Propagation of a Magnetic Field into a Superconductor*

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The propagation of a magnetic field into a superconducting wire of circular cross section is analyzed theoretically. Upper and lower bounds are obtained on t_0, the time required for the field to reach the axis, and on $R(t)$, the inner radius of the normally conducting region. The lower bound is just the approximate result obtained by Sixtus and Tonks, Pippard, and Lipshitz. It has been verified experimentally by Faber for applied fields less than 1.04 times the critical field. It is believed that for large applied fields the actual results should be closer to the upper bounds.

1. INTRODUCTION

It is generally agreed that the magnetic field strength is zero inside a superconducting material even though there may be a nonzero field in the region adjacent to the superconductor. However, if the field strength in the adjacent region exceeds a critical value H_c the external field gradually penetrates into the superconductor and converts it to a normal conductor. While the penetration is occurring the material consists of a diminishing superconducting region in which a nonzero field exists. This conclusion has been tested by Faber[1] and by Burwick[2] who used the following experiment. A field $H_0>H_c$ was switched on instantaneously parallel to the axis of a superconducting circular cylindrical wire. The resistance of the wire was measured and found to remain at the value zero until a certain time t_0 after the field had been switched on, when it became positive. Presumably t_0 was the time required for the field to penetrate to the axis of the cylinder and convert the entire wire to the normally conducting state.

In this article we consider the theoretical problem of determining t_0 as a function of H_0 and H_c. We obtain both upper and lower bounds on t_0 for all values of H_0 and H_c. We also obtain bounds on $R(t)$, the radius of the inner boundary of the normally conducting region at the time t. It seems likely that the actual values of t_0 and of $R(t)$ are close to the lower bounds for small values of $p=(H_0-H_c)/H_c$ and close to the upper bounds for large values of p. This has been verified by Faber for t_0 when $p\leq 0.04$. Our lower bound is just the approximate result obtained by Sixtus and Tonks[3] in their study of the propagation of Barkhausen discontinuities. The same formula was also found by Pippard[4] and Lipshitz[5] in analyzing the present problem.

2. FORMATION OF THE PROBLEM

The problem which we consider is that of determining an axial magnetic field $H(r,t)$ in the region $R(t)\leq r\leq a$. Here a is the radius of the wire and $R(t)$ is the inner boundary of the normally conducting region, which must also be found. The time t_0 at which $R(t_0)=0$ is the "switching time", i.e., the time at which the superconducting region disappears. The field H is zero in the superconducting region $0\leq r<R(t)$, while at the boundary $R(t)$, H has the critical value H_c. At the outer boundary $r=a$, H has the value H_0 of the applied field. At $t=0$, $R(0)=a$ since initially there is no normally conducting region. The velocity R_t of the boundary is equal to $-(4\pi\sigma\mu/c^2H_c)H_r(R,t)$. This follows from conservation of flux. The constants σ and μ are the conductivity and permeability of the normally conducting material and c is the velocity of light. In the normally conducting region, H satisfies the diffusion equation $\nabla^2H=4\pi\sigma\mu c^{-2}H_t$ provided that σ is large enough for the term H_{tt} to be neglected.

It is convenient to use the radius a as the unit of length and the quantity $4\pi\sigma\mu a^2c^{-2}$ as the unit of time. Then the equations of our problem become (see Fig. 1)

$$H_{rr}+r^{-1}H_r=H_t,\quad R(t)\leq r\leq 1 \tag{1}$$

$$H(1,t)=H_0, \tag{2}$$

$$H(R,t)=H_c, \tag{3}$$

$$R_t=-H_c^{-1}H_r(R,t), \tag{4}$$

$$R(0)=1. \tag{5}$$

We must find $H(r,t)$ and $R(t)$ satisfying (1)–(5). Then t_0 can be found from the equation $R(t_0)=0$.

3. A LOWER BOUND—THE QUASI-STATIC SOLUTION

The solution of the above problem depends upon the the parameter $p=(H_0-H_c)H_c^{-1}$. When p is small and

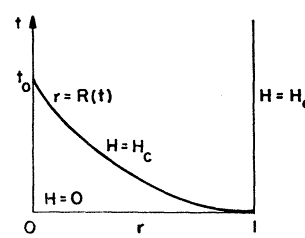

FIG. 1. The formulation of the problem in the r, t plane. The interface at $R(t)$ starts at $R=1$ when $t=0$ and reaches $R=0$ at $t=t_0$. The magnetic field $H=0$ for $r<R(t)$, $H=H_c$ at $r=R(t)$ and $H=H_0$ at the outer boundary $r=1$. In the normally conducting region H satisfies the diffusion equation. The velocity of the interface is given by $R_t=-H_c^{-1}H_r(R,t)$.

