

## Conductivity of Nonpolar Crystals in Strong Electric Field. II\*

JIRO YAMASHITA†

Radiation Laboratory, University of Pittsburgh, Pittsburgh, Pennsylvania

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The distribution function of the "hot electron" is investigated in germanium for the case of a small applied electric field, where both acoustical- and optical-mode scattering exists. The electron energy loss due to the interaction with the optical mode is found to be quite large and it eliminates large parts of the quantitative discrepancy between the previous theory and experiment.

### I. INTRODUCTION

IN the previous paper,<sup>1</sup> which we shall denote as I, we investigated the conductivity of nonpolar crystals in a strong electric field. We used a simplified model in which the energy surfaces are assumed to be spherical and the conduction electrons interact only with acoustical modes of vibrations. We solved the Boltzmann equation approximately for this model and found that the conduction electron distribution is not Maxwellian, and nonohmic current characteristics follow from the new distribution of electrons in the strong electric field. Although the simplified theory can give a qualitative description of observed nonohmic current in germanium and silicon, the result is by no means satisfactory in the quantitative sense.

Let us denote the distribution function by

$$f(\mathbf{k}) = f_0(E) + K_z g(E),$$

as usual. Here  $K$  and  $E$  are the wave number vector and energy of conduction electrons, respectively, and  $z$  is the direction of the applied field. Then, as seen in I,  $f_0(E)$  is given by

$$f_0(E) = (E/kT + p)^p \exp(-E/kT). \quad (1)$$

Here  $p$  is defined by

$$p = \frac{3\pi \mu_0^2}{16 c^2} F^2, \quad (2)$$

where  $\mu_0$  is the low-field mobility for pure lattice scattering,  $c$  is velocity of sound waves,  $F$  is the intensity of the applied field,  $k$  is the Boltzmann constant, and  $T$  is the absolute temperature. If the field is sufficiently weak, we can expand the distribution function and keep only terms of the first order in  $p$ :

$$f_0(y) = N_p e^{-y} [1 + p \ln(y+p)], \quad (3)$$

where  $N_p$  is the normalization constant and  $y = E/kT$ . Now, we define a quantity  $\sigma$  by

$$\sigma = \{\mu(F) - \mu_0\} / \mu_0. \quad (4)$$

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† On leave from the Institute of Science and Technology, University of Tokyo, Tokyo, Japan.

<sup>1</sup> J. Yamashita and M. Watanabe, *Progr. Theoret. Phys. Japan* **12**, 443 (1954).

By using (2) and the usual current expression, we have

$$\sigma = -p \left\{ 1 + \frac{2}{\sqrt{\pi}} \int_0^\infty \ln(y+p) y^{3/2} e^{-y} dy - \int_0^\infty \ln(y+p) y e^{-y} dy \right\}. \quad (5)$$

According to observations for  $n$ -type germanium samples,  $\sigma$  is proportional to  $F^2$  and the absolute value of  $\sigma/F^2$  is about  $-1.5 \times 10^{-7} \text{ cm}^2 \text{ v}^{-2}$  at room temperature.<sup>2</sup> Equation (5), on the other hand, gives a value about two hundred times larger than that observed. As we shall see later, the large discrepancy should be ascribed to the oversimplification of the model.

An assembly of conduction electrons which is in equilibrium with a lattice in a strong electric field would receive more energy from the field than it can dissipate by interaction with acoustical modes of vibration. Since the dissipation rate of energy by the interaction with acoustical modes increases with increasing electron energy, the average energy of the assembly of electrons must increase in the strong electric field so as to transfer more energy to the lattice and maintain the electron in an equilibrium state. Therefore, the discrepancy mentioned previously suggests that the average energy loss of an electron in colliding with a phonon is greater than has been considered in paper I. On the other hand, we know that the temperature dependence of mobility in  $n$ -type germanium is given by a  $T^{-1.66}$  instead of a  $T^{-1.5}$  law. We can imagine three possible mechanisms underlying this deviation from a  $T^{-1.5}$  law: (1) interaction with optical modes of vibration, (2) intervalley scattering, and (3)  $K$  dependence of the electron effective mass. However, the second effect can be eliminated as too small, judging from the acousto-electric effect observed by Weinreich.<sup>3</sup> In the following we shall consider the effects upon the hot-electron problem of scattering with the optical modes, assuming that the deviation from the  $T^{-1.5}$  law depends only upon conduction electron-optical mode interaction.<sup>4</sup> Thus we are considering effect (1) as most important and yet we cannot deny the possible importance of (3).

