Remarks on Pion Production in Pion-Nucleon Collisions at Moderate Energies*

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Some remarks are made concerning pion production in pion-nucleon collisions at incident kinetic energies of from 300 Mev to about 600 Mev. We discuss a certain representation of the matrix element which involves explicitly a term that is dependent upon a pion-pion scattering amplitude off the energy shell. A partial-wave decomposition of the rest of the transition operator is attempted upon the assumption that the amplitudes so introduced contain the effect of strong pion-nucleon interactions in the P wave. We discuss certain quantities, such as the angular distribution and the polarization of the recoil nucleon, which may be determined largely by the interference between the amplitudes involving specific pion-pion and pion-nucleon effects, respectively.

I. INTRODUCTION

IT appears likely that in the near future a considerable experimental effort will be expended with the new cyclotrons, as well as with the larger machines, in the detailed study of the properties of pion-nucleon collisions at bombarding energies from 300 Mev to about 600 Mev.[1–5] This region would seem to be the natural extension of the region in which the low-energy scattering experiments have been carried out. It will be interesting to determine the extent to which present descriptions of the S- and P-wave elastic scattering can be extended to the higher energies and also to study the behavior of the scattering in the higher orbital angular momentum states. Above 300 Mev preliminary experimental work indicates that pion production by pions begins to occur to the extent of several millibarns of cross section.[5] In this energy region the pion production will still be small compared to the elastic scattering (5–20% of the total interaction cross section) but will be quite measurable. A separate study of the pion production reactions may be a useful manner of gaining further insight into the dynamics of the pion-nucleon system.

In anticipation of the current experimental studies of these processes a number of calculations have been made.[6–12] These calculations have to a large extent been directed at determining the order of magnitude of the total cross section for pion production at these moderate bombarding energies. With the exception of an application of the Tamm-Dancoff approximation to the relativistic theory,[11] the theoretical work has been mainly carried out using the fixed-source theory of P-wave pions which has been useful in correlating the low-energy elastic scattering in the isotopic spin $\frac{3}{2}$, angular momentum $\frac{3}{2}$ state.[13,14] Some S-wave effects have also been calculated within the framework of a fixed-source model.[12] It is interesting that whereas several of the calculations have given cross sections of the order of magnitude of the current measurements,[6–8] several others, approximating the fixed-source equations in a different manner, have given results one or two orders of magnitude smaller.[9,10] It is of further interest that the very preliminary Russian measurements[5] of the reactions $\pi^-+p\rightarrow\pi^++\pi^-+n$ and $\pi^-+p\rightarrow\pi^-+\pi^0+p$ at bombarding energies from about 300 to 370 Mev seem to indicate cross sections about twice those given in the more optimistic of the calculations with the fixed-source P-wave theory.[7]

It is the purpose of this note to discuss briefly the pion production from the following viewpoint: It may be possible, under certain simplifying assumptions, to discuss the transition operator for these reactions in a manner which exhibits explicitly the quantities involving the various dynamical effects which we might expect to be present, and then to suggest certain types of measurements which might be useful in determining the extent to which these hypothesized dynamical effects are present. It is quite likely that the interaction in the final state of either or both of the pions with the nucleon will be an important dynamical factor. It is also possible that a specific pion-pion interaction may play a role in these reactions. By a specific pion-pion effect we mean an interaction of the incident pion with the meson cloud of the nucleon, the result being the creation of two real pions with the dynamics of the situation being determined largely by the pion-pion interaction, apart from the factor involving the nucleon as the source of the meson cloud. The tremendous difficulty with the present approach is that it may not be at all possible to separate the observable effects of

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two such physically distinct factors in the reaction. It is quite possible that a pion-pion effect might be operative in the production mechanism, but that its specific effects are lessened or obscured in the observable quantities by subsequent pion-nucleon interactions. In the following we have assumed that at least to some extent these effects are separable. Upon writing out the formal expression for the matrix element for the process, we observe that a portion of the "dispersive" part of this element may be identified with a specific pion-pion effect. A partial-wave analysis of the remainder of the matrix element is made and its usefulness involves the implicit assumption that the amplitudes introduced are largely determined by the dynamics of the strong pion-nucleon interactions in the P wave. Formulas for certain types of measurements are discussed in their dependence upon the amplitudes describing these different dynamical effects. We are well aware that limited statistics and techniques may make rather detailed studies quite futuristic. However, the formulas may be useful in discussing the manner in which such future detailed experiments on these reactions may be capable of further probing the rather complicated dynamics of the pion-nucleon system at moderate energies.

II. REPRESENTATION OF THE MATRIX ELEMENT

We denote the S-matrix element for the process in which a pion and a nucleon of four-momenta $(\mathbf{q}_1,\omega_1)$ and $(\mathbf{p}_1,E_1)$, respectively, collide and produce two pions and a nucleon of four-momenta $(\mathbf{q}_2,\omega_2)$, $(\mathbf{q}_3,\omega_3)$, and $(\mathbf{p}_2,E_2)$, respectively (the isotopic indices are included in these symbols), by

$$S=2\pi i\delta(\omega_2+\omega_3+E_2-\omega_1-E_1)\times\delta(\mathbf{q}_2+\mathbf{q}_3+\mathbf{p}_2-\mathbf{q}_1-\mathbf{p}_1)T. \quad (1)$$

