Effects of Vacuum Polarization Scattering in the Treatment of Proton-Proton Scattering Data*

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The effects of including vacuum polarization scattering in angular momentum states with $L>0$ and of employing the relativistic value of the Coulomb scattering parameter η_r rather than its nonrelativistic value η have been considered in the analysis of low-energy p-p scattering data. It has been found that about half of the apparent mean P-wave phase shift present in published data, in the energy range 1.8–4.2 Mev, should be attributed to this vacuum polarization scattering. A further decrease in magnitude of the P wave is obtained when η_r is employed, rather than η. Both the vacuum polarization scattering in states with $L>0$ and the employment of η_r rather than η result in a contribution to the f function which does not vary linearly with energy. The effects of these contributions on the coefficients of a polynomial expansion of f in powers of E are ascertained with reference to p-p scattering data in the energy range 0.2 to 7.5 Mev. The object of the work is more that of investigating the possible magnitude of effects on conclusions drawn from data due to applying the various corrections than that of deriving final phase-shift values.

1. INTRODUCTION

IT was found by Foldy and Eriksen[1] that S-wave vacuum polarization scattering produces appreciable effects on the interpretation of proton-proton scattering data at the lowest energies. Eriksen, Foldy, and Rarita[2] found furthermore that indications of P-wave scattering anomalies are appreciably modified through the inclusion of vacuum polarization effects, about half of the apparent anomaly being explicable as P-wave scattering caused by the vacuum polarization potential. The inclusion of vacuum polarization anomalies with $L>1$ was shown by Durand[3] to give contributions to the differential cross section comparable with those caused by $L=1$. The changes in the cross section, calculated in the work just referred to, are small, amounting at most to 0.7% of the experimental cross sections. Nevertheless they have to be considered in the interpretation of the experimental work of Worthington, McGruer, and Findley[4] which has probable errors of comparable magnitude. The additions to the p-p scattering amplitudes caused by relativistic and magnetic modifications of the Coulomb interaction between protons[5,6] drop out at low energies,[7,3] with the exception of the change in the Coulomb scattering parameter η. Rather than the nonrelativistic value $\eta=(e^2/\hbar)(M/2E)^{\frac{1}{2}}$ one should use the value[5] $\eta=e^2/\hbar v$, where v is now the laboratory velocity of the incident proton calculated relativistically. The change in the p-p cross section caused by this change in η is appreciable, resulting in an increase of the small-angle cross section of about 0.3% at 1.8-Mev incident proton energy and about 0.7% at 4.2 Mev.

The object of the present work is a reanalysis of the pertinent low-energy p-p scattering data intended primarily to show how much conclusions regarding nuclear P-wave phase shifts may be affected as a result of the inclusion in such an analysis of the vacuum polarization scattering in angular momentum states with $L>0$ and of the employment of the relativistic Coulomb scattering parameter η_r rather than η. Conclusions regarding probable phase-shift values are only a secondary objective, complete consistency in the fits being difficult to obtain.

The following notation will be used throughout:

E = kinetic energy of the incident particle in the laboratory system.

$k=(ME/2\hbar^2)^{\frac{1}{2}}=2\pi$ times the reciprocal of the wavelength in the center-of-mass system.

$\eta=e^2/\hbar v$ = the Coulomb scattering parameter, with v denoting the velocity of the incident particle in the laboratory system. If the velocity is calculated relativistically this parameter is written as η_r.

$K_0{}^a$ = apparent value of the S-wave phase shift obtained by fitting data with neglect of vacuum polarization and of the difference between the relativistic and nonrelativistic η.

$K_0{}^N$ = the phase shift for the relativistic wave function if vacuum polarization effects outside the nuclear range of force were absent.

δ_J = triplet P-wave phase shift for the state with total angular momentum $J\hbar$.

$\delta=(1/9)(\delta_0+3\delta_1+5\delta_2)$ = mean P-wave phase shift used in fitting data.

σ = theoretical p-p scattering cross section in the

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[1] L. L. Foldy and E. Eriksen, Phys. Rev. **98**, 775 (1955).

[2] Eriksen, Foldy, and Rarita, Phys. Rev. **103**, 781 (1956).

[3] L. Durand, III, Phys. Rev. **108**, 1597 (1957). Some of the material reported on in the present paper appears in a preliminary form in the dissertation of L. Durand, III, Yale University, 1957 (unpublished).

[4] Worthington, McGruer, and Findley, Phys. Rev. **90**, 899 (1953). Referred to in text as WMF.

[5] G. Breit, Phys. Rev. **99**, 1581 (1955).

[6] A. Garren, Phys. Rev. **96**, 1709 (1954); **101**, 419 (1956).

[7] G. Breit, Phys. Rev. **106**, 314 (1957).

center-of-mass system, including effects of vacuum polarization.

$(\Delta\sigma)_{vp}$ = contribution to σ due to vacuum polarization scattering in orbital angular momentum states with $L>0$.

$(\Delta\sigma)_\eta = (\partial\sigma/\partial\eta)(\eta_r - \eta)$.

$(\Delta\sigma)_s$ = contribution to σ due to small change ΔK_0 in the S-wave phase shift, linear in ΔK_0.

$(\Delta\sigma)_p$ = contribution to σ due to a small mean P-wave phase shift δ, linear in δ.

σ_e = experimental p-p scattering cross section in the center-of-mass system obtained from data, employing relativistic kinematics.

σ_{el} = experimental p-p scattering cross section in the laboratory system.

f = the function introduced by Breit, Condon, and Present,[8] which varies nearly linearly with energy for short-range specific nucleon-nucleon forces.