<sup>2</sup> J. B. Gunn, in *Progress in Semiconductors* (John Wiley and Sons, Inc., New York, 1957), Vol. 2, p. 213.

<sup>3</sup> G. Weinreich, Bell Telephone Laboratories (private communication).

<sup>4</sup> T. Morgan, *Bull. Am. Phys. Soc. Ser. II*, **3**, 13 (1958).

## II. BOLTZMANN EQUATION

We use a simplified assumption that the shape of the energy surface is spherically symmetric in  $k$  space, although the assumption is not correct in  $n$ -type germanium and silicon. We neglect also the Coulomb interaction between conduction electrons.<sup>5</sup> Then we can describe the Boltzmann equation,

$$[\partial f/\partial t]_F + [\partial f/\partial t]_c = 0, \quad (6)$$

for a system which consists of conduction electrons and both acoustical and optical modes of vibration. Here the operator  $[\partial f/\partial t]_F$  is given by

$$\left[\frac{\partial f}{\partial t}\right]_F = \frac{eF}{\hbar} \left( g(E) + \frac{2}{3}E \frac{dg}{dE} + \frac{\hbar^2}{m} K_z \frac{df_0}{dE} \right), \quad (7)$$

and the operator  $[\partial f/\partial t]_c$  consists of two terms:

$$[\partial f/\partial t]_c = [\partial f/\partial t]_{ac} + [\partial f/\partial t]_{op}. \quad (8)$$

Evaluation of the first term is not difficult, because we can expand the functions  $f_0(E+\hbar\omega_a)$ ,  $f_0(E-\hbar\omega_a)$ , and  $\exp(\hbar\omega_a/kT)$  with respect to  $\hbar\omega_a$  ( $\hbar\omega_a$  is a frequency of the acoustical mode of vibrations), and retain terms through second order in  $\hbar\omega_a$ . After some computations we get the following expression (see I):

$$\left[\frac{\partial f}{\partial t}\right]_{ac} = \frac{A}{\sqrt{E}} \left[ \frac{d^2 f_0}{dE^2} + \left( \frac{E^2}{kT} + 2E \right) \frac{df_0}{dE} + \frac{2E}{kT} f_0 - \left( \frac{K_z}{2mc^2} \right) E g(E) \right], \quad (9)$$

where  $A$  is defined by

$$A = 9VC^2 m^3 (kT) / \pi M \hbar^4 (2m)^{3/2}; \quad (10)$$

$V$  is the volume of the crystal,  $m$  is the effective mass of the conduction electron,  $M$  is the mass of the ion, and  $C$  is a coupling constant between conduction electrons and the acoustical mode of vibration.<sup>6</sup> On the other hand, the operator  $[\partial f/\partial t]_{op}$  is given by<sup>6</sup>

$$\begin{aligned} \left[\frac{\partial f}{\partial t}\right]_{op} = & \frac{B}{\hbar\omega\sqrt{E}} \{ [E(E+\hbar\omega)]^{3/2} \\ & \times [f_0(E+\hbar\omega) \exp(\hbar\omega/kT) - f_0(E)] \\ & + \epsilon(E) [E(E-\hbar\omega)]^{3/2} [f_0(E-\hbar\omega) - f_0(E) \exp(\hbar\omega/kT)] \\ & - K_z g(E) [ [E(E+\hbar\omega)]^{3/2} + \epsilon(E) \exp(\hbar\omega/kT) \\ & \times [E(E-\hbar\omega)]^{3/2} ] \}. \quad (11) \end{aligned}$$

<sup>5</sup> According to the experiment of K. Seeger,  $\sigma$  depends on the number of conduction electrons [K. Seeger, Bull. Am. Phys. Soc. Ser. II, 3, 112 (1958)]. We wish to express our gratitude to him for helpful discussions about the hot-electron problem.