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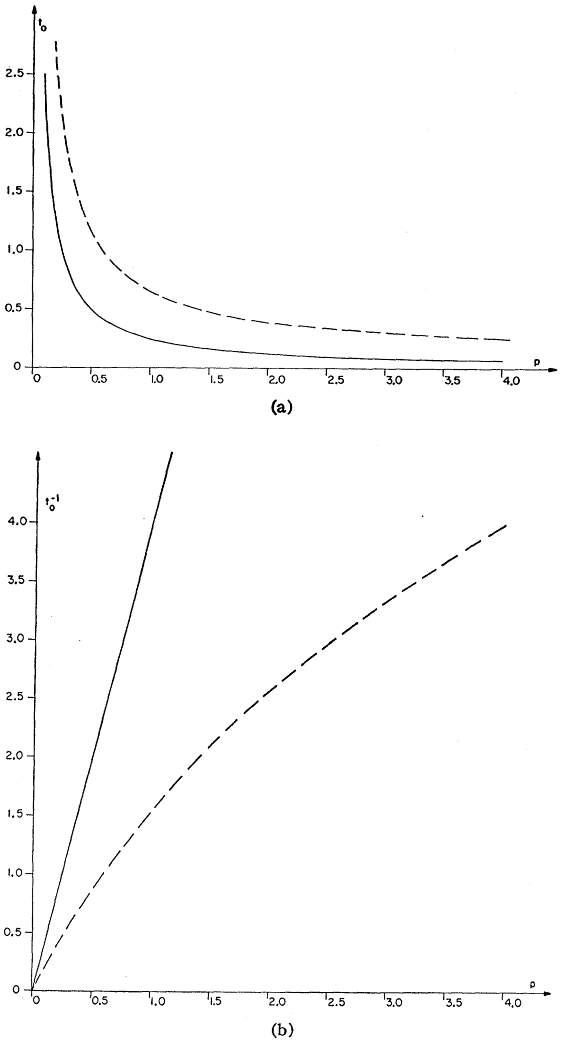

FIG. 2. (a) Graphs of bounds on the switching time t_0 at which the superconducting region disappears as functions of the fractional excess applied magnetic field $p=(H_0-H_c)/H_c$. The lower bound is given by (9) and the upper bound by (13). The unit in which t_0 is measured is $4\pi a^2\sigma\mu c^{-2}$. (b) Graphs of bounds on t_0^{-1}.

positive the field must diffuse slowly, so we may neglect the term H_t in (1). Then the solution of (1) is

$$H=H_0+\frac{(H_c-H_0)}{\ln R}\ln r. \tag{6}$$

The two constants in (6) have been chosen to satisfy (2) and (3). Now we insert (6) into (4), which becomes

$$R_t=p/(R\ln R). \tag{7}$$

The solution of (7) which satisfies (5) is given by

$$t=\frac{1}{4p}+\frac{R^2}{2p}(\ln R-\tfrac{1}{2}). \tag{8}$$

The "switching time" t_0 at which $R=0$ is, from (8),

$$t_0=1/(4p). \tag{9}$$

As we expect, (9) shows that t_0 increases as p decreases. In terms of dimensional quantities,

$$t_0=\pi a^2\sigma\mu H_c/[c^2(H_0-H_c)]. \tag{9$'$}$$

In the Appendix it is shown that (9) yields a lower bound on t_0 and that (8) gives a lower bound on $R(t)$. These results may be explained physically by observing that the omission of H_t from (1) is equivalent to assuming that diffusion occurs instantaneously. This will obviously result in a too rapid motion of the boundary.

4. AN UPPER BOUND—THE ONE-DIMENSIONAL SOLUTION

To obtain an upper bound on t_0 and on $R(t)$, let us consider the one-dimensional problem in which a field H_0 is applied on the two sides of a superconducting slab of thickness $2a$. It is physically obvious that it will take longer for the superconducting region to disappear in this case than in the case of the circular cylinder. It is also clear that it will take longer for the interface to travel any specified distance than in the cylinder problem, so that this problem yields an upper bound on $R(t)$ as well as on t_0. These facts can be proved mathematically, but it does not seem necessary to give the proof.