Then the T-matrix element is given in pseudoscalar meson theory by the following expression[15]:

$$T_{q1}=(m/2\omega_1E_2)^{\frac{1}{2}}\times\Big[\frac{1}{2}\int dx\, e^{iQx}\delta(x)\langle^{(-)}q_2q_3|\,\bar{u}(p_2)\tau_{q1}\gamma_5\psi(x)\,|p_1\rangle -i\int dx\, e^{iQx}\langle^{(-)}q_2q_3|\,\mathcal{P}\{j_{q1}(-x/2),\bar{u}(p_2)\Lambda(x/2)\}|p_1\rangle\Big], \quad (2)$$

where[16] $Q=\frac{1}{2}(q_1+p_2)$; m is the nucleon mass;

$$j_{q1}=g\bar{\psi}\gamma_5\tau_{q1}\psi+\lambda\phi_j\phi_j\phi_{q1}-\delta\mu^2\phi_{q1};$$
$$\Lambda=g\tau_j\gamma_5\psi\phi_j-\delta m\psi;$$

$u(p_2)$ is a positive-energy Dirac spinor for a particle of four-momentum p_2; and $\mathcal{P}$ denotes the time-order product. All operators are in the Heisenberg representation and $|p_1\rangle$ and $|q_2q_3^{(-)}\rangle$ are exact single-nucleon and two-meson (with incoming wave) eigenstates of the total Hamiltonian, respectively. The second term in the square bracket may be rewritten as

$$+i\int dx\, e^{iQx}\eta(x)\times\langle^{(-)}q_2q_3|[j_{q1}(-x/2),\bar{u}(p_2)\Lambda(x/2)]|p_1\rangle, \quad (3)$$

where the bracket here denotes the anticommutator and $\eta(x)$ is the step function. The equivalence of the two terms may be shown by noting that the difference between the terms, which is given by

$$-i\int dx\, e^{iQx}\langle^{(-)}q_2q_3|\,j_{q1}(-x/2),\bar{u}(p_2)\Lambda(x/2)\,|p_1\rangle \quad (4)$$

vanishes since it contains a delta function of four momenta, $\delta(p_n+p_2-p_1)$, where $p_n^2\geq\mu^2$ and $p_1^2=p_2^2=m^2$. We now write the T-matrix element as $iT=D+iA$, where[17]

$$D_{q1}=(m/2\omega_1E_2)^{\frac{1}{2}}\times\Big\{\frac{1}{2}\int dx\, e^{iQx}[\delta(x)\langle^{(-)}q_2q_3|\,\bar{u}(p_2)\tau_{q1}\gamma_5\psi(x)\,|p_1\rangle +i\epsilon(x)\langle^{(-)}q_2q_3|[j_{q1}(-x/2),\bar{u}(p_2)\Lambda(x/2)]|p_1\rangle]\Big\}, \quad (5)$$
$$A_{q1}=(m/2\omega_1E_2)^{\frac{1}{2}}\times\Big\{\frac{1}{2}\int dx\, e^{iQx}\langle^{(-)}q_2q_3|[j_{q1}(-x/2),\bar{u}(p_2)\Lambda(x/2)]|p_1\rangle,$$

with

$$\epsilon(x)=2\eta(x)-1 \quad\text{and}\quad \eta(x)=1 \text{ for } x_0>0,\quad \eta(x)=0 \text{ for } x_0<0.$$

It can be shown that invariance under time reversal implies the following relations:

$$D_{+,-,0}^{\dagger}(\mathbf{Q},\mathbf{q}_2,\mathbf{q}_3,\mathbf{p}_1)=D_{+,-,0}(-\mathbf{Q},-\mathbf{q}_2,-\mathbf{q}_3,-\mathbf{p}_1),$$
$$A_{+,-,0}^{\dagger}(\mathbf{Q},\mathbf{q}_2,\mathbf{q}_3,\mathbf{p}_1)=A_{+,-,0}(-\mathbf{Q},-\mathbf{q}_2,-\mathbf{q}_3,-\mathbf{p}_1), \quad (6)$$

where the subscripts refer to the charge state of the incident pion.

Instead of dealing with the matrix elements T, D, or A we may deal with Lorentz-invariant operators in the

[15] The symbol q_1 denotes the four-vector of energy and momentum whereas $\mathbf{q}_1$ denotes the momentum three-vector.

[16] The symbols have their usual meaning: τ is the nucleon isotopic spin operator, $\delta\mu^2$ and δm are the pion and nucleon mass renormalization constants, respectively; and g and λ are the pion-nucleon and pion-pion coupling constants, respectively.

[17] It should be remarked that A has singularities of the delta function type arising from the single-pion intermediate state in the first term [this state also gives rise to a singularity in Eq. (4)] and from the single-nucleon intermediate state in the second term. These terms vanish for physical values of the momenta, which are those with which we are concerned with in this discussion of the physics of the problem. The singularities would be significant in a dispersion integral over the relevant variable in A.

nucleon spin space defined by

$$iT = C\chi_f \mathcal{T}\chi_i = C\chi_f(\mathfrak{D}+i\mathfrak{A})\chi_i,$$

where

$$C=(m^2/8\omega_1\omega_2\omega_3E_1E_2)^{\frac{1}{2}},$$

and χ_i and χ_f are two-component spin functions for the initial and final nucleons at rest. The most general form of T is given by

$$T=\bar{u}(p_2)\gamma_5(\beta_0+\beta_1\gamma\cdot\lambda/\mu+\beta_2\gamma\cdot\lambda'/\mu +\beta_3\gamma\cdot\lambda\gamma\cdot\lambda'/\mu^2)u(p_1), \quad (7)$$

when μ is the pion mass; $\lambda'=q_2+q_3$; $\lambda=q_2-q_3$; and the β are complex functions of the five scalars $(\lambda)^2$, $\lambda\cdot K$, $\lambda\cdot Q$, $\lambda'\cdot K$, and $\lambda'\cdot Q$ with $K=q_1-p_2$. The $u(p)$ are positive-energy Dirac spinors of four momentum p. By reducing this expression to its equivalent in terms of the two-by-two nucleon spin operators, we obtain the general forms for the operators $\mathfrak{D}$ and $\mathfrak{A}$:

$$\begin{aligned}\mathfrak{D}=\nu_1\nu_2\{&d_0[\boldsymbol{\sigma}\cdot\mathbf{p}_2\eta_2-\boldsymbol{\sigma}\cdot\mathbf{p}_1\eta_1]\\ &+d_1[\boldsymbol{\sigma}\cdot\mathbf{p}_1\eta_1\lambda_0+\boldsymbol{\sigma}\cdot\mathbf{p}_2\eta_2\lambda_0+\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}+\boldsymbol{\sigma}\cdot\mathbf{p}_1\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}\boldsymbol{\sigma}\cdot\mathbf{p}_2\eta_1\eta_2]\\ &+d_2[\boldsymbol{\sigma}\cdot\mathbf{p}_1\eta_1\lambda_0'+\boldsymbol{\sigma}\cdot\mathbf{p}_2\eta_2\lambda_0'\\ &\qquad+\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}'+\boldsymbol{\sigma}\cdot\mathbf{p}_1\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}'\boldsymbol{\sigma}\cdot\mathbf{p}_2\eta_1\eta_2]\\ &+d_3[(\boldsymbol{\sigma}\cdot\mathbf{p}_2\eta_2-\boldsymbol{\sigma}\cdot\mathbf{p}_1\eta_1)\lambda_0\lambda_0'-\boldsymbol{\sigma}\cdot\mathbf{p}_2\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}'\eta_1\eta_2\\ &+\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}'\boldsymbol{\sigma}\cdot\mathbf{p}_1\eta_1\eta_2+\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}\lambda_0'+\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}'\lambda_0\\ &\qquad+\boldsymbol{\sigma}\cdot\mathbf{p}_2\boldsymbol{\sigma}\cdot(\boldsymbol{\lambda}'\lambda_0-\boldsymbol{\lambda}\lambda_0')\boldsymbol{\sigma}\cdot\mathbf{p}_1\eta_1\eta_2]\}, \quad (8)\end{aligned}$$

and similarly for $\mathfrak{A}$ with d_i replaced by a_i. Here $\eta_i=(E_i+m)^{-1}$ and $\nu_i=[(E_i+m)/2m]^{\frac{1}{2}}$ with $i=1, 2$. Equation (6) requires that the d's and a's be real functions of the five scalars enumerated above.

Before going on to a phenomenological discussion of the pion production, we remark upon the structure of D and A. If we open up the commutator by introducing a complete set of intermediate states between the two operators we note that in A (often called the "absorptive" part of the matrix element), the first term in the commutator contributes nothing, while the contributions from the second term start with the nucleon-one pion intermediate state, i.e., $p_n^2 \geqslant (\mu+m)^2$.[17] In D (the "dispersive" part of the matrix element) both terms in the commutator contribute; the lowest-mass intermediate state contributing to the first term is that of a single pion, and to the second term that of a single nucleon. These terms involve the product of two matrix elements; one, in each case, is related to a pion-nucleon vertex function, and the second is related to an off the energy shell scattering amplitude. In the first term this is a pion-pion scattering amplitude, and in the second term, a pion-nucleon scattering amplitude. In the following we wish to discuss briefly the former term, and then to attempt a partial-wave description of the remainder of the pion-production matrix element.

III. DISCUSSION OF THE TRANSITION OPERATOR

The contribution to $\mathfrak{D}$ from the term involving the pion-pion scattering amplitude is given by the following expression:

$$\frac{iF((\boldsymbol{\lambda}_1)^2,\boldsymbol{\varrho}^2,(\boldsymbol{\lambda}-\boldsymbol{\varrho})^2)\langle q|\bar{u}(p_2)\Lambda(0)|p_1\rangle}{(2\omega_q)^{\frac{1}{2}}(\omega_q+E_2-E_1)}, \quad (9)$$

where $\mathbf{q}=\mathbf{p}_1-\mathbf{p}_2$, $\omega_q=(\mathbf{q}^2+\mu^2)^{\frac{1}{2}}$, and the quantity F is a Lorentz invariant real scalar function of the indicated scalar variables, with $\boldsymbol{\varrho}$ and $\boldsymbol{\lambda}_1$ being the relative momenta of the two pions before and after the pion-pion interaction, respectively, as measured in the center-of-mass system of the two pions; $\boldsymbol{\lambda}_1-\boldsymbol{\varrho}$ is then the momentum transfer in the pion-pion interaction. The collision conserves 3-momentum but not energy, i.e., $\boldsymbol{\lambda}_1^2\neq\boldsymbol{\varrho}^2$. In what follows we make the approximation

$$\langle q|\bar{u}(p_2)\Lambda(0)|p_1\rangle \sim -i\boldsymbol{\sigma}\cdot\mathbf{q}\tau_q/(2\omega_q)^{\frac{1}{2}}(f_r/\mu) \quad \text{for} \quad q^2\ll m^2,$$

where f_r is the rationalized, renormalized pseudovector coupling constant, $f_r^2/4\pi\sim 0.08$.