<sup>6</sup> Here  $C$  is defined according to F. Seitz, Phys. Rev. 73, 549 (1948).

Here  $B$  is defined by

$$B = \frac{VD^2 m^3}{\pi M \hbar^4 (2m)^{3/2}} \left( \frac{\hbar^2 \rho^2}{2m} \right) \left( \frac{1}{\exp(\hbar\omega/kT) - 1} \right), \quad (12)$$

where  $\hbar\omega$ , the characteristic energy of an optical phonon, is assumed to be a constant,  $D$  is a coupling constant between conduction electrons and the optical mode of vibrations,  $\rho$  is the first nonvanishing reciprocal vector of the lattice, and  $\epsilon(E)$  is a step function equal to one for  $E > \hbar\omega$  and equal to zero otherwise.

Inserting (9) and (11) into (6) and equating the terms of the same angular dependence, we have a pair of equations,

$$\begin{aligned} \frac{A}{\sqrt{E}} \left[ E^2 f_0'' + \left( \frac{E^2}{kT} + 2E \right) f_0' + \frac{2E}{kT} f_0 \right] + \frac{B}{\hbar\omega\sqrt{E}} \\ \times \{ [E(E+\hbar\omega)]^{3/2} \{ f_0(E+\hbar\omega) \exp(\hbar\omega/kT) - f_0(E) \} \\ + \epsilon(E) [E(E-\hbar\omega)]^{3/2} \{ f_0(E-\hbar\omega) - f_0(E) \exp(\hbar\omega/kT) \} \} \\ = \frac{eF}{\hbar} \left( g(E) + \frac{2}{3}E \frac{dg}{dE} \right) \quad (13) \end{aligned}$$

and

$$\left\{ \frac{A\sqrt{E}}{2mc^2} + \frac{B}{\hbar\omega} [E(E+\hbar\omega)]^{3/2} + \epsilon(E) \exp(\hbar\omega/kT) (E-\hbar\omega)^{3/2} \right\} g(E) = \left( \frac{eF\hbar}{m} \right) \frac{df_0}{dE}. \quad (14)$$

In general we cannot expand the function  $f_0(E+\hbar\omega)$  or  $f_0(E-\hbar\omega)$  with respect to  $\hbar\omega$ , because  $\hbar\omega$  is not necessarily small as compared with  $E$ . Therefore, by including interaction with the optical mode of vibrations the problem becomes more difficult to solve. However, if the intensity of the applied field is small enough, we need to consider only terms linear in  $p$ , and we can solve the problem approximately.

## III. SMALL QUADRATIC CHANGES IN MOBILITY

Let us assume that

$$f_0(E) = \chi(E) N_m \exp(-E/kT)$$

and

$$g(E) = \zeta(E) N_m \exp(-E/kT),$$

where  $N_m$  is the normalization constant of the usual Maxwell distribution function:

$$N_m \int \exp(-E/kT) \sqrt{E} dE = 1,$$

and  $f_0(E)$  is assumed to be normalized in the sense

$\int f_0(E)\sqrt{E}dE=1$ . Then, Eqs. (13) and (14) become

$$\begin{aligned} & \frac{A}{\sqrt{E}} \left[ E\chi'' + \left( 2E - \frac{E^2}{kT} \right) \chi' \right] \\ & + \frac{B}{\hbar\omega\sqrt{E}} \left[ [E(E+\hbar\omega)]^{\frac{1}{2}} \{ \chi(E+\hbar\omega) - \chi(E) \} \right. \\ & \left. + \epsilon(E) [E(E-\hbar\omega)]^{\frac{1}{2}} \{ \chi(E-\hbar\omega) - \chi(E) \} \exp(\hbar\omega/kT) \right] \\ & = -\frac{eF\hbar}{\hbar} \left( \zeta(E) + \frac{2}{3} \left[ \frac{d\zeta}{dE} - \frac{\zeta}{kT} \right] E \right) \end{aligned} \quad (15)$$

and

$$\begin{aligned} & \left\{ \frac{A\sqrt{E}}{2mc^2} + \frac{B}{\hbar\omega} [(E+\hbar\omega)^{\frac{1}{2}} \right. \\ & \left. + \epsilon(E) \exp(\hbar\omega/kT) (E-\hbar\omega)^{\frac{1}{2}}] \right\} \zeta(E) \\ & = \left( \frac{eF\hbar}{m} \right) \left( \frac{d\chi}{dE} - \frac{\chi}{kT} \right), \end{aligned} \quad (16)$$

respectively.