We shall let r denote distance from the median plane of the slab, and we shall consider only the region $r\geq 0$. The solution is obviously symmetric in the median plane. The formulation of this problem is exactly the same as that given above in (1) to (5) provided the term $r^{-1}H_r$ is omitted from (1). It has the explicit solution

$$H=H_c+(H_0-H_c)\left(\int_0^b \exp(-z^2)dz\right)^{-1} \times\int_{(1-r)/2t^{\frac{1}{2}}}^b \exp(-z^2)dz, \tag{10}$$

$$R(t)=1-2bt^{\frac{1}{2}}. \tag{11}$$

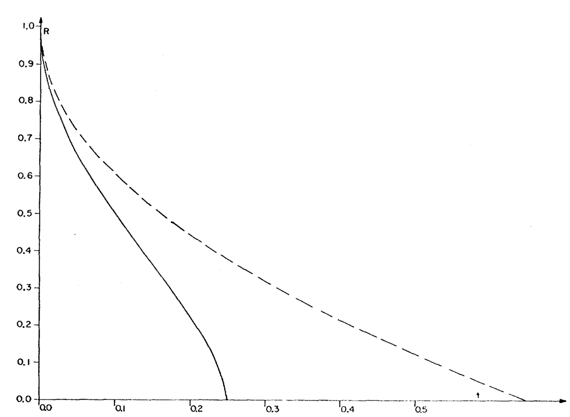

FIG. 3. Graphs of bounds on $R(t)$ as functions of t for $p=1$. The lower bound is given by (8) and the upper bound by (11).

The constant b in (10) and (11) is the solution of the equation

$$2b\exp(b^2)\int_0^b \exp(-z^2)dz=p. \tag{12}$$

From (11) the switching time is

$$t_0=1/(4b^2). \tag{13}$$

Equations (12) and (13) give t_0 as a function of p in terms of the parameter b, which can easily be eliminated. However, it is just as convenient to keep the parametric form.

For $p\ll 1$, Eq. (12) yields $b^2\approx p/2$ and then (13) becomes

$$t_0=1/(2p),\quad 0<p\ll 1. \tag{14}$$

On the other hand, for $p\gg 1$, Eq. (12) yields $b^2\approx \ln p$ and (13) becomes

$$t_0=1/(4\ln p),\quad p\gg 1. \tag{15}$$

In Fig. 2 graphs of the upper and lower bounds on t_0 are shown as functions of p. To restore the units the right sides of (13) to (15) must be multiplied by $4\pi a^2\sigma\mu c^{-2}$. Figure 3 shows the bounds on $R(t)$ for $p=1$.

APPENDIX—PROOF CONCERNING THE LOWER BOUND

We shall now prove that the values of $R(t)$ given by (8) and of t_0 given by (9) are lower bounds on the exact values of $R(t)$ and of t_0. To do this we first rewrite (1) as

$$r^{-1}(rH_r)_r=H_t. \tag{A1}$$

Now we multiply by r and then integrate (A1) from R to r, obtaining

$$rH_r-RH_r(R,t)=\int_R^r rH_t dr. \tag{A2}$$

We divide by r in (A2) and integrate again from R to 1, which yields

$$H(1)-H(R)=RH_r(R,t)\ln\frac{1}{R}+\int_R^1 r_1^{-1}\int_R^{r_1} rH_t dr dr_1. \tag{A3}$$

Making use of (2), (3), and (4) in (A3) gives

$$(H_0-H_c)H_c^{-1}=R_tR\ln R+H_c^{-1}\int_R^1 r_1^{-1}\int_R^{r_1} rH_t dr dr_1. \tag{A4}$$

Finally we integrate (A4) with respect to t from $t=0$, obtaining

$$pt=\tfrac{1}{4}+\frac{R^2}{2}(\ln R-\tfrac{1}{2})+H_c^{-1}\int_0^t\int_R^1 r_1^{-1}\int_R^{r_1} rH_t dr dr_1. \tag{A5}$$

If the integral in (A5) were absent, (A5) would agree with the quasi-static result (8). We shall show that the integral in (A5) is positive because H_t is non-negative. Then it will follow that

$$pt\leq\tfrac{1}{4}+\frac{R^2}{2}(\ln R-\tfrac{1}{2}). \tag{A6}$$

From (A6) it follows that the value of R given by (8) is a lower bound on the true value of R and that the value of t_0 given by (9) is also a lower bound on the true value of t_0.

To prove that $H_t\geq 0$, we differentiate (1) with respect to t and find that H_t is also a solution of (1). The value of H_t at $r=1$ is zero since $H=H_0$ at $r=1$. To obtain the value of H_t at $r=R$, we differentiate (3) with respect to t and obtain $H_r(R,t)R_t+H_t(R,t)=0$. Then using (4) yields $H_t(R,t)=-H_r(R,t)R_t=H_c^{-1}H_r^2(R,t)>0$. Thus we find that H_t is a solution of the heat equation (1) in the region $R(t)\leq r\leq 1$ and that H_t has non-negative values on the boundary of the region. By the maximum principle for solutions of the heat equation, $H_t\geq 0$. This completes the proof.