$f_a = f(\eta, K_0^a) = f(\eta_r, K_0^N) + (\Delta f)_\eta + (\Delta f)_{vp}$ = the apparent f function.

$(\Delta f)_{vp}$ = contribution to f_a due to the vacuum polarization interaction.

$(\Delta f)_\eta$ = contribution to f_a when employing η_r rather than η in analyzing the data.

2. LOW-ENERGY p-p SCATTERING DATA

In the present section the accuracy of the low-energy p-p scattering data will be considered. The most accurate published data are believed to be those of Worthington, McGruer, and Findley,[4] spanning the energy range 1.8–4.2 Mev. The differential cross section has been measured for scattering angles from 12° to 90° or more in the center-of-mass system; nonsystematic uncertainties in the cross section are typically less than 0.3–0.4% A number of other high-precision experiments have been performed below 10 Mev,[9] but the data of WMF are of special interest because they cover a wide angular range and also because many precautions were incorporated. A thorough discussion of the systematic corrections applied to their data was given by WMF.

The transformations relating cross section and scattering angle as measured in the laboratory system to those in the center-of-mass system were used in the nonrelativistic approximation by WMF. These transformations are

$$\sigma_{nr}(\theta_{nr}) = \tfrac{1}{4}\sigma_l(\Theta)/\cos\Theta, \quad \theta_{nr} = 2\Theta. \tag{2.1}$$

The nonrelativistic approximation of a quantity is here designated by the subscript nr. It has been found that the kinematic corrections resulting from a relativistic refinement of these relations are of a magnitude comparable with the vacuum polarization corrections, and the values of σ_e and θ have been corrected therefore for the difference between θ and θ_{nr} and the associated difference in the cross section. At low energies the relativistic quantities are connected with the nonrelativistic ones by[10]

$$\begin{aligned} \sigma(\theta) &= \sigma_{nr}(\theta_{nr})[1-(E/2Mc^2)\cos\theta_{nr}+\cdots], \\ \theta &= \theta_{nr} + (E/4Mc^2)\sin\theta_{nr}+\cdots, \end{aligned} \tag{2.2}$$

where E is the nonrelativistic kinetic energy of the incident proton in the laboratory system and Mc^2 is the proton rest-mass energy. The quantity E is the usual incident proton energy obtainable as the product of the charge and the potential difference of the accelerator. The corrections to the data were made by applying (2.2) to the experimental and theoretical values of the quantities θ, $\sigma(\theta)$. In the calculations the theoretical values were computed directly for the exact value of θ corresponding to the experimental Θ. The theoretical cross section needs no further correction if $\sigma(\theta)$ is calculated with the correct θ. If it is calculated, however, in the approximation of using θ_{nr} for θ then according to (2.2) it requires a correction, which is shown in Fig. 1 as Graph A for $E=3.037$ Mev and $K_0=51°$. The ratio σ/σ_e is thus affected in this case by the factor $[1-(E/2Mc^2)]/[1-(E/Mc^2)] = 1+(E/2Mc^2)$ on account of the combined effect of having omitted the kinematic relativistic correction in obtaining σ_e and the difference $\theta-\theta_{nr}$ in calculating the theoretical σ. At 3.5 Mev this correction to $\sigma_e-\sigma$ is 0.2% of σ. In 9 out of 12 available cases the employment of $\sigma_e(\theta)$ in place of $\sigma_{enr}(\theta_{nr})$ somewhat improved the symmetry of the WMF cross sections about 90° in the c.m. system and only for one case was the symmetry poorer after the correction. The fractional contribution to $\sigma(\theta_{nr})$ resulting from the change of θ_{nr} to θ is shown in Fig. 1 as Graph A for $E=3.037$ Mev and $K_0=51°$. The fractional contribution to the experimental value $\sigma_{enr}(\theta_{nr})$ resulting from the use of the exact transformation of σ_{el} to σ_e is shown as Graph B in that figure.

An error in the measurement of the energy of the incident protons would introduce some uncertainty in the interpretation. The beam energy in the WMF experiments was calibrated against the $Li^7(p,n)Be^7$ threshold energy of 1.882 ± 0.002 Mev (abs.).[4,11] The energy resolution of the electrostatic analyzer used was about 0.1%.[4] An error in the beam energy corre-

[8] Breit, Condon, and Present, Phys. Rev. **50**, 825 (1936).

[9] The following low-energy experiments were considered in a least-squares fit to the data, with the weight w per point assigned to each group of observers; Heydenburg and Little, see reference 23, $w=0.0216$; Cooper, Frisch, and Zimmerman, Phys. Rev. **94**, 1209 (1954), $w=0.0263$; Heydenburg, Hafstad, and Tuve, Phys. Rev. **56**, 1078 (1939), $w=0.00633$; Herb, Kerst, Parkinson, and Plain, Phys. Rev. **55**, 998 (1939), $w=0.0663$; Blair, Frier, Lampi, Sleator, and Williams, Phys. Rev. **74**, 553 (1948), $w=0.0659$; J. Rouvina, Phys. Rev. **81**, 593 (1951), $w=0.00349$; R. E. Meagher, Phys. Rev. **78**, 667 (1950), $w=0.00028$; E. J. Zimmerman and P. G. Kreuger, Phys. Rev. **83**, 218 (A) (1951), $w=0.00450$; K. B. Mather, Phys. Rev. **82**, 133 (1951), $w=0.00116$. These weights were obtained by means of the $\mathcal{I}\mathcal{E}$ criterion of reference 23, employing in addition some preliminary data from Professor R. G. Herb and the WMF data. The f-function values were corrected for $(\Delta f)_\eta$, $(\Delta_2 f)_{vp}$ and $(\Delta_3 f)_{vp}$ in this calculation. The effect of the difference between the weights used here and those used in this laboratory earlier[23] is not serious.