We see that a term proportional to a specific pion-pion interaction appears in a first approximation to the "dispersive" part of the transition operator for the process $\pi+N\rightarrow 2\pi+N$ and its effects are largely imbedded in the structure of the function F. We do not at this time go into a discussion of the possible theoretical structure of F. It is known that such structure may arise from an interaction term involving the fourth power of the pion field, or by the intermediary of baryon-antibaryon pairs,[18] or possibly by the intermediary of pairs of K mesons.[19] What we should like to note at present is that this function, for a given energy of the incident pion, depends in general upon the following four variables as measured in the center of mass of the pion production reaction (with an orthogonal coordinate system defined conveniently by the momentum of the incident pion as polar direction and that of the recoil nucleon): the polar angle and magnitude of the recoil nucleon momentum (these determine the pion-pion Q values before and after the collision), and the polar and azimuthal angles of the relative momentum of the two final state pions (these determine the momentum transfer in the pion-pion interaction). What may well be emphasized here is that $F(\boldsymbol{\lambda}_1^2,\boldsymbol{\varrho}^2,(\boldsymbol{\lambda}_1-\boldsymbol{\varrho})^2)$ may have important dependence on each of the three indicated variables. Measurements of pion-pion Q values in pion production events measure a dependence upon $|\boldsymbol{\lambda}_1|$ after integration over the three angles enumerated above. A marked clustering of such measurements about some value might be a spectacular indication of a pion-pion interaction. However, the absence of such gross behavior may mean that one must study events in which some of the variables are fixed, and look for interesting behavior in the dependences upon one or two free variables, in order to gain an indication of the possible presence of these effects. To make such an approach at all reasonable, it would be well to try to

[18] A. M. Mitra and R. P. Saxena, Phys. Rev. **108**, 1083 (1957).
[19] S. Barshay, Phys. Rev. **109**, 2160 (1958).

TABLE I. Angular momentum configurations for initial and final states. In this table, L is the angular momentum of one pion and the nucleon; j is the total angular momentum of this system; l is the angular momentum of the second pion and the latter system about the total center of mass; and J is the total angular momentum.

	Final configuration			Incident orbital state	
l	L	$J(j=\frac{1}{2})$	$J(j=\frac{3}{2})$	$J(j=\frac{1}{2})$	$J(j=\frac{3}{2})$
s	S	$\frac{1}{2}$	$\cdots$	P	$\cdots$
s	P	$\frac{1}{2}$	$\frac{3}{2}$	S	D
p	S	$\frac{1}{2}, \frac{3}{2}$	$\cdots$	S, D	$\cdots$
p	P	$\frac{1}{2}, \frac{3}{2}$	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$	P	P, F

construct a phenomenological expression for the differential cross section for pion production in terms of parameters which describe the various dynamical effects which may contribute and interfere. We have discussed a specific pion-pion effect in the transition operator, $\mathcal{T}$. Because of the nature of the energy denominator in Eq. (9) this term is not likely to be well represented in a partial-wave analysis of the transition operator. However, such an analysis may be useful for the rest of this operator, (which we denote by $\mathcal{R}$), in which the effects of specific pion-nucleon interactions may predominate. In Table I are given some of the lower configurations defined in terms of the orbital angular momentum of one pion and the nucleon, and the orbital angular momentum of the second pion and the center of mass of the pion-nucleon system. The incident states which contribute to these configurations are also given.

A partial-wave expansion of $\mathcal{R}$ in the total center-of-mass system may be defined by the following expression[20]:

$$\mathcal{R}=\textstyle\sum_j A_j(\omega_2,\omega_3,E_0)B_j(\boldsymbol{\sigma},\mathbf{k},\mathbf{p},\boldsymbol{\lambda}), \tag{10}$$

where the A_j are complex functions of the total available energy E_0 and the energies of the two final-state pions, ω_2 and ω_3; these functions are symmetric in ω_2 and ω_3. The quantities B_j are given by the following pseudoscalar quantities invariant under rotations:

$$\begin{aligned}
&(1)\ \ (\mathbf{a}\cdot\mathbf{b})^\nu(\mathbf{k}\cdot\mathbf{b})^\rho(\mathbf{k}\cdot\mathbf{a})^\eta\boldsymbol{\sigma}\cdot\mathbf{k}/(\mu)^{2(\nu+\rho+\eta)+1}\\
&(2)\ \ (\mathbf{a}\cdot\mathbf{b})^\nu(\mathbf{k}\cdot\mathbf{b})^\rho(\mathbf{k}\cdot\mathbf{a})^\eta\boldsymbol{\sigma}\cdot\mathbf{a}/(\mu)^{2(\nu+\rho+\eta)+1}\\
&(3)\ \ (\mathbf{a}\cdot\mathbf{b})^\nu(\mathbf{k}\cdot\mathbf{b})^\rho(\mathbf{k}\cdot\mathbf{a})^\eta\boldsymbol{\sigma}\cdot\mathbf{b}/(\mu)^{2(\nu+\rho+\eta)+1}\\
&(4)\ \ (\mathbf{a}\cdot\mathbf{b})^\nu(\mathbf{k}\cdot\mathbf{b})^\rho(\mathbf{k}\cdot\mathbf{a})^\eta\mathbf{k}\cdot(\mathbf{a}\times\mathbf{b})/(\mu)^{2(\nu+\rho+\eta)+3}.
\end{aligned} \tag{11}$$

The ν, ρ, η are zero or any positive integer, $\mathbf{k}$ is the beam momentum,

$$\mathbf{a}=\mathbf{q}_2-\mathbf{p}_2=(3\mathbf{p}+\boldsymbol{\lambda})/2,\quad \mathbf{b}=\mathbf{q}_3-(\mathbf{q}_2+\mathbf{p}_2)=\mathbf{p}-\boldsymbol{\lambda},$$

where $-\mathbf{p}=-(\mathbf{q}_2+\mathbf{q}_3)$ is the recoil nucleon momentum, and $\boldsymbol{\lambda}=\mathbf{q}_2-\mathbf{q}_3$ is the relative momentum of the two pions. If we consider only the configurations in Table I, then for even orbital states of relative motion of the two pions we obtain

$$\begin{aligned}
\mathcal{R}=&A_0\boldsymbol{\sigma}\cdot\mathbf{k}\mu^{-1}+A_1\mathbf{p}\cdot\mathbf{k}\boldsymbol{\sigma}\cdot\mathbf{k}\mu^{-3}\\
&+A_2\boldsymbol{\sigma}\cdot\mathbf{p}\mu^{-1}+A_3(3p^2-\lambda^2)\boldsymbol{\sigma}\cdot\mathbf{k}\mu^{-3}\\
&+A_4[3(\mathbf{k}\cdot\mathbf{p})^2-(\mathbf{k}\cdot\boldsymbol{\lambda})^2]\boldsymbol{\sigma}\cdot\mathbf{k}\mu^{-5}\\
&+A_5(3\mathbf{k}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}-\mathbf{k}\cdot\boldsymbol{\lambda}\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})\mu^{-3}.
\end{aligned} \tag{12}$$

