Improved Sum Rule for Electron-Deuteron Scattering*

R. BLANKENBECLER†
Stanford University, Stanford, California
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A qualitative discussion of the approximations implicit in the present theoretical treatment of the deuteron is presented. An improved sum rule which relates the total elastic and inelastic scattering of electrons from the deuteron to the free electron-nucleon cross sections is derived. It has the property of reducing to the correct relativistic result upon neglect of binding. It is proved that the use of folded nucleon and nuclear form factors is correct. It is shown that the finite nucleon size does not affect any real photon process.

I. INTRODUCTION

THE sum rule which relates the total elastic and inelastic scattering of electrons from nuclei to the scattering from free nucleons is well known. It has been derived for the particular case of the deuteron by Jankus[1] with certain approximations. His result is that the total scattering from the deuteron is equal to the elastic scattering from a free neutron and proton. Recent experimental results on electrodisintegration of the deuteron[2] promise to yield information on the electromagnetic structure of the neutron if the corrections to this approximate result can be evaluated. These corrections may be divided into two classes: kinematic and mesonic.

The kinematic corrections may be further divided into two classes. A knowledge of the free scattering cross sections over a finite energy range is required by the width of the momentum distribution in the deuteron. This suggests that the total cross section be written in terms of the free cross sections and their derivatives evaluated at the free kinematic values. This correction will be treated in detail in this paper. Since the nucleons are bound, their energy and momentum are not related in the usual way ($E^2 \neq p^2+M^2$). This allows the addition of terms to the nucleon current operator which will not contribute to free-particle scattering, and hence are not susceptible to a phenomenological analysis. The estimation of these terms with perturbation theory is completely unreliable. A dimensional argument of questionable validity will be used to get a qualitative estimate of these terms.

The mesonic corrections may be arbitrarily divided into three overlapping classes: the exchange of charged mesons which yields an additional current in the deuteron, the exchange of mesons which affects the structure of the electromagnetic vertex for any single nucleon, and the exchange of mesons by the outgoing nucleons which cannot be described satisfactorily by a static potential. The second effect might be thought of as a "warping" of the nucleon form factors although it depends on the structure of the bound state. The last effect might be estimated by using a "physical" argument such as used in an estimate of the mesonic contribution to the photodisintegration of the deuteron.[3] Just as in this case, the meson-nucleon scattering resonance will certainly affect the electrodisintegration, but because of the nucleon electromagnetic form factors, its relative effect should be considerably reduced. These effects cannot be calculated with perturbation theory due to the strong coupling and the resonance. Dispersion theory or a Chew-Low approach might prove useful. These mesonic corrections will not be dealt with in this paper in detail. A qualitative estimate of these effects is discussed in Sec. V.

In order to "see" the neutron, a large momentum transfer q is required. The nucleon magnetic moment scattering is then large compared to the proton charge scattering. Hence, we shall evaluate the sum rule in the limit of large q. Any attempt to describe such a situation with the Schrödinger equation is questionable. For example, at the experimentally interesting values of energy and q, the difference between relativistic and nonrelativistic kinematics is about ten percent. The cross sections for a Pauli nucleon and a Dirac nucleon differ by the same amount in this range because of the difference between the three- and four-momentum transfer which occurs squared as the coefficient of the magnetic-moment term. To try to patch up a Schrödinger result by the *ad hoc* introduction of relativistic corrections is ambiguous and, therefore, unsatisfactory. We shall consider a Bethe-Salpeter deuteron with an instantaneous potential which, with the neglect of nucleon pairs, reduces to a Breit deuteron. A survey of the approximations that are made will be carried out.

First, we will examine a simple model in order to make a qualitative estimate of the kinematic effects. Consider the deuteron to be a superposition of six states—the nucleons moving with a velocity $\pm V$ parallel to the incident electron momentum, perpendicular to the incident and final electron momenta, and perpendicular to the incident but in the plane of the incident and final momenta. The total cross section

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[1] V. Z. Jankus, Phys. Rev. 102, 1586 (1956).
[2] R. Hofstadter and M. Yearian, Phys. Rev. 110, 552 (1958).
[3] R. Wilson, Phys. Rev. 104, 218 (1956); N. Austern, Phys. Rev. 100, 1522 (1955).

will be an average of these six situations and an average over the velocity distribution.[4]