[10] O. Chamberlain and C. Wiegand, Phys. Rev. **79**, 81 (1950).

[11] Herb, Snowdon, and Sala, Phys. Rev. **75**, 246 (1949).

sponding to the 0.1% uncertainty in the threshold energy would produce a change of 0.2% in the theoretical value of the cross section in the angular region in which Coulomb scattering predominates and a change of 0.1% at larger angles. Such an error may matter in a precision analysis.

In the WMF experiments the sizes of the slits defining the scattered beam were adjusted so as to make the corrections for scattering geometry given by Breit, Thaxton, and Eisenbud[12] vanish if the cross section is assumed to be of Rutherford form.[4] The use of a cross section containing both Coulomb and S-wave scattering, however, introduces additional corrections to the cross section of about 0.1% in the angular region $30°\leq\theta\leq 60°$ at energies $E>3$ Mev.

In view of the possibility of such uncertainties an exhaustive analysis of the data was not attempted. The WMF data are studied here only as an example of the influence of vacuum polarization scattering and the use of the relativistic value of the Coulomb scattering parameter on the apparent P-wave phase shift, but the results regarding this aspect of the general problem are, at best, preliminary. Nuclear S waves, however, are fairly well determined so that the f function[8] can be considered meaningful and both the WMF and other low-energy data[9] will be used to clarify the changes in nuclear S-wave scattering resulting from the application of vacuum polarization and other corrections.

3. ANALYSIS OF THE WMF DATA

The WMF data were analyzed assuming that Coulomb and nuclear S-wave scattering are dominant and that contributions from higher orbital angular momentum states of nuclear origin are small. The experimental cross sections as given by WMF correspond to $\sigma_{enr}(\theta_{nr})$ of (2.1) and for the present work they were transformed to $\sigma_e(\theta)$, employing the first of Eqs. (2.2). Similarly, the theoretical cross sections were calculated at the exact angles θ.

Preliminary analyses were made employing the nonrelativistic η and assuming only Coulomb and S-wave scattering.[13] Nuclear P-wave contributions to the cross section can then be introduced, using the fact that small P waves have little effect for θ close to 90°. The value of K_0 obtained in the preliminary analysis was therefore adjusted to give a best least-squares fit to the data in the angular region $\theta\geq 50°$. This adjustment was carried out using the formula for the change in the cross section linear in ΔK_0, the change in K_0, *viz.*,

$$(\Delta\sigma)_s=(2/k^2)\,\mathrm{Re}\{[\exp(-2iK_0)][{}^S a_c(\theta)+(\sin K_0)\exp(iK_0)]\}\Delta K_0, \quad (3.1)$$

where ${}^S a_c(\theta)$ is the singlet Coulomb scattering amplitude.[14] Since the change ΔK_0 was always less than 0.06°, the linear approximation used here is well justified. A good fit to the cross section for $\theta\cong 90°$ is thus obtained with an apparent S wave which would give a fit neglecting the presence of vacuum polarization and of dynamic relativistic corrections and of P-wave effects between $\theta=50°$ and $\theta=90°$. This apparent phase shift is called

$$K_0^a=K_0^N+K_0^{vp}+(\Delta K_0)_{vp}+(\Delta K_0)_\eta+(\Delta K_0)_p.$$

It consists of a specifically nuclear part K_0^N which is the phase shift for the relativistic wave function which would be caused in the absence of vacuum polarization effects outside the nuclear range of force, a vacuum polarization S wave K_0^{vp}, and three further contributions: $(\Delta K_0)_{vp}$, $(\Delta K_0)_\eta$, and $(\Delta K_0)_p$ due to the presence of vacuum polarization scattering with $L>0$ for $\theta\cong 90°$, the employment of η rather than η_r in obtaining K_0^a from the experimental cross section, and the P-wave scattering contribution to σ at angles $\theta\cong 90°$. The magnitude of the contribution $(\Delta K_0)_p$ depends on the size of the P wave. The contribution $(\Delta K_0)_p$ was neglected in the present work, but its possible effect on the vacuum polarization contribution to the f function is discussed in Sec. 4. The fractional deviations $[\sigma_e-\sigma+(\Delta\sigma)_{vp}+(\Delta\sigma)_\eta]/\sigma$ of the theoretical S-wave fit from the data are shown in Graph A of Fig. 2 for $E=3.037$ Mev. Deviations larger than the experimental uncertainties appear generally at angles $\theta\leq 40°$ for all the WMF experiments. These deviations have roughly the form of a small mean nuclear P-wave contribution to the cross section. Nuclear P-wave scattering was therefore introduced and was described by a mean P-wave phase shift $\delta=(1/9)(\delta_0+3\delta_1+5\delta_2)$, with neglect of quadratic contributions in δ. In this linear approximation the contribution to the theoretical cross section is[15]

$$(\Delta\sigma)_p=(18/k^2)\delta\cos\theta\,\mathrm{Re}[{}^T a_c(\theta)e_{1,0}^*], \quad (3.2)$$

where ${}^T a_c$ is the triplet Coulomb scattering amplitude[14]

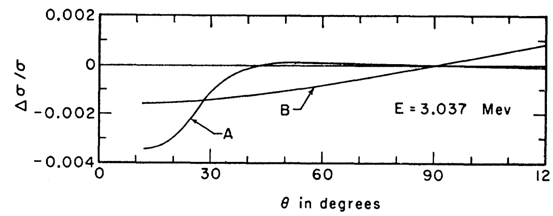

FIG. 1. Graph A: The fractional contribution to the theoretical cross section $\sigma(\theta_{nr})$ when changing θ_{nr} to θ. Graph B: The fractional contribution to the experimental value $\sigma_{enr}(\theta_{nr})$ when the exact transformation of σ_{el} to σ_e is used.