The terms in A_0 and A_3 may be combined, the former arises from the sS configuration, the latter from the pP configuration. For odd orbital states of relative motion of the two pions we have

$$\begin{aligned}
\mathcal{R}=&A_0'\mathbf{k}\cdot\boldsymbol{\lambda}\boldsymbol{\sigma}\cdot\mathbf{k}\mu^{-3}+A_1'\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}\mu^{-1}+A_2'\mathbf{p}\cdot\boldsymbol{\lambda}\boldsymbol{\sigma}\cdot\mathbf{k}\mu^{-3}\\
&+A_3'\mathbf{k}\cdot\boldsymbol{\lambda}\mathbf{k}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{k}\mu^{-5}+A_4'\mathbf{k}\cdot\boldsymbol{\lambda}\boldsymbol{\sigma}\cdot\mathbf{p}\mu^{-3}\\
&+A_5'\mathbf{k}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}\mu^{-3}+A_6'\mathbf{k}\cdot(\mathbf{p}\times\boldsymbol{\lambda})\mu^{-3}.
\end{aligned} \tag{13}$$

In certain of the pion production reactions the Bose principle for the two final-state pions affords a considerable simplification in the analysis. In the reaction $\pi^++p\rightarrow2\pi^++n$ the pions are in a pure isotopic spin two state and in the reaction $\pi^-+p\rightarrow2\pi^0+n$ the pions are in states of isotopic spin zero and two. In both reactions the pions are in even orbital states of relative motion and the appropriate $\mathcal{R}$ is simply given by Eq. (12). Experimental study of these processes might be amenable to a phenomenological analysis which attempts to discuss specific dynamical effects. For these processes we write the complete phenomenological transition operator in the total center-of-mass system:

$$\begin{aligned}
\mathcal{T}=&\frac{F(f_r/\mu)\boldsymbol{\sigma}\cdot(\mathbf{p}-\mathbf{k})\tau_q}{2\omega_q[\omega_q+E(p)-E(k)]}+A_1\mathbf{p}\cdot\mathbf{k}\boldsymbol{\sigma}\cdot\mathbf{k}\mu^{-3}\\
&+A_2\boldsymbol{\sigma}\cdot\mathbf{p}\mu^{-1}+A_3(3p^2-\lambda^2)\boldsymbol{\sigma}\cdot\mathbf{k}\mu^{-3}\\
&+A_4[3(\mathbf{k}\cdot\mathbf{p})^2-(\mathbf{k}\cdot\boldsymbol{\lambda})^2]\boldsymbol{\sigma}\cdot\mathbf{k}\mu^{-5}\\
&+A_5(3\mathbf{k}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\mathbf{p}-\mathbf{k}\cdot\boldsymbol{\lambda}\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})\mu^{-3},
\end{aligned} \tag{14}$$

where $\omega_q=[(\mathbf{p}-\mathbf{k})^2+\mu^2]^{\frac{1}{2}}$ and we may consider effects of the sS configuration to be included in the function A_3. On the latter we must remark further. It has recently been pointed out[21] in connection with a similar three-body final-state problem that whereas the angular dependence which is extracted by the partial-wave expansion is exact, the momentum dependence (p^L) is approximate. In the analysis of a two-body final state, this dependence arises from the penetration of the wave function from the region of primary interaction into the asymptotic region. In a three-body final state, the sS configuration may be formed in the primary interaction region or it may be formed as follows: a pP configuration is formed in the primary collision; then, since the relative orbital momentum of the two mesons is indefinite, these particles may approach closely and interact, throwing the complete system into the asymptotically observed sS configuration. Similarly the pP configuration may arise from an sS configuration formed

[20] K. M. Watson and K. A. Brueckner, Phys. Rev. 83, 1 (1951).

[21] V. N. Gribov, Nuclear Phys. 5, 653 (1958).

in the primary interaction followed by a pion-pion interaction. In such instances the functions A will be complicated functions of momenta that will likely obscure the p^L dependence which appears explicitly in Eq. (14). We assume that, apart from the term involving the specific pion-pion effect, the rest of the transition operator contains the effect of the strong pion-nucleon interaction in the P wave. The partial-wave expansion as written is then valid, the explicit momentum dependence being appropriate to the leading term and multiple scattering effects being imbedded in the functions A.