At large momentum transfers, the magnetic moment scattering is predominant. Neglecting terms which are small for backward scattering, the Rosenbluth cross section for point nucleons in this limit becomes

$$\sigma = C[1+k_0'(1-x')/M]^{-2}d\Omega'/v_R,$$

where x' is the cosine of the scattering angle and v_R is the relative velocity. The various quantities are evaluated in the frame in which the nucleon is initially at rest. Since this is a uniquely defined frame, the cross section can be written in an invariant form. The transformation of the solid angle is easily found by considering the invariant formed by the three-momentum volume element divided by the energy. The transformation of the incident flux is included. Invoking the conservation of four-momentum, we find

$$\sigma = C(k_0P_0/k_i\cdot P)(k_f\cdot P/k_i\cdot P)^2(d\Omega'/d\Omega)d\Omega, \quad (1)$$

where P is the initial four-momentum of the nucleon. This can be evaluated in the lab system to give

$$\sigma(v)/\sigma(0) = (1+v^2/2)(1-vt+v^2/2)^{-1} \times[1-vs/(1+y)+v^2/2(1+y)]^{-2}, \quad (2)$$

where

$$\mathbf{v}\cdot\mathbf{k}_f = vk_fs,$$
$$\mathbf{v}\cdot\mathbf{k}_i = vk_it,$$
$$y = k_i(1-x)/M.$$

The total point-nucleon cross section to order v^2 is

$$\sigma_T/\sigma_0 = 1+\tfrac{1}{3}\langle v^2\rangle_{\text{Av}}[1+3(1+y)^{-2}+(2x-3)(1+y)^{-1}]. \quad (3)$$

For $k_0=470$ Mev, $\theta=135°$, we find

$$\sigma_T/\sigma_0 = 1-0.17\langle v^2\rangle_{\text{Av}}.$$

Since $\langle v^2\rangle_{\text{Av}}\simeq 0.04$ for the deuteron, the kinematic corrections are small for point nucleons. This type of analysis can easily be extended to finite-sized nucleons by introducing the variation of the form factor. The algebra is cumbersome and only the result will be quoted. Assuming Gaussian nucleons with the proton rms radius, the total cross section for the above conditions is found to be

$$\sigma_T/\sigma_0 = 1+0.71\langle v^2\rangle_{\text{Av}}.$$

The kinematical corrections are essentially due to the curvature of the free cross section as a function of v. The introduction of form factors influences this curvature, and therefore affects the corrections. The variation due to the form factor is more important than the curvature of the Rosenbluth point cross section in this energy-angle range.

The calculations of this note will confirm these qualitative estimates and extend the considerations to interactions in the final state.

II. THEORY

In order to examine the lowest order relativistic corrections, we will assume a Breit deuteron; that is, the sixteen-component wave function of the two-nucleon system satisfies the equation

$$[G_p^{-1}+G_n^{-1}-V]\phi(p,P)=0, \quad (4)$$

where V is the (assumed) instantaneous binding potential and the G's are the inverse Dirac operators, $(\gamma\cdot P-M)$. This is a one-particle equation and neglects the effects of nucleon pairs as well as the retardation of the binding potential.

The Breit wave functions are assumed to be factorable into a product of the proton and neutron spinors and a scalar wave function. The spinors are further assumed to be the same as free-particle spinors. This, coupled with the assumption that the current operator is the same as for a free particle, reduces the current of the bound two-nucleon system to the current of the free two-nucleon system with a weighting factor roughly corresponding to the momentum distribution in the deuteron. This assumption on the current operator will be examined in the next section. The binding certainly alters the small components of the spinors, but this is neglected although it is of the same order as the correction terms which will be retained. This effect could certainly be estimated via perturbation theory, but because of the approximations of the same order already made on the nucleon current operator, it was not considered entirely consistent to retain these corrections.

The matrix element for the transition is (see Sec. III)

$$M=\delta(P'-P-q)\frac{J_\mu{}^e}{q^2}\int d^3L[\bar{\chi}(\mathbf{L},\mathbf{k};P')j_\mu{}^p\phi(\mathbf{L}-\tfrac{1}{2}\mathbf{q};P) +\text{neutron}], \quad (5)$$

where P and P' are the initial and final center-of-mass momenta of the two nucleons, K is their final asymptotic relative momentum, χ is the scattering-state solution, and ϕ is the bound-state solution of Eq. (4). It is important to notice that the initial and final wave functions are defined with different c.m. momenta. This leads to a final-state wave function relativistically contracted with respect to the initial state. This effect has been calculated for elastic scattering and found to be quite large (the elastic form factor for a zero-range potential is reduced by 15% for $q=3\times10^{13}$ cm^{-1}).[5] This effect in the case of inelastic scattering might be expected to be less important, but has not been calculated.