[12] Breit, Thaxton, and Eisenbud, Phys. Rev. **55**, 1018 (1939).

[13] The authors wish to thank Dr. J. Shapiro and Mr. K. D. Pyatt for coding this work for an IBM 650 digital computer.

[14] The Coulomb scattering amplitudes as used in the present work are equivalent to scattering matrix elements S^{sc}, S^{ac} of G. Breit and M. H. Hull, Jr., Phys. Rev. **97**, 1047 (1955) after multiplying the latter by $(1/k\sqrt{2})e^{-i\Phi}$.

[15] Equation (3.2) is equivalent to Eqs. (6.3) and (6.6) of reference 8 and Eq. (1) of Breit, Kittel, and Thaxton, Phys. Rev. **57**, 255 (1940), if quadratic effects are neglected.

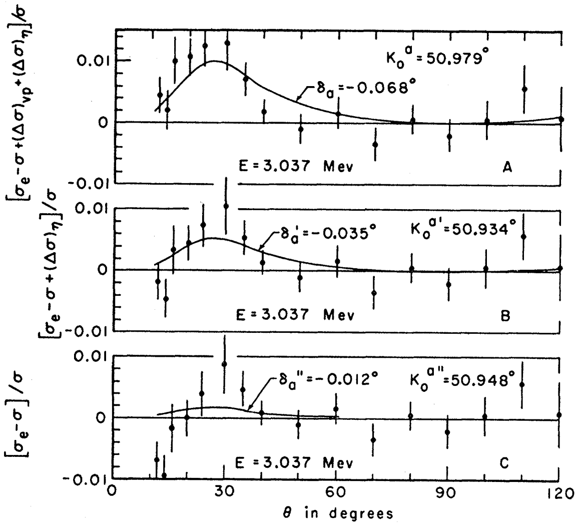

FIG. 2. The results of the analysis of the 3.037-Mev experiment of WMF. In Graph A is shown the fractional difference between the experimental and theoretical cross sections; the latter was calculated neglecting vacuum polarization scattering and using η rather than η_r. The S wave in the present and the following graphs is obtained from a fit to the data in the angular range $50°\leq\theta\leq 120°$, and is here the apparent S wave $K_0^a=K_0^N+K_0^{vp}+(\Delta K_0)_{vp}+(\Delta K_0)_\eta+(\Delta K_0)_p$. The solid curve is the fractional contribution $(\Delta\sigma)_p/\sigma$ to σ due to the apparent mean P wave $\delta_a=\delta_N+(\Delta\delta)_{vp}+(\Delta\delta)_\eta$. Graph B shows the fractional difference when S-wave vacuum polarization scattering is neglected and η is used rather than η_r in calculating σ. The S wave is now $K_0^{a'}=K_0^a-(\Delta K_0)_{vp}$ and the vacuum polarization S wave has been considered part of the total S wave temporarily. The solid curve shows the fractional contribution to σ of an apparent mean P wave $\delta_a'=\delta_a-(\Delta\delta)_{vp}$. Graph C shows the fractional difference when only the S-wave vacuum polarization scattering is neglected and considered part of the total S wave. The S wave is now $K_0^{a''}=K_0^a-(\Delta K_0)_{vp}-(\Delta K_0)_\eta$. The solid curve shows the contribution to σ of the mean P wave $\delta_a''=\delta_a-(\Delta\delta)_{vp}-(\Delta\delta)_\eta$.

and $e_{L,0}$ is the Coulomb factor.[14] These apparent mean P-wave phase shifts were considered to consist of three parts

$$\delta_a=\delta_N+(\Delta\delta)_{vp}+(\Delta\delta)_\eta,$$

where δ_N is the specifically nuclear P-wave phase shift and $(\Delta\delta)_{vp}$ and $(\Delta\delta)_\eta$ are the contributions due to vacuum polarization scattering and the employment of η rather than η_r in obtaining δ_a from the experimental cross section. The mean P-wave contribution $(\Delta\sigma)_p/\sigma$ thus obtained at 3.037 Mev is shown as a solid line in Graph A of Fig. 2. The results for K_0^a and δ_a at the five energies used by WMF are shown in Table I in columns two and five, respectively, and do not differ substantially from those obtained previously by Hall and Powell.[16] The present values of K_0^a should be readjusted again for the contribution $(\Delta K_0)_p$, resulting from the small P-wave contributions to σ at the large angles. When this was done, agreement was obtained with the results of Hall and Powell, to within the statistical uncertainties.

The contributions to the theoretical cross sections, $(\Delta\sigma)_{vp}$ or $(\Delta\sigma)_{vp}/\sigma$, resulting from vacuum polarization scattering in angular momentum states with $L>0$ are given in Table II[17] and Fig. 7 of reference 3, for the WMF energies. The vacuum polarization S-wave phase shift of Foldy and Eriksen[1] was temporarily considered to be part of the total S-wave phase shift K_0 necessary to fit the data because conclusions regarding presence of P waves are not affected by the origin of K_0 but only by its magnitude. After including $(\Delta\sigma)_{vp}$ in σ a different S-wave phase shift is necessary to fit the data around $\theta=90°$. The deviations $[\sigma_e-\sigma+(\Delta\sigma)_\eta]/\sigma$ are shown for the 3.037-Mev data in Graph B of Fig. 2 with S wave $K_0^{a'}=K_0^a-(\Delta K_0)_{vp}$. The changes $(\Delta K_0)_{vp}$ are given in column three of Table I for the WMF experiments. At all energies there still occurred deviations of the theoretical fit from the data at angles $\theta\leq 40°$, and these were attributed to a mean P wave, denoted by $\delta_a'=\delta_a-(\Delta\delta)_{vp}$. The quantities $(\Delta\delta)_{vp}$ and $\delta_a-(\Delta\delta)_{vp}$ are given in columns six and seven of Table I, and the solid curve in Graph B of Fig. 2 shows this P-wave contribution as $(\Delta\sigma)_p/\sigma$ for the 3.037-Mev experiment.