At this point let us remark that, within the framework of our originally stated viewpoint, the equations enumerated thus far are exact consequences of the representation of the matrix element, Eq. (2), and of the kinematics of the reaction. The matrix element can be formally written in other ways,[6] but we have chosen to write it in a manner which puts in evidence the two-pion to two-pion amplitude. In the present very early stage of the experimental and theoretical study of these inelastic processes, the purpose of this analysis is only to indicate the phenomenological equations that may be of use in probing the new dynamical situations. The complex system of strong interactions and our present limited calculational techniques which rest on perturbation theory and possible slight improvements thereof[6–10] preclude our calculating convincingly the functions in an equation like (14) and thereby predicting beforehand the phenomena in this new energy range of pion experiments. An analysis based on examining the analytic properties of the matrix element as a function of one or more of its variables may be useful. However, such an analysis, as applied to elastic pion-nucleon scattering, has shown its power in utilizing (rather than predicting) detailed experimental information in the determination of the basic parameters of the theory (which, of course, at present must be inserted into any detailed dynamical theory), and in testing the principle of microscopic causality.[22]

For the present we shall simply consider expressions for several experimental quantities and shall remark upon how certain kinds of observations might arise from interference effects between the first term and the remaining terms of the phenomenological transition operator given by Eq. (14). The interpretation of such observations as will be remarked upon is not unique, and therefore these comments are merely speculative.

IV. DIFFERENTIAL CROSS SECTION AND NUCLEON POLARIZATION

The differential cross section in the center of mass of the pion-production event is given by the following expression:

$$d\sigma=(2\pi)^{-5}\tfrac{1}{2}\,\mathrm{tr}(\mathcal{T}^\dagger\mathcal{T})(m^2/8\omega_1\omega_2\omega_3E_1E_2)\times(k/\omega_1+k/E_1)^{-1}q_3\omega_3q_2\omega_2d\omega_3d\Omega_2d\Omega_3, \quad (15)$$

where $\omega_1=(k^2+\mu^2)^{\frac{1}{2}}$, $E_1=(k^2+m^2)^{\frac{1}{2}}$, and energy conservation requires $\omega_1+E_1=\omega_2+\omega_3+[(\mathbf{q}_2+\mathbf{q}_3)^2+m^2]^{\frac{1}{2}}$. The symbol tr denotes the trace of the matrix $\mathcal{T}^\dagger\mathcal{T}$. It is sometimes useful to rewrite the differential cross section in a way which emphasizes its dependence upon the recoil nucleon momentum (as measured in the total center-of-mass system) and the relative momentum of the two final-state pions (as measured in the center-of-mass system of the two pions):

$$d\sigma=(2\pi)^{-5}\tfrac{1}{2}\,\mathrm{tr}(\mathcal{T}^\dagger\mathcal{T})(m^2/8\omega_1\omega_2\omega_3E_1E_2)\times(k/\omega_1+k/E_1)^{-1}\tfrac{1}{8}Jp^2dp\lambda_1{}^2d\lambda_1d\Omega_pd\Omega_{\lambda_1}, \quad (16)$$

where J is the Jacobian of the transformation between the relative momentum of the pions as measured in their center-of-mass system and in the total center-of-mass system. These vectors are related by

$$\boldsymbol{\lambda}=\boldsymbol{\lambda}_1+[\mathbf{p}\lambda_1\cdot\mathbf{p}(\gamma-1)/p^2],$$

where $\gamma=(1-\beta^2)^{-\frac{1}{2}}$ and $\beta=p/(\omega_2+\omega_3)$ for the transformation. Energy conservation requires that $\omega_1+E_1=(4\omega^2+p^2)^{\frac{1}{2}}+(p^2+m^2)^{\frac{1}{2}}$, where 2ω is the total energy of the two pions in the final state and $2\omega\gamma=\omega_2+\omega_3$.

Early experiments give an indication that at moderate energies the nucleons tend to go preferentially into the backward hemisphere.[4] If the term in $\mathcal{T}$ involving the specific pion-pion effect were to be dropped and only the pP configuration were considered to be important in the rest of the operator, there would be fore-aft symmetry. It may be that specific pion-nucleon effects enhance the sP and pS configurations as well. This may be true in the energy range under consideration at incident pion kinetic energies of from about 300 to 400 Mev. From Table I we note that these configurations are fed by the D wave in the incident state if one pion couples to the nucleon or to the pion-nucleon system to form a total angular momentum of $\frac{3}{2}$. At about 450 Mev when one pion and the nucleon have a Q value of 160 Mev, the second pion and the nucleon may have a Q value of as much as 70 Mev. At this incident pion kinetic energy and above, specific final-state pion-nucleon interactions in the P wave will likely enhance the pP configuration predominantly. We give here the expression for the difference between the square of the matrix element, $\frac{1}{2}\,\mathrm{tr}(\mathcal{T}^\dagger\mathcal{T})$, evaluated for recoil nucleons at 180° and 0° to the incident beam (neglecting the sP and pS configurations):

$$\begin{aligned}2\,\mathrm{Re}\{&[A_3{}^*p(3p^2-\lambda^2)\mu^{-3}+A_4{}^*p(3p^2k^2-(\boldsymbol{\lambda}\cdot\mathbf{k})^2)\mu^{-5}]\\&\times[G(0)(p-k)+G(\pi)(p+k)]\\&+A_5{}^*G(0)[p^2k(p-k)+(\boldsymbol{\lambda}\cdot\mathbf{k})^2(1-p/k)]\mu^{-3}\\&+A_5{}^*G(\pi)[p^2k(p+k)-(\boldsymbol{\lambda}\cdot\mathbf{k})^2(1+p/k)]\mu^{-3}\}\\&+|G(0)|^2(p^2+k^2-2pk)\\&-|G(\pi)|^2(p^2+k^2+2pk), \quad (17)\end{aligned}$$

where

$$G(\theta_{\mathbf{p}\cdot\mathbf{k}})=(f_r/\mu)F\tau_q/2\omega_q[\omega_q+E(p)-E(k)].$$

[22] S. Mandelstam (Phys. Rev., to be published) has made a recent attempt at formulating an approximation scheme, capable of predicting the dynamics, on the basis of the analytic property approach.