The spinor functions have the usual orthonormal properties. The Breit wave functions can then be

[4] A more detailed and significant calculation using this type of approach has been performed by Mr. A. Goldberg (private communication). His results agree with the ones presented here.

[5] R. Blankenbecler (to be published).

written as

$$\bar{\chi}=\bar{u}_p(\mathbf{L}+\tfrac{1}{2}\mathbf{P}')\bar{u}_n(-\mathbf{L}+\tfrac{1}{2}\mathbf{P}')\bar{\chi}(\mathbf{L},\mathbf{k}),$$

and (6)

$$\phi=u_p(\tfrac{1}{2}\mathbf{P}+\mathbf{L}-\tfrac{1}{2}\mathbf{q})u_n(\tfrac{1}{2}\mathbf{P}-\mathbf{L}+\tfrac{1}{2}\mathbf{q})\phi(\mathbf{L}-\tfrac{1}{2}\mathbf{q}).$$

The matrix element becomes

$$M=\delta(P'-P-q)\frac{J_\mu{}^e}{q^2}\int d^3L[\langle j_\mu{}^p\rangle\bar{\chi}(\mathbf{L},\mathbf{k})\phi(\mathbf{L}-\tfrac{1}{2}\mathbf{q})+\text{neutron}]. \quad (7)$$

The structure of the current operators to be used in this equation will be discussed in the next section. It is to be noted that they are not indentical with the free-particle current operators.

Since the initial proton and neutron spins are correlated in the deuteron, the spin sums must be performed with care. The only effect of this correlation is in the cross term between the neutron and proton current, since in the individual nucleon terms, all possible orientations between the electron and nucleon are possible whether bound or free. The cross term is negligible in any case because it will involve a term of the form

$$\int \langle j_\mu{}^p\rangle\langle j_\nu{}^n\rangle\delta(P'-P-q)\bar{\chi}(\mathbf{L},\mathbf{k})\phi(\mathbf{L}-\tfrac{1}{2}\mathbf{q})\bar{\chi}(\mathbf{L}',\mathbf{k})\times\phi(\mathbf{L}'+\tfrac{1}{2}\mathbf{q})d^3L\, d^3L'\, d^3P_p\, d^3P_n,$$

which for free final states becomes approximately

$$\int \langle j_\mu{}^p\rangle\langle j_\nu{}^n\rangle\bar{\phi}(\mathbf{k}+\tfrac{1}{2}\mathbf{q})\phi(\mathbf{k}-\tfrac{1}{2}\mathbf{q})d^3k\, d^3P'\delta(P'-P-q).$$

The elastic-scattering form factor corresponding to this value of q is

$$F(q)_{\text{elastic}}=\int d^3k\, \bar{\phi}(\mathbf{k}+\tfrac{1}{2}\mathbf{q})\phi(\mathbf{k}).$$

Neglecting the k dependence in the nucleon currents, we see that the cross term is approximately proportional to the deuteron elastic scattering form factor at twice the q value in question. This is completely negligible at the experimentally interesting values of q. The effect of final-state interaction could hardly be expected to change this result radically in this range of q. The physical reason is that any q which is large enough to expose the nucleon will certainly "ignore" a structure as large as the deuteron.

Upon defining the inelastic deuteron form factor as

$$F_\pm(\mathbf{k},\mathbf{q})=\int d^3L\, \bar{\chi}(\mathbf{L},\mathbf{k})\phi(\mathbf{L}\pm\tfrac{1}{2}\mathbf{q}), \quad (8)$$

the inelastic cross section becomes

$$d\sigma/d\Omega_e=(2\pi)^{-8}\sum_{\text{spin}}\int d^3k\, d^3P'\, P_e^2dP_e\delta(P'-P-q)\times|F_\pm(\mathbf{k},\mathbf{q})|^2\frac{\langle J_\mu{}^e\rangle\langle J_\nu{}^e\rangle}{q^4}[\langle j_\mu{}^p\rangle\langle j_\nu{}^p\rangle+\langle j_\mu{}^n\rangle\langle j_\nu{}^n\rangle].$$