The effect of the contribution $(\Delta\sigma)_\eta$ to the theoretical cross section which results from employing the relativistic value of the Coulomb scattering parameter rather than its nonrelativistic value was treated in the same manner as $(\Delta\sigma)_{vp}$. The contribution is

$$\begin{aligned}(\Delta\sigma)_\eta&=(\partial\sigma/\partial\eta)(\eta_r-\eta)\\&=\tfrac{3}{4}(E/Mc^2)\eta(\partial\sigma/\partial\eta)\end{aligned}\tag{3.3}$$

to order E/Mc^2, where Mc^2 is the proton rest-mass energy, $\eta=(e^2/\hbar)(M/2E)^{\frac{1}{2}}$, and k^2 is considered independent of η in the partial derivative. Over the energy range of the WMF experiments $(\Delta\sigma)_\eta/\sigma$ is isotropic for large angles[18] and its contribution to σ required the small change $(\Delta K_0)_\eta$ in K_0^a, which is given in column four of Table I. The deviations $[\sigma_e-\sigma]/\sigma$, with S wave $K_0^{a''}=K_0^a-(\Delta K_0)_{vp}-(\Delta K_0)_\eta$, are shown in Graph C of Fig. 2 for $E=3.037$ Mev. These deviations were in general negative at angles $\theta\leq 20°$ for all the WMF experiments, the maximum deviation of 1% occurring for the 3.037-Mev experiment. It was not found to be possible to fit the data simultaneously at large and small angles by changing the S-wave phase shift and adding small P- and D-wave contributions to the cross section. Nevertheless, at each energy, a fit to these deviations was made using a mean P-wave phase shift denoted by $\delta_a''=\delta_a-(\Delta\delta)_{vp}-(\Delta\delta)_\eta$. The quantities $(\Delta\delta)_\eta$ and $\delta_a-(\Delta\delta)_{vp}-(\Delta\delta)_\eta$ are given in columns eight and nine of Table I. The solid line in Graph C of Fig. 2 is the contribution of this P wave to σ at $E=3.037$ Mev. Although the fit to the data could thus be improved somewhat at intermediate angles, $20°\leq\theta\leq 40°$, the fit at small angles, $\theta\leq 20°$, became worse for all the WMF experiments.

[16] H. H. Hall and J. L. Powell, Phys. Rev. **90**, 912 (1953).

[17] $[\Delta\sigma(12°)]_{vp}=59.0$ mb at 1855 kev instead of 77.1 mb as given in Table II of reference 3.

[18] See Fig. 5 of reference 3 for $(\Delta\sigma)_\eta/\sigma$ at 1.855 Mev.

TABLE I. The S- and P-wave phase shifts[a] fitting the p-p scattering data,[b] from 1.8 to 4.2 Mev.

E (Mev)	$K_0{}^a$ (deg)	$(\Delta K_0)_{vp}$ (deg)	$(\Delta K_0)_\eta$ (deg)	δ_a (deg)	$(\Delta\delta)_{vp}$ (deg)	$\delta_a-(\Delta\delta)_{vp}$ (deg)	$(\Delta\delta)_\eta$ (deg)	$\delta_a-(\Delta\delta)_{vp}-(\Delta\delta)_\eta$ (deg)
1.855	44.249	0.051	−0.010	−0.041	−0.023	−0.018	−0.014	−0.004
1.858	44.304	0.051	−0.010	−0.051	−0.030	−0.021	−0.013	−0.008
2.425	48.345	0.047	−0.013	−0.072	−0.030	−0.042	−0.016	−0.026
3.037	50.979	0.045	−0.014	−0.068	−0.033	−0.035	−0.023	−0.012
3.527	52.534	0.043	−0.015	−0.079	−0.030	−0.049	−0.022	−0.027
3.899	53.342	0.042	−0.015	−0.085	−0.030	−0.055	−0.024	−0.031
4.203	53.848	0.041	−0.015	−0.066	−0.021	−0.045	−0.019	−0.026

[a] The statistical uncertainty in $K_0{}^a$ was in the range 0.020° to 0.070°, while the statistical uncertainty in δ_a was typically 0.020°.
[b] See reference 4.

The effect on the present results of using split P-wave phase shifts in the analysis has been considered only qualitatively[19] employing results obtained by Hull and Shapiro[20] with split P- and D-wave phase shifts. It is apparent, however, that the present fits cannot be improved by a useful amount unless the δ_J are sufficiently large ($|\delta_J|>1°$), so that the mean phase-shift approximation is inapplicable. A real determination of the presence or absence of split P- and D-wave phase shifts will require a more complete analysis than has been attempted here.