From this expression we see that, for $p \sim p_{\max}$ (i.e., collisions with small absolute value of the momentum transfer to the nucleon), if $G(0) \sim G(\pi)$ and $\mathrm{Re}(GA_{3,4,5}^*) > 0$, the interference terms will give rise to a fore-aft asymmetry in favor of backward moving recoil nucleons. These conditions are met by the functions A discussed in the appendix if the real function $F > 0$. Of course, at a given incident pion energy, the asymmetry is a function of three further variables, the two-pion Q value, $2\omega - 2\mu$, and the polar and azimuthal angles of their relative momentum. An important dependence upon these angles may be in the function F since they determine the momentum transfer in the pion-pion interaction, both in the $\mathbf{k}-\mathbf{p}$ plane, and in the direction of the normal, $\mathbf{k}\times\mathbf{p}$.

It would be useful to have a quantity which depends upon the amplitudes through $\mathrm{Im}(GA_{3,4,5}^*)$. Such a quantity is the polarization of the recoil nucleon, defined by $\langle\boldsymbol{\sigma}\rangle \frac{1}{2}\,\mathrm{tr}(\mathcal{T}^\dagger\mathcal{T}) = \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}\mathcal{T}\mathcal{T}^\dagger)$. In terms of the amplitudes describing the pP configuration and the specific pion-pion effect, we have

$$\begin{aligned}\tfrac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}\mathcal{T}\mathcal{T}^\dagger)&\\ = -2\,\mathrm{Im}\{G[&A_3^*(3p^2-\lambda^2)\mu^{-3}\\ &+A_4^*(3(\mathbf{p}\cdot\mathbf{k})^2-(\mathbf{k}\cdot\boldsymbol{\lambda})^2)\mu^{-5}+A_5^*\mathbf{p}\cdot\mathbf{k}\mu^{-3}](\mathbf{p}\times\mathbf{k})\\ &+\mathbf{p}\cdot\mathbf{k}A_5[A_3^*(3p^2-\lambda^2)\mu^{-6}\\ &+A_4^*(3(\mathbf{p}\cdot\mathbf{k})^2-(\mathbf{k}\cdot\boldsymbol{\lambda})^2)\mu^{-8}](\mathbf{p}\times\mathbf{k})\\ &+\mathbf{k}\cdot\boldsymbol{\lambda}A_5[-G^*\mu^{-3}+A_3^*(3p^2-\lambda^2)\mu^{-6}\\ &+A_4^*(3(\mathbf{p}\cdot\mathbf{k})^2-(\mathbf{k}\cdot\boldsymbol{\lambda})^2)\mu^{-8}](\mathbf{k}\times\boldsymbol{\lambda})\\ &-GA_5^*\mathbf{k}\cdot\boldsymbol{\lambda}\mu^{-3}(\mathbf{p}\times\boldsymbol{\lambda})\}. \quad (18)\end{aligned}$$

In those events for which $\mathbf{p}\cdot\mathbf{k}=0$ and $p\sim p_{\max}$, we have simply

$$\tfrac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}\mathcal{T}\mathcal{T}^\dagger) = -2\,\mathrm{Im}(GA_3^*)3p^2\mu^{-3}\mathbf{p}\times\mathbf{k}. \quad (19)$$

The polarization is in the direction of the cross product between the nucleon recoil momentum and the incident beam momentum, as measured in the laboratory, i.e., it is completely normal to the nucleon motion in the laboratory. At about 450 Mev incident pion kinetic energy, the recoil nucleons at 90° in the center-of-mass system can already have as much as 100-Mev kinetic energy in the laboratory. Another type of event for which the polarization is completely normal to the nucleon motion in the laboratory is that for which $\omega\sim\omega_{\max}$ ($\mathbf{p}=0$). Then the nucleons recoil forward in the laboratory (with only about 50-Mev kinetic energy for 450-Mev incident pions) and the polarization is in the direction of the cross product between the recoil momentum and the two-pion relative momentum, as measured in the laboratory system.

V. CONCLUDING REMARKS

We have described certain effects in pion production by pions, such as the fore-aft asymmetry and the polarization of the recoiling nucleon which depend, in part, upon the interference of the amplitude describing a specific pion-pion effect with amplitudes which have been assumed to be determined largely by the strong pion-nucleon interaction in the P wave. The discussion may be useful in the consideration of future refined experiments to probe the pion-nucleon dynamics at moderate energies. To this end it is helpful to have a possible theoretical form for the amplitudes describing the pP configuration. This configuration may be partially calculable with the well-known fixed-source theory which was so useful in correlating the P-wave elastic scattering. In the Appendix we give expressions for the amplitudes $A_{3,4,5}$ defined in the above discussion, as obtained from a recently published calculation with the fixed-source P-wave theory.[6] There is some hope that these rather simple functions may partially represent the enhancement in pion production at moderate energies arising from the strong pion-nucleon interaction in the P wave.

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APPENDIX

There is some ambiguity in going from the general discussion to quantities defined in the static limit, where the nucleon absorbs momentum but does not move (nor does the center of mass of the nucleon and one meson). The appropriate relative momenta in the partial-wave expansion go over to the momenta of each of the two mesons with respect to the total center of mass which is sitting on the nucleon. For our purpose, this has the effect, when using amplitudes calculated with the static model, of allowing in Eq. (14):

$$(3p^2-\lambda^2)\rightarrow(p^2-\lambda^2);$$
$$[3(\mathbf{p}\cdot\mathbf{k})^2-(\mathbf{k}\cdot\boldsymbol{\lambda})^2]\rightarrow[(\mathbf{p}\cdot\mathbf{k})^2-(\mathbf{k}\cdot\boldsymbol{\lambda})^2];$$
$$(3\mathbf{p}\cdot\mathbf{k}\boldsymbol{\sigma}\cdot\mathbf{p}-\mathbf{k}\cdot\boldsymbol{\lambda}\boldsymbol{\sigma}\cdot\boldsymbol{\lambda})\rightarrow(\mathbf{p}\cdot\mathbf{k}\boldsymbol{\sigma}\cdot\mathbf{p}-\mathbf{k}\cdot\boldsymbol{\lambda}\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}).$$