Defining a cross section for nucleon scattering

$$\sigma^i=2\pi\sum_{\text{spin}}\left|\frac{\langle j_\mu{}^i\rangle\langle J_\mu{}^e\rangle}{q^2}\right|^2\rho$$

allows the inelastic cross section to be written as

$$d\sigma/d\Omega=\int P_e^2dP_e\left(\frac{d^3k}{(2\pi)^3}\right)\delta(P_0'-P_0-q_0)|F_\pm(\mathbf{k},\mathbf{q})|^2\times[(\sigma^p+\sigma^n)/\rho].$$

For free final states, $F_\pm$ becomes the Fourier transform of the deuteron wave function, which for any reasonable model is peaked at $k\simeq q/2$. This value of the relative momentum corresponds to the free scattering value and will be used as an expansion point. To simplify the algebra, it is convenient to assume nonrelativistic kinematics in evaluating the correction terms, while treating the zero-order term exactly. The entire problem will be treated nonrelativistically and at the end of the calculation, the correct kinematics will be restored to the zero-order term. One must confirm the consistency of this scheme at each stage in the calculation.

The energy-conserving delta function can be written as

$$\delta(P_0'-P_0-q_0)=\delta(P_e-\bar{P}_e(k^2))[\partial\bar{E}_f/\partial\bar{P}_e]^{-1},$$

where $\bar{P}_e(k^2)$ is the positive root of the argument of the delta function. This enables us to define an inelastic density of final states as

$$\rho_{\text{I}}=\bar{P}_e^2[\partial\bar{E}/\partial\bar{P}_e]^{-1}(2\pi)^{-3},$$

and to write the total inelastic cross section as

$$d\sigma/d\Omega_e=\int\left(\frac{d^3k}{(2\pi)^3}\right)|F_\pm(\mathbf{k},\mathbf{q})|^2[B], \quad (9)$$

where

$$B=(\sigma^p+\sigma^n)\rho_I/\rho|_{P_e=\bar{P}_e(k^2)}.$$

The term in the brackets varies slowly with k compared to the inelastic form factors. Since $F_\pm$ is a sharply peaked function of k, it is convenient to expand the slowly varying factors in a Taylor's series about this maximum.

In order to evaluate the correction terms of interest, an explicit form for χ must be used. This weight function is approximated by assuming that it satisfies a Schrödinger type equation. This ignores the contraction effect in the final state and the retardation of the potential. At large momentum transfers, the once iterated Born expansion should be adequate. From the nonorthogonality of the initial and final states, an error is being made of the order of the elastic scattering form factor compared to one. This is completely negligible for large q.

Writing V as the potential that binds the deuteron, we find

$$\chi(\mathbf{L},\mathbf{k})=\delta(\mathbf{L}-\mathbf{k})-(\mathbf{L}^2-\mathbf{k}^2-i\epsilon)^{-1}V(\mathbf{L}-\mathbf{k}). \quad (10)$$

The ingoing boundary condition must be used in the final states. Inserting Eq. (10) into the inelastic cross section, keeping only linear terms in V, we find the expression

$$\frac{d\sigma}{d\Omega_e}=\int\frac{d^3k}{(2\pi)^3}\bigg[|\phi(\mathbf{k}-\tfrac{1}{2}\mathbf{q})|^2 +2\mathrm{P}\int d^3L\,\phi(\mathbf{k}-\tfrac{1}{2}\mathbf{q})\frac{V(\mathbf{k}-\mathbf{L})}{\mathbf{k}^2-\mathbf{L}^2}\phi(\mathbf{L}-\tfrac{1}{2}\mathbf{q})\bigg]B, \quad (11)$$

where P means principal value. To the order to which we shall evaluate this expression, $L-\frac{1}{2}q(k^2)$ can be replaced by $L-\frac{1}{2}q(L^2)$ in the second term, owing to the peak of V and the wave functions at $L\simeq k$.

The first term is evaluated in the following way: performing the angular integrations, we find

$$\int d\Omega_k|\phi(\mathbf{k}-\tfrac{1}{2}\mathbf{q})|^2=[kq(k^2)]^{-1}[G(k-\tfrac{1}{2}q)-G(k+\tfrac{1}{2}q)].$$

The second term is much smaller than the first throughout most of the range of k for any reasonable wave function. The first term is strongly peaked when

$$k=\tfrac{1}{2}q(k^2). \quad (12)$$

This defines the expansion point k_0. Neglecting the binding energy, this is easily shown to be the free scattering value. Expanding q about this value, we find

$$k-\tfrac{1}{2}q(k^2)=\alpha_0(k-\tfrac{1}{2}q_0);\quad \alpha_0=[1-\tfrac{1}{2}\partial q/\partial k]_0. \quad (13)$$