4. NUCLEAR S-WAVE SCATTERING

The present section is concerned with the f function of Breit, Condon, and Present,[8] which is known to vary nearly linearly with energy over a considerable range of energies, if the non-Coulomb interaction between the protons has a short range. Any small additional long-range interaction between protons will contribute to the f function a part which does not vary linearly with energy especially at the lowest scattering energies. The vacuum polarization potential represents just such a long-range interaction. The contribution to the f function resulting from vacuum polarization scattering in the $L=0$ state has been considered by Foldy and Eriksen.[1] The present work supplements their treatment by considering both the contributions to f due to vacuum polarization scattering in angular momentum states with $L>0$ and the employment of the relativistic value of the Coulomb scattering parameter rather than its nonrelativistic value.

The f function is defined as

$$f=(C_0{}^2/\eta)\cot K_0+4C-2-2\ln\eta \\ +2\,\mathrm{Re}\{\Gamma'(-i\eta)/\Gamma(-i\eta)\}, \quad (4.1)$$

where $C_0{}^2=(2\pi\eta)/[\exp(2\pi\eta)-1]$, $C=0.5772\cdots$ is Euler's constant and $\Gamma(x)$ is the gamma function. The vacuum polarization scattering contributes to f through its contributions $K_0{}^{vp}$ and $(\Delta K_0)_{vp}$ in the apparent S-wave phase shift $K_0{}^a$. The quantity $K_0{}^{vp}+(\Delta K_0)_{vp}$ corresponds to a small contribution $(\Delta f)_{vp}$ to f_a which was split into three parts,

$$(\Delta f)_{vp}=(\Delta_1 f)_{vp}+(\Delta_2 f)_{vp}+(\Delta_3 f)_{vp}.$$

The first two parts are due to the vacuum polarization S-wave phase shift $K_0{}^{vp}$, and were calculated by Foldy and Eriksen as $\Delta_1\mathbf{K}=\frac{1}{2}(\Delta_1 f)_{vp}$ and $\Delta_2\mathbf{K}=\frac{1}{2}(\Delta_2 f)_{vp}$. The first part is caused by the interaction at distances less than e^2/mc^2 and varies linearly with energy. The corresponding contribution to the total S-wave phase shift was therefore considered[21] to be part of the nuclear phase shift $K_0{}^N$. The second part is caused by the interaction at distances greater than e^2/mc^2. The third part is caused by the vacuum polarization interaction in states with $L>0$, and was obtained from $(\Delta K_0)_{vp}$ as calculated in Sec. 3 for the WMF energy range, with the formula

$$(\Delta_3 f)_{vp}=(\partial f/\partial K_0)(\Delta K_0)_{vp}. \quad (4.2)$$

For energies outside the range 1.8–4.2 Mev the quantity $(\Delta K_0)_{vp}$ was obtained on the assumption that the S-wave phase shift is determined by the data at $\theta=90°$. This assumption is equivalent to assuming that $[\Delta\sigma(\theta)]_{vp}/[\Delta\sigma(\theta)]_s$ is sufficiently constant over the angular range which is used to determine $K_0{}^a$. This was found to be the case for $50°\leq\theta\leq 90°$ in the energy range 1.4–4.2 Mev, but for energies below 1 Mev the validity of the assumption was not investigated. For energies above $\sim$2 Mev it was also assumed that nuclear P-wave scattering may be neglected for the present purpose. The reason for this was that in the present work the change $(\Delta K_0)_{vp}$ in the S-wave phase shift, after $(\Delta\sigma)_{vp}$ was included in σ, was taken to be the difference between the two *initial* S-wave fits to the data in the angular range $90°\geq\theta\geq 50°$, *before* the introduction of a mean P wave in the analysis. When P-wave scattering was introduced, these initial S-wave phase shifts required a readjustment $(\Delta K_0)_p$ which depends on the sizes of the respective P-wave phase shifts δ_a and $\delta_a-(\Delta\delta)_{vp}$. Since the P-wave phase shift became smaller in magnitude after the inclusion of $(\Delta\sigma)_{vp}$, the readjustment $(\Delta K_0)_p$ of the initial S-wave

[19] L. Durand, III, dissertation, Yale University, 1957 (unpublished).
[20] M. H. Hull, Jr., and J. Shapiro, Phys. Rev. **109**, 846 (1958). Effects of split P-wave phase shifts on p-p scattering analysis have been considered for older data by Breit, Kittel, and Thaxton, Phys. Rev. **57**, 225 (1940).

[21] The authors wish to thank Professor G. Breit for suggesting this procedure.

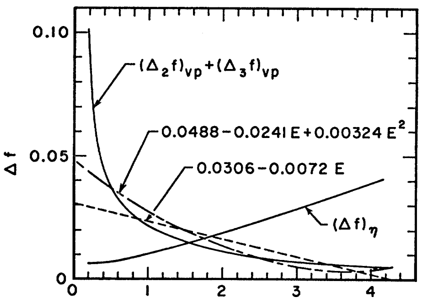

FIG. 3. The contributions to f which do not vary linearly with energy. The vacuum polarization contribution $(\Delta_2 f)_{vp}+(\Delta_3 f)_{vp}$ is due to scattering in all angular momentum states *except* for the interaction at $r<e^2/mc^2$ in the $L=0$ state, which is not included. The dashed curves are the least-squares fits to this contribution. The employment of η_r rather than η in the determination of the S wave from experiment and in the calculation of f results in the contribution $(\Delta f)_\eta$.

phase shift was different, depending on whether or not $(\Delta\sigma)_{vp}$ was included in σ. For example, at 3.899 Mev, considering the two initial S-wave fits, $(\Delta K_0)_{vp}=0.042°$. After the addition of the mean P-wave phase shifts δ_a and $\delta_a-(\Delta\delta)_{vp}$ of Table I, and subsequent readjustment of the initial S-wave phase shifts, $(\Delta K_0)_{vp}=0.031°$. Below 1 Mev, $[\Delta\sigma(90°)]_{vp}$ was calculated according to Eq. (24.3) of reference 3, with the vacuum polarization scattering amplitude to order η^2. In Table II the quantity $(\Delta_3 f)_{vp}$ and the results of Foldy and Eriksen for $(\Delta_2 f)_{vp}$ are listed over the energy range 0.2–4.0 Mev for comparison, and Fig. 3 shows $(\Delta_2 f)_{vp}+(\Delta_3 f)_{vp}$ *vs* E.