The main assumption of the calculation is that at moderate energies the process is calculable as an off-the-energy-shell pion-nucleon scattering in the $(\frac{3}{2},\frac{3}{2})$ state (as represented by the one-meson approximation of Chew and Low[14]) with prior or subsequent emission of the second pion. The unitarity condition in the one-meson approximation is not quite satisfied. However, this is connected with the neglect of a certain Feynman diagram contributing to the process [diagram (d) of Fig. 2 in reference 6] and it is argued that this diagram contributes a small effect compared to those diagrams that are included.[6] This uncertainty is likely to be no greater than the already present uncertainty in the damping effect of the Fourier transform of the source function evaluated at the rather high momentum of the incident pion.

For the process $\pi^+ + p \rightarrow 2\pi^+ + n$, we have the following functions derived from the static model:

$$A_4=0$$
$$A_3=A_5=\tfrac{1}{4}[h(\omega_2)+h(\omega_3)], \quad (1)$$

where

$$h(\omega_2)=[(4\pi)^{\frac{1}{2}}(\sqrt{2}/3)]V(k)\lambda_3{}^2\mu^2 \times\left\{\frac{f}{\omega_2(1-\omega_2/\bar{\omega})-i\lambda_3{}^2k_2{}^3}\right\}(1/\omega_3),$$

and for $h(\omega_3)$ interchange ω_2 and ω_3. In this expression $\lambda_3{}^2=16f^2/9\mu^2$, where f is the unrationalized, renormalized pseudovector coupling constant, $f^2\sim0.08$; $\bar{\omega}$ is the resonance energy $\sim$0.3 Bev; and $V(k)$ is the Fourier transform of the source function evaluated at the center-of-mass momentum of the incident pion. In terms of the previously defined vectors $\boldsymbol{\lambda}$ and $\mathbf{p}$, we have $\omega_{2,3}=(\frac{1}{4}\lambda^2+\frac{1}{4}p^2\pm\frac{1}{2}\boldsymbol{\lambda}\cdot\mathbf{p}+\mu^2)^{\frac{1}{2}}$ where the plus sign refers to ω_2, the minus sign to ω_3. Thus the functions $A_{3,4,5}$ derived from the static model contain higher-configuration angular corrections to the angular dependence explicitly extracted by the partial-wave expansion, Eq. (14), in terms of the vectors $\boldsymbol{\lambda}$, $\mathbf{p}$, and $\mathbf{k}$. Additional contributions from higher configurations may be approximated by the addition to the transition operator of

$$\tfrac{3}{4}[h(\omega_2)-h(\omega_3)](\mathbf{k}\cdot\boldsymbol{\lambda}\boldsymbol{\sigma}\cdot\mathbf{p}-\mathbf{k}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}). \quad (2)$$

For the process $\pi^-+p\rightarrow2\pi^0+n$, one must add to the amplitude A_3, given above, the following:

$$\tfrac{1}{2}[g(\omega_2)+g(\omega_3)], \quad (3)$$

where

$$g(\omega_2)=-[(4\pi)^{\frac{1}{2}}(\sqrt{2}/3)]V(k)\lambda_3{}^2\mu^2 \times\left\{\frac{f}{\omega_2(1-\omega_2/\bar{\omega})-i\lambda_3{}^2k_2{}^3}\right\}[1/(\omega_2+\omega_3)],$$

and for $g(\omega_3)$ interchange ω_2 and ω_3. A further contribution in this case from higher configurations may be approximated by addition to the transition operator of

$$\tfrac{1}{2}[g(\omega_2)-g(\omega_3)](i\boldsymbol{\lambda}\times\mathbf{p}\cdot\mathbf{k}+\mathbf{k}\cdot\mathbf{p}\boldsymbol{\sigma}\cdot\boldsymbol{\lambda}-\mathbf{k}\cdot\boldsymbol{\lambda}\boldsymbol{\sigma}\cdot\mathbf{p}). \quad (4)$$

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Charge Independence and the Low-Energy Parameters in Pion Physics

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It is pointed out that the striking discrepancy among the very low-energy parameters in pion physics might be resolved by considering deviations from charge independence. An accurate set of measurements of the total cross section for charge exchange scattering at low energies is needed to prove or disprove this contention.

RECENTLY much attention has been directed to pion phenomena at very low energies. In particular, one of the puzzling problems is the threshold value of the negative-to-positive ratio in pion photoproduction (R), the Panofsky ratio (P), and the relationship between these two quantities.

As far as R is concerned, there has been an apparent discrepancy between the "experimental" value[1] ($R\sim1.9$) and the theoretical prediction ($R\sim1.4$). While this discrepancy will not be completely resolved without an improved set of experiments, it appears now[2,3] that the "experimental" value is closer to 1.6 and that the remaining discrepancy between that and the theoretical 1.4 can be explained by corrections due to the structure of the deuteron[3] on which the experiments are performed. In particular, the final-state Coulomb effects[4] have to be taken into account. Thus it seems that the discrepancy between the "experimental" and theoretical values of R is not as sharp as it appeared a year ago.[5]

The experimental value of P is also slightly in doubt, since the three recent experiments[6–8] do not quite agree among themselves. It is likely, however, that the value of P is 1.7 with an error of 0.1 or so.

The two quantities R and P are not independent of each other. A relationship can be derived between them[8,9] which predicts a numerical value for the

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