As seen from (16),  $\zeta$  is of first order in  $F$ , so that the right-hand side of (15) is quadratic in  $F$ . If we neglect terms of the order of the magnitude of  $F^2$ , we see from (15) and (16) that  $\chi(E)$  is constant and  $K_s\zeta(E)$  is also constant. Inserting  $\zeta(E)$  into (15), we have

$$\begin{aligned} & \left( E^2\chi'' - \frac{E^2}{kT}\chi' + 2E\chi' \right) + \left( \frac{B}{A\hbar\omega} \right) \\ & \times \{ [E(E+\hbar\omega)]^{\frac{1}{2}} [\chi(E+\hbar\omega) - \chi(E)] \\ & + \epsilon(E) \exp(\hbar\omega/kT) [E(E-\hbar\omega)]^{\frac{1}{2}} [\chi(E-\hbar\omega) - \chi(E)] \} \\ & = -p \left[ EkT\chi'' + (kT-2E)\chi' + \left( \frac{E}{kT} - 1 \right) \chi \right]. \end{aligned} \quad (17)$$

Let us now put

$$\chi(E) = 1 + p\xi(E) - p\langle\xi\rangle \quad (18)$$

for small values of  $p$ ; then we have the following equations:

$$\begin{aligned} & \left( E^2\xi'' - \frac{E^2}{kT}\xi' + 2E\xi' \right) + \left( \frac{B}{A\hbar\omega} \right) \\ & \times \{ [E(E+\hbar\omega)]^{\frac{1}{2}} [\xi(E+\hbar\omega) - \xi(E)] \\ & + \epsilon(E) \exp(\hbar\omega/kT) [E(E-\hbar\omega)]^{\frac{1}{2}} \\ & \times [\xi(E-\hbar\omega) - \xi(E)] \} + \left( \frac{E}{kT} - 1 \right) \xi = 0, \end{aligned} \quad (19)$$

and

$$\begin{aligned} g(E) &= -\frac{eF\hbar}{m} \frac{df_0}{dE} / \left( \frac{A}{2mc^2} \right) \left\{ E^{\frac{1}{2}} + \frac{B}{A} \frac{2mc^2}{\hbar\omega} \right. \\ & \left. \times [(E+\hbar\omega)^{\frac{1}{2}} + \epsilon(E) \exp(\hbar\omega/kT) (E-\hbar\omega)^{\frac{1}{2}}] \right\}, \\ \frac{df_0}{dE} &= \left( p \frac{d\xi}{dE} - \frac{1}{kT} [1 + p\xi(E) - p\langle\xi\rangle] \right) \exp(-E/kT). \end{aligned} \quad (20)$$

Here  $\langle\xi\rangle$  means the average value of  $\xi(E)$ , which is defined by

$$\langle\xi\rangle = N_m \int \xi(E) \exp(-E/kT) \sqrt{E} dE. \quad (21)$$

Since Eq. (19) still has to be solved, we introduce here a rather crude approximation. We expand  $\xi(E+\hbar\omega)$  and  $\xi(E-\hbar\omega)$  with respect to  $\hbar\omega$ , and take terms through the second derivative of  $\xi(E)$  into account. Then we obtain finally a simple second-order differential equation:

$$\begin{aligned} & \left[ x^2 + \frac{B}{A} \left\{ \frac{1}{2} [x(x+1)]^{\frac{1}{2}} + \epsilon(x) e^{s/2} [x(x-1)]^{\frac{1}{2}} \right\} \right] \frac{d^2\xi}{dx^2} \\ & + \left[ -sx^2 + 2x + \frac{B}{A} [x(x+1)]^{\frac{1}{2}} \right. \\ & \left. - \epsilon(x) e^s [x(x-1)]^{\frac{1}{2}} \right] \frac{d\xi}{dx} + (sx-1)\xi = 0, \end{aligned} \quad (22)$$