For values of k large enough so that this expansion is not valid, the wave function is near zero. Using the same expansion point in the second term and undoing the angular integrations, we get

$$\int d\Omega_k|\phi(\mathbf{k}-\tfrac{1}{2}q(\mathbf{k}^2))|^2=\frac{\alpha_0^2q_0}{q(k^2)}\int d\Omega_k|\phi(\alpha_0\mathbf{k}-\tfrac{1}{2}\alpha_0\mathbf{q}_0)|^2.$$

Expanding the brackets in Eq. (1) with the displacement operator

$$B(\mathbf{k})=\exp[(\mathbf{k}-\tfrac{1}{2}\mathbf{q}_0)\cdot\boldsymbol{\delta}][B(\tfrac{1}{2}q_0)],$$

where $\boldsymbol{\delta}$ is the gradient with respect to $\frac{1}{2}q_0$ and does not operate on the q_0 in the exponential, the first term of Eq. (11) becomes after a change of variable

$$\frac{q_0}{\alpha_0}\int\frac{d^3P}{(2\pi)^3}|\phi(\mathbf{P})|^2\exp\bigg[\frac{\mathbf{P}\cdot\boldsymbol{\delta}}{\alpha_0}\bigg]\bigg[\frac{B}{q}\bigg]_{\bar{P}(\frac{1}{2}q_0)}.$$

It should be kept in mind that q is a function of q_0. Upon expanding the exponential operator to second order in P, this becomes

$$\frac{q_0}{\alpha_0}\bigg[1+\langle P^2\rangle\frac{\delta^2}{6\alpha_0^2}\bigg]\bigg[\frac{B}{q}\bigg]_{\bar{P}(\frac{1}{2}q_0)}, \quad (14)$$

where δ^2 is the Laplacian with respect to $\frac{1}{2}q_0$.

Since the bracket in Eq. (11) is a function of $q(k^2)$ only for large momentum transfer, it is convenient to expand the second term in another form:

$$[B]_{\bar{P}(k)}=\exp[(k^2-\tfrac{1}{4}q_0^2)\bar{\Delta}][B]_{\bar{P}(\frac{1}{2}q_0)};\quad \bar{\Delta}=\partial/\partial(\tfrac{1}{4}q_0^2).$$

It then becomes

$$-2\mathrm{P}\int\frac{d^3k\,d^3L}{L^2-k^2}\phi(k-\tfrac{1}{2}q(k^2))\frac{V(k-L)}{(2\pi)^3}\phi(L-\tfrac{1}{2}q(L^2)) \times\langle 1+(k^2-\tfrac{1}{4}q_0^2)\bar{\Delta}\rangle[B].$$

The term independent of $\bar{\Delta}$ vanishes because it is a symmetric integral of an antisymmetric function of L and k. By symmetrizing the term which is first order in $\bar{\Delta}$, we find

$$(2\pi)^{-3}\int d^3k\,d^3L\,\phi(\mathbf{k}-\tfrac{1}{2}\mathbf{q}(k^2))V(\mathbf{k}-\mathbf{L}) \times\phi(\mathbf{L}-\tfrac{1}{2}\mathbf{q}(L^2))\bar{\Delta}[B].$$

Expanding the arguments of the wave functions according to Eq. (13), we find

$$\cong\alpha^{-6}(2\pi)^{-3}\int d^3P\,d^3P'\,\phi(\mathbf{P})V\bigg(\frac{\mathbf{P}-\mathbf{P}'}{\alpha}\bigg)\phi(\mathbf{P}')\bar{\Delta}[B]$$
$$=\alpha^{-n}\langle V\rangle\bar{\Delta}[B], \quad (15)$$

where $N=4$ to 6 and depends on the shape of the potential. For example, if V is a zero-range potential, $N=6$; if V is a Yukawa potential, $N\simeq5$; if V is a square well, $N\simeq4$.

