_\rho^\dagger(\mathbf{l})]=k\delta_{\epsilon\rho}\delta(\mathbf{k}-\mathbf{l}), \tag{10.3}$$

where

$$\epsilon,\rho=\pm. \tag{10.4}$$

To prove the covariance of these commutation relations the transformation properties of the u's[11] and b's will be determined. Let us consider the infinitesimal rotation

$$\mathbf{k}'=\mathbf{k}-\boldsymbol{\omega}\times\mathbf{k}, \tag{10.5}$$

[10] The fermion fields have been treated in detail in Rarita-Schwinger form by C. G. Bollini, Nuovo cimento **8**, 39 (1958). Also, all nonzero spin fields have been quantized independently by C. L. Hammer and R. H. Good, Phys. Rev. **111**, 342 (1958).

[11] I am indebted to Dr. C. L. Hammer and Dr. R. H. Good, Jr., for a private communication indicating this line of derivation of the transformation property of $u_\epsilon(\mathbf{k})$.

where $\boldsymbol{\omega}$ is a real 3-vector, and

$$k'=k. \quad (10.6)$$

Then, by making use of the commutation relation (where $\mathbf{A}$ and $\mathbf{B}$ are 3-vectors)

$$[(\mathbf{A}\cdot\boldsymbol{\sigma}),(\mathbf{B}\cdot\boldsymbol{\sigma})]_-=i\mathbf{A}\times\mathbf{B}\cdot\boldsymbol{\sigma}, \quad (10.7)$$

together with (9.25), one easily finds that

$$(\mathbf{k}'\cdot\boldsymbol{\sigma})(1+i\boldsymbol{\omega}\cdot\boldsymbol{\sigma})u_\epsilon(\mathbf{k})=\epsilon sk'(1+i\boldsymbol{\omega}\cdot\boldsymbol{\sigma})u_\epsilon(\mathbf{k}). \quad (10.8)$$

Hence, $u_\epsilon(\mathbf{k}')$ is proportional to $(1+i\boldsymbol{\omega}\cdot\boldsymbol{\sigma})u_\epsilon(k)$. Since $(1+i\boldsymbol{\omega}\cdot\boldsymbol{\sigma})$ is unitary to first order in $\boldsymbol{\omega}$, it follows that $(1+i\boldsymbol{\omega}\cdot\boldsymbol{\sigma})u_\epsilon(k)$ is normalized. Consequently,

$$u_\epsilon(\mathbf{k}')=e^{i\varphi}(1+i\boldsymbol{\omega}\cdot\boldsymbol{\sigma})u_\epsilon(\mathbf{k}). \quad (10.9)$$

If one considers now the infinitesimal coordinate rotation

$$\delta\mathbf{x}=-\boldsymbol{\omega}\times\mathbf{x}, \quad (10.10)$$

$$\delta\psi=i\boldsymbol{\omega}\cdot\boldsymbol{\sigma}\psi, \quad (10.11)$$

one sees that the Fourier expansion (10.1) will be form-invariant if

$$\delta\mathbf{k}=-\boldsymbol{\omega}\times\mathbf{k},$$

so that $u_\epsilon(\mathbf{k}')$ is given by (10.9), and

$$b_+'(\mathbf{k}')=e^{-i\varphi}b_+(\mathbf{k}), \quad (10.12)$$

$$b_-'^\dagger(\mathbf{k}')=e^{-i\varphi}b_-^\dagger(\mathbf{k}). \quad (10.13)$$

An arbitrary Lorentz transformation L can be expressed in the form

$$L=R_1L_zR_2, \quad (10.14)$$

where R_1 and R_2 are rotations and L_z is a transformation to a parallel frame moving in the z direction. Since it has previously been shown[7] that the b's are scalars under L_z-transformations, it follows that under any infinitesimal Lorentz transformation the b's are transformed by a phase factor as in (10.12) and (10.13). As a result of this transformation property of the b's the commutation relations (10.2) are seen to be covariant. Furthermore, the commutator $[b_\epsilon(\mathbf{k}),b_\epsilon(\mathbf{l})]$ is easily seen to be a scalar.