where  $x = E/\hbar\omega$  and  $s = \hbar\omega/kT$ . A solution of (22) is easily found to be

$$\begin{aligned} \eta(x) &= \eta(x_0) + \exp \left\{ - \int_{x_0}^x Q(z) dz \right\} \int_{x_0}^x \frac{(1-su)}{R(u)} \\ & \times \exp \left\{ \int_{x_0}^u Q(w) dw \right\} du, \end{aligned} \quad (23)$$

where  $\eta = d\xi/dx$  and

$$\begin{aligned} Q(x) &= [-sx^2 + 2x + (B/A)\{\dots\}]/R(x), \\ R(x) &= [x^2 + (B/A)\{\dots\}], \end{aligned}$$

and  $x_0$  is any small value of  $x$ . In order to determine the value for small values of  $x$ , it is more convenient to start from (17). By expanding  $\chi(E+\hbar\omega)$  and  $\chi(E-\hbar\omega)$  with respect to  $\hbar\omega$ , we have

$$\begin{aligned} & \left\{ x^2 + \frac{B}{A} \frac{1}{2} [x(x+1)]^{\frac{1}{2}} + px \right\} \frac{d^2\chi}{dx^2} \\ & + \left\{ -sx^2 + 2x + \frac{B}{A} [x(x+1)]^{\frac{1}{2}} + p - 2px \right\} \frac{d\chi}{dx} \\ & + p(sx-1)\chi = 0. \end{aligned} \quad (24)$$

TABLE I. Computed values of  $[\xi(x) - \langle \xi \rangle]$  and  $\eta(x)$  at 320°K.

$x$	$\xi(x) - \langle \xi \rangle$	$\eta(x)$	Ratio at 320°K
0.1	-0.0176	0.0173	2.01
0.2	-0.0159	0.0210	1.49
0.3	-0.0138	0.0220	1.56
0.4	-0.0116	0.0217	1.26
0.5	-0.0095	0.0207	1.10
0.6	-0.0075	0.0193	1.00
0.8	-0.0049	0.0164	0.96
1.0	-0.0007	0.0134	0.85
1.2	+0.0005	0.0118	0.96
1.4	0.0028	0.0113	1.00
1.6	0.0050	0.0108	1.01
1.8	0.0071	0.0102	1.01
2.0	0.0091	0.0096	1.01
2.4	0.0126	0.0081	1.01
2.8	0.0155	0.0062	1.02
3.0	0.0166	0.0051	1.03
3.8	0.0183	-0.0017	1.05

We see easily that this equation has a solution for small values of  $x$ :

$$\chi(x) = 1 + p(8A/3B)x^{\frac{3}{2}}. \quad (25)$$

The parameter  $\sigma$  which shows the degree of departure from the ohmic law is also given by

$$\sigma = -p \int_0^\infty \left[ \frac{1}{s} \frac{d\xi}{dx} + \langle \xi \rangle - \xi(x) \right] \Gamma(x) dx / \int_0^\infty \Gamma(x) dx, \quad (26)$$

where  $\Gamma(x)$  is defined by

$$\Gamma(x) = x \sqrt{x} e^{-sx} / \left[ x^{\frac{3}{2}} + \frac{B}{A} \frac{2mc^2}{\hbar\omega} \times \{ [x+1]^{\frac{3}{2}} + \epsilon(x)e^s \times [x-1]^{\frac{3}{2}} \} \right]. \quad (27)$$

By using  $g(E)$  in (20) with the condition  $p=0$ , we can derive the usual formula for mobility. The two parameters  $A$  and  $B$  are determined so as to give the observed value of the mobility of  $n$ -type germanium at room temperature ( $3900 \text{ cm}^2 \text{ v}^{-1} \text{ sec}^{-1}$ ) and the observed value of the temperature dependence of mobility in the temperature region from 100°K to room temperature (that is,  $T^{-1.66}$ ). Since the parameters  $A$  and  $B$  involve temperature, it is more convenient to introduce a new parameter  $J$  by

$$(2mc^2/\hbar\omega)(B/A) = Js/(e^s - 1),$$

which is independent of temperature. By using observed values we determine  $J$  as about 0.14, so that the contribution from the optical mode of vibrations to electron scattering is rather small below room temperature. However, the effect of the optical-mode scattering for the energy loss of conduction electrons is remarkable even at room temperature, because it is proportional to the factor  $B/A$ , which is  $(\hbar\omega/2mc^2)$  times larger than  $Js/(e^s - 1)$ .<sup>7</sup> Therefore, an assembly of conduction elec-