Using the relation $1=\rho_{\mathrm{I}}(k)/\rho(k)\alpha(k)$, which follows from the definition of $\bar{P}$ and α, and also holds for relativistic kinematics if the appropriate changes are made, we find for the total inelastic cross section

$$d\sigma/d\Omega_e=(\sigma^p+\sigma^n)_0[1+\langle P^2\rangle\Delta], \quad (16)$$

where

$$\langle V\rangle\simeq-\langle P^2\rangle,$$

$$(\sigma^p+\sigma^n)\Delta=\bigg(\frac{2}{3\alpha_0^3}\bigg)\frac{\partial^2}{\partial q_0^2}\bigg(\frac{q_0(\sigma^p+\sigma^n)}{q(\frac{1}{2}q_0)}\alpha(q_0)\bigg) -\bigg(\frac{2}{\alpha_0^nq_0}\bigg)\frac{\partial}{\partial q_0}\langle\alpha_0(\sigma^p+\sigma^n)\rangle, \quad (17)$$

and $N\simeq4-6$.

This is our result if the nucleon cross sections in Eq. (17) are identified with the free-scattering cross sections. The approximations implicit in this identification are discussed in the next section. We shall now examine the deuteron current operator in detail.

III. CURRENT OPERATOR

The bound state of the two-nucleon system is described by the Bethe-Salpeter equation,[6] which can be written in the form

$$G_1^{-1}G_2^{-1}\psi = V_{12}\psi, \tag{18}$$

where V_{12} is an integral operator which is itself known only as an infinite series in the meson-nucleon coupling constant and represents the sum of all irreducible diagrams of the interacting system. The one-particle Green's functions contain their respective mass operators. The matrix element of interest in terms of the B-S amplitudes is[7]

$$M = \langle f|[j_p{}^{\mu}A_\mu(x_p)G_n{}^{-1} + j_n{}^{\mu}A_\mu(x_n)G_p{}^{-1} + (\delta V_{12}/\delta A_\mu)A_\mu(x)]|i\rangle, \tag{19}$$

where $A(x)$ is the Møller potential generated by the electron,

$$A_\mu(x) = \langle f|J_\mu{}^e|i\rangle q^{-2}\exp[iq\cdot x].$$

The B-S amplitudes are eigenfunctions of the c.m. momentum P. It is convenient to transform to momentum space, where the amplitudes are defined as

$$|i\rangle = (2\pi)^{-4}\exp[iP\cdot X]\int d^4p\,\psi(p,P)\exp[ip\cdot x],$$

$$\langle f| = (2\pi)^{-4}\exp[-iP'\cdot X] \times\int d^4L\,\Theta(L,k;P')\exp[-iL\cdot x].$$

Inserting these into the matrix element, neglecting the contributions due to the coupling operator V_{12}, and performing X, x, and p integrations, we find

$$M = \frac{\langle J_\mu{}^e\rangle}{q^2}\delta(P'-P-q)\int d^4L\left[\Theta j_p{}^{\mu}G_n{}^{-1}\psi(L-\tfrac{1}{2}q) + \text{neutron}\right]. \tag{20}$$

The omission of the current due to the coupling operator V_{12} neglects not only the meson exchange current but also the interaction with the nucleons while two or more mesons are being exchanged. The retardation of the lowest order ("ladder") potential is, of course, correctly described by Eq. (20).

[6] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452, 455 (1951).
[7] K. Nishijima, Progr. Theoret. Phys. (Japan) **14**, 203 (1955); A. Klein and C. Zemach, Phys. Rev. **108**, 126 (1957).

The current operators must now be constructed. The structure of the nucleon current operator has been investigated in fullest generality by Zemach.[8] By considering the Green's function for the dressed nucleon and invariance and conservation properties, he has derived the form of the operator. It can be written as

$$j_\mu = j_\mu{}^F + j_\mu{}^B, \tag{21}$$

where j^F is the current operator which contributes to the free-scattering process and j^B vanishes when its expectation value is taken between free spinors. The free part of the current contains the effect of nucleon structure by the inclusion of form factors. Some of these functions are measured in electron-proton scattering,[9] and agree roughly with those calculated from meson theory and dispersion relations.

The bound part of the current will, of course, contribute to free electron-proton scattering if more than one photon is exchanged. This effect for electron-proton scattering has been calculated by Drell and Ruderman[10] and Drell and Fubini[11] using dispersion relations and found to be small for the energies and angles of experimental interest. The destructive interference which led to the smallness of their result need not be present in the case of the deuteron. An examination of the photodisintegration cross section in the resonance region leads one to expect that the interference is constructive.[12] The bound-current contribution to electrodisintegration may well be large also.