To prove the covariance of the commutation relations it remains only to prove that $k\delta(\mathbf{k}-\mathbf{l})$ is a scalar. If

$$\delta\mathbf{k}=-\boldsymbol{\omega}\times\mathbf{k}+\epsilon\boldsymbol{\nu}k, \quad (10.15)$$

$$\delta k=\epsilon\boldsymbol{\nu}\cdot\mathbf{k}, \quad (10.16)$$

then[12]

$$\delta\{k\delta(\mathbf{k}-\mathbf{l})\}=\epsilon\boldsymbol{\nu}\cdot\mathbf{k}\delta(\mathbf{k}-\mathbf{l}) +k\{-\boldsymbol{\omega}\times(\mathbf{k}-\mathbf{l})+\epsilon\boldsymbol{\nu}(k-l)\}\cdot\boldsymbol{\nabla}\delta(\mathbf{k}-\mathbf{l}). \quad (10.17)$$

[12] Note that δ is used in two ways in (10.17).

Using the relations

$$\boldsymbol{\omega}\times(\mathbf{k}-\mathbf{l})\cdot\boldsymbol{\nabla}\delta(\mathbf{k}-\mathbf{l})=0, \quad (10.18)$$

$$(k-l)\boldsymbol{\nabla}\delta(\mathbf{k}-\mathbf{l})=-k^{-1}\mathbf{k}\delta(\mathbf{k}-\mathbf{l}), \quad (10.19)$$

together with (10.17), one finds that

$$\delta\{k\delta(\mathbf{k}-\mathbf{l})\}=0. \quad (10.20)$$

This completes the proof of the covariance of the commutation relations. These commutation relations reduce to the usual ones for the neutrino and photon fields.

The annihilation operators $a_\epsilon(\mathbf{k})$ are defined by

$$a_\epsilon(\mathbf{k})=k^{-\frac{1}{2}}b_\epsilon(\mathbf{k}), \quad (10.21)$$

the number operators by

$$N_\epsilon(\mathbf{k})=a_\epsilon^\dagger(\mathbf{k})a_\epsilon(\mathbf{k}), \quad (10.22)$$

and the Hamiltonian by

$$\mathcal{H}=\sum_{\epsilon=\pm}\int d^3k\, kN_\epsilon(\mathbf{k}). \quad (10.23)$$

The Hamiltonian satisfies the usual condition

$$\psi(\mathbf{x},t)=\exp(it\mathcal{H})\psi(\mathbf{x},0)\exp(-it\mathcal{H}). \quad (10.24)$$

From the preceding quantization it is clear that only transverse quanta enter this theory. Consequently, the theory is gauge independent.

The four-dimensional commutation relations in configuration space are

$$\frac{1}{|C|^2}[\psi(x),\psi^\dagger(y)]_\rho =|\nabla|^{2s-1}\mathcal{O}(-i\boldsymbol{\nabla})\{i|\nabla|\,\mathcal{O}^{(1-\rho)/2}(-i\boldsymbol{\nabla}) -\nabla_0\mathcal{O}^{(1+\rho)/2}(-i\boldsymbol{\nabla})\}\Delta(x-y), \quad (10.25)$$

where

$$\rho=\pm1, \quad (10.26)$$

$$\nabla_0=\partial/\partial x^0, \quad (10.27)$$

$$\mathcal{O}(\mathbf{k})=\frac{\mathbf{k}\cdot\boldsymbol{\sigma}}{sk} \quad \text{for} \quad s=\tfrac{1}{2},\,1, \quad (10.28)$$

$$\mathcal{O}(\mathbf{k})=\frac{\mathbf{k}\cdot\boldsymbol{\sigma}}{sk}\prod_m\left\{\frac{(\mathbf{k}\cdot\boldsymbol{\sigma}/sk)^2-(m/s)^2}{1-(m/s)^2}\right\} \quad \text{for} \quad s>1, \quad (10.29)$$

and the product is over all positive eigenvalues m of σ_z except $m=s$.