The employment of the relativistic value of the Coulomb scattering parameter rather than its nonrelativistic value resulted in a contribution $(\Delta f)_\eta$ to f_a,[22] which consisted of two parts

$$(\Delta f)_\eta=(\Delta_1 f)_\eta+(\Delta_2 f)_\eta.$$

The first part arose from the small change $(\Delta K_0)_\eta$ in $K_0{}^a$ when the contribution $(\Delta\sigma)_\eta$ in σ was incorporated in the data analysis. Assumptions similar to those used in obtaining $(\Delta_3 f)_{vp}$ were made for calculating $(\Delta_1 f)_\eta$. The quantity $(\Delta_1 f)_\eta$ was calculated from

$$(\Delta_1 f)_\eta=(\partial f/\partial K_0)(\Delta K_0)_\eta, \tag{4.3}$$

where $(\Delta K_0)_\eta$ was obtained from $[\Delta\sigma(90°)]_\eta$ and $[\Delta\sigma(90°)]_s$ for energies outside the range 1.8–4.2 Mev. The second part is

$$(\Delta_2 f)_\eta=(\partial f/\partial\eta)(\eta-\eta_r), \tag{4.4}$$

and is the difference between the two values of f calculated with nonrelativistic and relativistic values of η, for a given fixed value of K_0 in Eq. (4.1). These quantities are given in Table II and Fig. 3 over the energy range 0.2–4.0 Mev.

The effect on f resulting from the use of the exact transformations (2.2) for $\sigma_{el}(\Theta)$ and Θ instead of (2.1) was not investigated. The main effect appears in the P wave, which is more negative when Eqs. (2.2) are used. The effect on the S-wave phase shift is smaller than $(\Delta K_0)_\eta$ in magnitude. Equations (2.2) were only used for the WMF data.

The data determine an apparent f function

$$f_a=f(\eta,K_0{}^a)=f(\eta_r,K_0{}^N)+(\Delta f)_{vp}+(\Delta f)_\eta. \tag{4.5}$$

The f function was represented by an expansion in powers of the energy in the form

$$f=f^{(0)}+f^{(1)}E+f^{(2)}E^2+\cdots, \tag{4.6}$$

and the coefficients of the powers of E were determined from a least-squares analysis of the data. Sources of experimental data and the weight per point for each group of observers are given in reference 9. The WMF data were used with a weight of 0.37 per point. The f function as calculated from the data in previous work[23] corresponds to the present f_a, which is calculated employing η and $K_0{}^a$ in (4.1). A linear fit to f_a was first made in the form (4.6) with $f^{(2)}=0$. The result was

$$f_a=7.8073+0.9162E. \tag{4.7}$$

Expansions of f_a quadratic in E can be obtained employing

$$\partial f^{(0)}/\partial f^{(2)}=5.64,\quad \partial f^{(1)}/\partial f^{(2)}=-5.23. \tag{4.7'}$$

These derivatives are independent of the experimental values of f and the size of $f^{(2)}$. The coefficient $f^{(2)}$ is small and may be calculated for a particular well shape using approximate values of $f^{(0)}$ and $f^{(1)}$. The effect of $(\Delta_2 f)_{vp}$, $(\Delta_3 f)_{vp}$, and $(\Delta f)_\eta$ on the coefficients $f_a{}^{(0)}$ and $f_a{}^{(1)}$ was first considered. This was done by means of a least-squares fit of a polynomial linear in E to $(\Delta_2 f)_{vp}$, $(\Delta_3 f)_{vp}$, and $(\Delta f)_\eta$, at the experimental values of E and employing the weights assigned to the experimental points. The results of these least-squares fits can then be combined with the fit to f_a according to (4.5) and give the same result for $f(\eta_r,K_0{}^N)$, which would have been obtained if a least-squares fit had been made to

TABLE II. Nonlinear contributions to the f function.

E (Mev)	$(\Delta_2 f)_{vp}{}^a$	$(\Delta_3 f)_{vp}$	$(\Delta_1 f)_\eta$	$(\Delta_2 f)_\eta$
0.2	0.1398	−0.0391	0.0033	0.0028
0.4	0.0690	−0.0239	0.0019	0.0042
0.6	0.0488	−0.0153	0.0015	0.0058
0.8	0.0392	−0.0127	0.0014	0.0073
1.0	0.0334	−0.0115	0.0015	0.0088
1.2	0.0296	−0.0107	0.0016	0.0104
1.6	0.0248	−0.0103	0.0018	0.0134
2.0	0.0220	−0.0102	0.0021	0.0165
3.0	0.0180	−0.0105	0.0028	0.0254
4.0	0.0158	−0.0109	0.0037	0.0352

[a] These results are due to Foldy and Eriksen, reference 1.

[22] The authors wish to thank Professor G. Breit for communicating to them the results of some unpublished work in which a partial justification for the use of η_r in the calculation of f was obtained.