<sup>7</sup> Since we use a simplified model, it is rather difficult to define the effective mass uniquely. Here we assume that it is determined

trons can dissipate much more energy to the lattice by interaction with the optical mode of vibrations than expected from the previous computation in I. After some numerical computation we have obtained the theoretical value of  $\sigma/F^2$  for 320°K and 90°K. They are  $-1.3 \times 10^{-7}$  and  $-1.6 \times 10^{-5} \text{ cm}^2 \text{ v}^{-2}$ , respectively.<sup>8</sup> These are to be compared with the corresponding experimental values of<sup>9</sup>  $-1.5 \times 10^{-7}$  and  $-2.8 \times 10^{-5} \text{ cm}^2 \text{ v}^{-2}$ .<sup>10</sup> In Table I we show some values of  $[\xi(x) - \langle \xi \rangle]$  and  $\eta(x)$  at 320°K. Next, we must estimate the error which is introduced with the replacement of  $[\xi(x+1) - \xi(x)]$  by  $\eta(x) + \frac{1}{2} d\eta/dx$ . By solving (23) we obtain numerical values of  $\xi(x)$  and  $\eta(x)$  for many points along  $x$ , so that we can examine the accuracy of the approximation by computing the ratio

$$\left[ \xi(x) + \eta(x) + \frac{1}{2} \frac{d\eta}{dx} \right] / [\xi(x+1)].$$

If this ratio is close to unity, the expansion of  $\xi(E + \hbar\omega)$  with respect to  $\hbar\omega$  may be justified. The result of this computation is depicted in Table I, and we see that the ratio is fairly close to unity except for small values of  $x$ . Thus we have found that the nonohmic current in a small applied field is given approximately by

$$j = eF\mu_0(1 - \kappa p),$$

where  $\kappa$  is about 0.00425 and 0.057 at 320°K and 90°K, respectively. Although the present solution is meaningful only for small applied fields, we may regard it as the first part of a series expansion representing the true solution of (17). Thus, we suppose that it exhibits the nonohmic features of the current in  $n$ -type germanium fairly well, as long as  $\kappa p$  is much smaller than one. Since  $p=1$  corresponds to an applied voltage of 185 volts/cm at room temperature, the value of  $\kappa p=0.2$  at room temperature corresponds to an applied voltage of 1000 volts/cm. Therefore, the theory predicts that the mobility at room temperature is reduced to 0.8 of the low-field mobility, when the applied voltage becomes 1000 volts/cm. This prediction is in reasonable agreement with observations.<sup>10</sup>

Since the value of  $\sigma$  predicted with the previous theory is about two hundred times larger than observed, the agreement of the present theory with experiment is rather surprising. Although the model used here is still too simple, we may say that a large part of the quantitative discrepancy between the theory and experiments is eliminated by considering both acoustical and optical modes of scattering.

by a relation  $1/m = (1/3)(1/m_1 + 1/m_2 + 1/m_3)$ , and the observed values of the cyclotron mass;  $m_1=1.58$  and  $m_2=m_3=0.082$ . By using the values  $c=5.4 \times 10^5 \text{ cm sec}^{-1}$  and  $\theta=320^\circ\text{K}$ , we obtain the value  $(\hbar\omega/2mc^2)=800$ .

<sup>8</sup> By using another value  $(\hbar\omega/2mc^2)=400$ , we have found that the value of  $\sigma$  becomes about twice the value obtained by the assumption  $(\hbar\omega/2mc^2)=800$ . As long as we discuss the order of magnitude of  $\sigma$ , the ambiguity in the effective mass is not serious.

<sup>9</sup> Arthur, Gibson, and Graville, J. Electron 2, 145 (1956).

<sup>10</sup> J. B. Gunn, J. Electronics 2, 87 (1956).