We shall now use a dimensional argument to get an estimate of these terms for a bound nucleon. A typical term in the bound-current operator has the form

$$e[\gamma_\mu \text{ or } \sigma_{\mu\nu}q^\nu]X(q,p)G^{-1}/M, \tag{22}$$

where $X(q,p)$ is a matrix function of the four-momentum transfer and the nucleon momentum. X is a dimensionless function and is assumed to be of order unity for small q and reasonable p.[8] Inserting this into Eq. (6) and using the B-S equation, we find a contribution to the current of the form

$$e\langle f|[\gamma_\mu \text{ or } \sigma_{\mu\nu}q^\nu]X(q,p)V_{12}/M|i\rangle.$$

It is assumed that as a function of q, X is similar to the free form factors. At least for small q this seems to be a reasonable assumption because the characteristic length involved in both terms is the meson Compton wavelength. This means that the bound-current terms are roughly of order (V/M) relative to the free ones. For an instantaneous potential, this reduces immediately to the average potential energy in units of

[8] C. Zemach, Phys. Rev. **104**, 1771 (1956).
[9] E. E. Chambers and R. Hofstadter, Phys. Rev. **103**, 1454 (1956).
[10] S. D. Drell and M. Ruderman, Phys. Rev. **106**, 561 (1957).
[11] S. D. Drell and S. Fubini (unpublished).
[12] The essential difference is that in the proton case, the nucleon must act twice with the virtual photon field of the electron in order to get a proton line off the mass shell. In the deuteron, the nucleon need act only once because it is already off the mass shell.

the nucleon mass. For a reasonable deuteron model, this is an effect of a few percent. This effect is compensated at least for small q by the normalization condition to be imposed upon the two-nucleon amplitudes, which is a requirement on the total charge of the system.[7] If $X(q)$ does not fall off with q as fast as the free-form factors, then these terms will become more important. The only practical course to take is to neglect the contribution of the bound terms to the current operator. This allows us to express the deuteron scattering in terms of the free-scattering cross sections.

The form of the matrix element given by Eq. (20) is quite untractable. Assuming an instantaneous potential and neglecting nucleon pairs in the initial and final state, the matrix element reduces upon performance of the L_0 integration to Eq. (5). Since the approximations made in going from Eq. (20) to Eq. (5) involve only the binding potential and nucleon pairs, *not* the scattering kinematics, it is clear that the four-momentum transferred by the electron is the argument of the nucleon form factor. A nonrelativistic treatment is ambiguous on this point. The difference between the value of the nucleon form factor with a three- and a four-momentum transfer is crucial in the interesting range of q. This treatment is unambiguous on this point.

By considering a trivial generalization to the n-nucleon system, this approach justifies the use of the "folded" charge distributions if the three- and four-momentum transfer are approximately equal. It is clear that the finite size of the nucleon will *not* affect any real photon process. There have been errors in the literature on this point, and they have been based on the results of the "folded" charge distribution, which is seen to be incorrect for real photons.

This treatment does not justify the use of the four-momentum in the Schrödinger wave functions. It seems to be impossible to do so. However, one has an intuitive feeling that a comparison with the three-momentum result yields a measure of some of the relativistic uncertainties in the problem, since the same transcription in the case of elastic scattering does not spoil the experimental agreement. That this prescription is not to be taken seriously can be seen from its application to photodisintegration where the four-momentum transfer is zero, and the energy dependence of the cross section is lost. The calculation with the four-dimensional momentum transfer has been carried out by Hofstadter and Yearian.[2]

One may legitimately ask the question, why introduce the B-S approach if one is not going to improve on a Breit (or even the Schrödinger) treatment of the problem. The only defense is that the B-S approach simplifies the discussion of the bound-current terms, settles unambiguously the question of three- *vs* four-momentum transfer in the nucleon form factors, and makes clear the approximations implicit in Eq. (4). With the neglect of these bound-current terms, the cross section σ_F may be identified with the free-scattering cross section, given by the Rosenbluth formula.

TABLE I. Values of Δ.

Nucleon model	N	75°	90°	120°	135°
Point	4	−0.15	−0.19	−0.24	−0.26
Gaussian	4	0.22	0.55	1.24	1.56
	6	0.40	0.86	1.78	2.22
Exponential	4	0.28	0.54	1.06	1.30
	6	0.45	0.80	1.47	1.78

IV. NUMERICAL RESULTS

Since Δ defined by Eq. (17) involves a ratio of cross sections, a small error is introduced in its evaluation by using nonrelativistic kinematics. The proton charge scattering has been neglected. The neutron was assumed to have the same moment form factor and radius as the proton.