[23] Yovits, Smith, Hull, Bengston, and Breit, Phys. Rev. **85**, 540 (1952).

the experimental values of this latter quantity. The values of $(\Delta_2 f)_{vp}$, $(\Delta_3 f)_{vp}$, and $(\Delta f)_\eta$ above 4.2 Mev were obtained from graphs extrapolated from the calculated values. These results are summarized in Table III. The results of Foldy and Eriksen for the short-range contribution of the vacuum polarization potential in the $L=0$ state can be represented by

$$(\Delta_1 f)_{vp}=0.0160+0.0004E, \qquad (4.8)$$

since it varies linearly with energy. The effect of $(\Delta_2 f)_{vp}$, $(\Delta_3 f)_{vp}$, and $(\Delta f)_\eta$ on the coefficients of an expansion linear in E will depend on the weights employed and the energy range considered, because they do not vary linearly with energy. It was found possible, however, to represent $(\Delta f)_\eta$ as a quadratic in E over the energy range 0.4–8.0 Mev

$$(\Delta f)_\eta=0.0035+0.0062E+0.00067E^2, \qquad (4.9)$$

which fits the calculated values to better than 2% of $(\Delta f)_\eta$ in this energy range, but is too small by 23% at 0.2 Mev. The effect of $(\Delta_2 f)_{vp}+(\Delta_3 f)_{vp}$ on the coefficients $f_a^{(0)}$, $f_a^{(1)}$, and $f_a^{(2)}$ of a quadratic fit was also considered. This was done by means of a least-squares fit of a polynomial quadratic in E to $(\Delta_2 f)_{vp}+(\Delta_3 f)_{vp}$ at the experimental values of E and employing the weights assigned to these points.[9] The result was

$$(\Delta_2 f)_{vp}+(\Delta_3 f)_{vp}=0.0488-0.0241E+0.00324E^2. \quad (4.10)$$

The linear and quadratic fits to $(\Delta_2 f)_{vp}+(\Delta_3 f)_{vp}$ are shown in Fig. 3, as dotted curves, for comparison with the calculated values.

The present results for the vacuum polarization contributions to f give a decrease in $f_a^{(0)}$ of about 0.8% on account of the $L=0$ state interaction and an increase of 0.2% on account of the interaction in states with $L>0$ resulting in a net decrease in $f_a^{(0)}$ of 0.6% when a linear fit to $(\Delta f)_{vp}$ is used. The use of a quadratic fit to $(\Delta f)_{vp}$ gives a decrease in $f_a^{(0)}$ of 0.8%. The effect on $f_a^{(1)}$ is an increase of about 0.8% for a linear fit to $(\Delta f)_{vp}$ and an increase of 2.6% for a quadratic fit to $(\Delta f)_{vp}$. The effect of $(\Delta f)_\eta$ on $f_a^{(0)}$ is less than 0.04% and gives a decrease in $f_a^{(1)}$ of 1% for a linear fit and a decrease of about 0.7% for a quadratic fit to $(\Delta f)_\eta$.

TABLE III. The contributions to $f_a^{(0)}$ and $f_a^{(1)}$ of Eq. (4.7) due to vacuum polarization scattering and due to employing η rather than η_r.

Quantity	$\Delta f^{(0)}$	$\Delta f^{(1)}$
$(\Delta_2 f)_{vp}$	0.0450	−0.0084
$(\Delta_3 f)_{vp}$	−0.0144	0.0012
$(\Delta f)_\eta$	−0.0002	0.0097

5. SUMMARY

The vacuum polarization scattering with $L>0$ accounts for about half of an apparent mean P-wave phase shift needed to fit the WMF data in the angular range $12°\leq\theta\leq 40°$, and in this angular range the apparent mean P-wave contribution to σ is typically twice the quoted, nonsystematic, experimental uncertainty. Below 2 Mev, mean P waves determined after introducing this vacuum polarization scattering are of the same magnitude as their statistical uncertainty.

The subsequent employment of the relativistic value of the Coulomb scattering parameter rather than its nonrelativistic value resulted in a theoretical cross section which was larger than the WMF cross section, but not by more than 1%, at angles $\theta\leq 20°$. It was not found possible to fit the data over their complete angular range using small mean P and D waves in addition to Coulomb, S-wave, and vacuum polarization scattering. Below 3 Mev, the mean P-wave phase shift has the same magnitude as its statistical uncertainty. A better determination of nuclear P waves would probably require data of higher precision, over the angular range $12°\leq\theta\leq 40°$, than was achieved in the WMF work.

The vacuum polarization scattering in states with $L>0$ also gave a small contribution to the S-wave phase shift, which is approximately equal to the statistical uncertainty in K_0^a. On account of its systematic character it was, nevertheless, applied and its effect on the f function was ascertained. It was found to give a contribution to f which does not vary linearly with energy, and its sign is opposite to that of the contribution to f owing to vacuum polarization scattering in the $L=0$ state, which had been considered previously by Foldy and Eriksen. The part $(\Delta_2 f)_{vp}+(\Delta_3 f)_{vp}$ of the total vacuum polarization contribution to f does not vary linearly with energy and is ~ 0.1 at 0.2 Mev decreasing to 0.02 at 0.9 Mev. For comparison one may quote the minimum uncertainty in f in this energy range as determined from the Heydenburg and Little data as between 0.01 and 0.04.[23] In the energy range 1.8–4.2 Mev this contribution varies from 0.01 to 0.005, whereas the uncertainty in f for the WMF data is between 0.008 and 0.010.

The contribution to f resulting from employing η_r rather than η in an analysis of the data was also considered, but in view of the difficulty in fitting the WMF data at small angles the use of this result would need a more thorough investigation than was attempted in the present work.

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