The algebra of evaluating the derivatives of the various quantities is straightforward but somewhat lengthy. The numerical results are presented in Table I for an incident electron energy of 470 Mev.

It is seen that the introduction of form factors changes the sign of the corrections and completely dominates its magnitude. An attractive potential in the final state is found to decrease the result, because it allows the nucleon to scatter from the electron with an effectively larger q, and hence a smaller cross section. With form factors, this decrease has roughly one-half the magnitude of the effect due to the variation of the cross section over the momentum distribution.

Since $\langle p^2\rangle = 15$–40 Mev for any reasonable deuteron model, an upper limit to the correction is $\simeq 9\%$ at 135°. It is smaller at the other angles considered. The approximations considered in this paper break down for small q, or forward scattering, so that the result at 75° might be viewed with caution.

V. DISCUSSION AND CONCLUSIONS

This development of the sum rule is perhaps better than the previous ones because it reduces with the neglect of binding to the correct relativistic limit. In this calculation we have essentially replaced the Pauli spinors by free Dirac spinors in order to achieve this limiting behavior. The kinematic corrections evaluated in this paper are found to be relatively small. However, the omission of many other terms of the same order seems to preclude a satisfactory interpretation of the data in terms of a neutron moment distribution to better than about 20% in the total cross section. This is reflected as a smaller error in the "radius" determination.

A covariant description of the bound state would seem essential in such a high-energy process as we are attempting to examine. The effect of nucleon pairs and the retardation of the binding potential has been neglected and may well be important in this energy range. There seems to be no satisfactory method for estimating the bound-current terms. A calculation of these terms would be equivalent to the discussion of the free nucleon current form factors from first principles, and dispersion relations may offer a fruitful approach.

Some information about this effect and the mesonic corrections might be found by performing the experiment at a small momentum transfer where the proton charge contributes most of the scattering. Since the magnetic moment form factors are roughly known from from the large-q experiments, their contribution can be evaluated quite accurately at small q. A Schrödinger description of the deuteron should also be more accurate in this region. A comparison with the free-proton charge scattering results would shed some light on these effects.

The determination of the neutron size by measuring the total inelastic cross section suffers from the mesonic and kinematic corrections which are most important when the electron has endured a large energy loss. Experimentally, it is also more difficult to make measurements under these conditions.

The mesonic corrections have been crudely estimated in the following way. The photodisintegration cross section is known as a function of the photon energy. This cross section is reduced by multiplying by nucleon form factors having a four-momentum transfer roughly corresponding to the photon energy involved. The electromagnetic field produced by the electron is assumed to have a distribution of equivalent photons which is zero for a photon energy greater than the energy loss suffered by the electron. The electrodisintegration total cross section is then an integral over the product of the reduced photo cross section and distribution function. By assuming reasonable distribution functions, the effect of the meson resonance was found to be of order 5–15% at 135° and falls off rapidly for smaller angles. For larger angles or q's, the form factors damp the meson resonance and the effect saturates at 10–20%. This approach may grossly underestimate the contribution from the meson exchange current if the meson form factor is much larger than the nucleon form factor for large momentum transfer. This term is relatively small below the resonance region, however.

Another method for the interpretation of the experiments is suggested by this work,[13] which would seem better than the total cross-section technique. One calculates the spectrum of inelastically scattered electrons near its peak, which corresponds to the expansion point of this paper. The neutron form factor is found by matching the measured peak value with the calculated value which is appropriately weighted with the proton and neutron form factors. The advantage of this procedure over measuring the total inelastic cross section is that the mesonic and kinematic corrections are presumably important only if the electron has lost a large amount of energy, and hence should appear on the low-energy side of the peak. The mesonic corrections in the final state should also be hampered by the relatively narrow width of the resonance. The static final-state-interaction corrections are also small near the peak and may be safely neglected. Unfortunately, the height of the peak is somewhat sensitive to the deuteron model assumed. The peak height is very insensitive to the substitution of the four-momentum transfer for the three-momentum transfer. This should not be taken to mean that the relativistic corrections are necessarily small here, because the wave-function normalization must also be changed in some (unknown) manner.

Once the neutron form factor has been determined by this method, the total cross-section measurements and the results of this paper can yield a measure of the unknown but extremely interesting relativistic and mesonic corrections. It seems that this experiment can supply information not only on the neutron but also one the structure of the two-nucleon bound state.

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[13] S. D. Drell (private communication).