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Meissner Effect*

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The magnetic properties of the Bardeen-Cooper-Schrieffer model of a superconductor are investigated by means of Bogoliubov's mathematical method. A derivation of the Meissner effect is given which is strictly gauge invariant in every step.

1. GENERAL DESCRIPTION OF METHOD

In their paper on the theory of superconductivity,[1] Bardeen, Cooper, and Schrieffer ("BCS") analyze the magnetic properties of their model of a superconductor and arrive at a result indicating a "Meissner effect." The correctness of this result has been doubted. Indeed the Hamiltonian used lacks gauge invariance,[2] and a longitudinal vector potential ($\mathbf{A}\sim\mathbf{q}\cos\mathbf{q}\cdot\mathbf{r}$, $\mathbf{B}=\nabla\times\mathbf{A}=0$) appears to induce a supercurrent. But Bardeen[3] has argued that the virtual states excited by a longitudinal $\mathbf{A}$ have a more complicated character, involving collective modes, such that the choice of the gauge $\nabla\cdot\mathbf{A}=0$ is a physically meaningful simplification. Anderson[4] has elaborated this argument; he points out that the sum rule which guarantees gauge invariance is, in the BCS theory, violated only by very small terms. Still, the question remains whether even with the gauge $\nabla\cdot\mathbf{A}=0$ the naive calculation can be trusted.

To make a more convincing argument it is necessary, not only to use a strictly gauge-invariant Hamiltonian which yields the equations of motion of the model accurately, but also to carry out all calculations in a gauge-invariant fashion. We propose to do this, following Bogoliubov's[5] approach and omitting all less essential features of the BCS model, such as temperature effects and Coulomb interactions of the electrons which, after all, can only impair the tendency of the electron-phonon interaction to make the electron gas behave as a superconductor.

The Hamiltonian of the model is

$$H=H_0+H_g+H_A+H_{AA}, \tag{1}$$

where H_0= energy of the free electron and phonon gas, H_g= electron-phonon interaction (g= coupling strength), and H_A+H_{AA}= magnetic energy (terms linear and quadratic in the vector potential $\mathbf{A}$, respectively).

All interactions will be regarded as perturbations, except that some of the effects of H_g are to be treated rigorously by means of Bogoliubov's method. This is essential, since we know from Schafroth's work[6] that the ordinary perturbation approach is insufficient to derive the Meissner effect. It will even be helpful and clarifying to compare the two methods, in other words, to compare the magnetic properties of the Bogoliubov ground state of the system with those of the "ordinary" ground state (i.e., Fermi gas perturbed by H_g).

The gauge-invariant treatment will be achieved as follows. For $g=0$, the term linear in $\mathbf{A}$ of the Hamiltonian can be eliminated by a unitary transformation:

$$H'=\exp(-K_A)H\exp(K_A),$$

where (2)

$$K_A^*=-K_A,\quad [H_0,K_A]=-H_A,$$

and the new terms quadratic in $\mathbf{A}$, together with H_{AA}, give for the ground state just the ordinary diamagnetic energy of the electron gas, $-\frac{1}{2}\chi\mathbf{B}^2V$ (with Landau's χ value). If $g\neq0$, however, the transformation of H_g, *viz.*,

$$\exp(-K_A)H_g\exp(K_A)=H_g+H_{gA}+H_{gAA}+\cdots, \tag{3}$$

$$H_{gA}=[H_g,K_A],\cdots \tag{4}$$

leads to new terms linear and of higher orders in $\mathbf{A}$. It is easily seen that, as a consequence of charge conservation, a longitudinal $\mathbf{A}$ gives no contribution to (3) if H_g depends on the electron coordinates through the density operator $\psi^*\psi$ only. This will be assumed. Hence, the transformed (total) Hamiltonian is "manifestly gauge invariant", and so is any calculation based on it. In particular, the quantity to be investigated, in view of the London equation, is the expectation value of the current density $\mathbf{j}=-V^{-1}\delta H/\delta\mathbf{A}=\mathbf{j}_0+\mathbf{j}_A$. After the transformation (2), the current operator is

$$\mathbf{j}'=\exp(-K_A)\mathbf{j}\exp(K_A)=\mathbf{j}_0+\mathbf{j}_A'. \tag{5}$$

Here, $\mathbf{j}_A'$ vanishes for a longitudinal $\mathbf{A}$, and reduces to the diamagnetic current ($\nabla\times\chi\mathbf{B}$) for the unperturbed ($g=0$) ground state.

Terms of the order $g^2\mathbf{A}$ in $\langle\mathbf{j}\rangle$ are the lowest which can possibly describe a Meissner effect. Their calculation is facilitated by two further unitary transformations (note that all operators K are anti-Hermitian). To

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[1] Bardeen, Cooper, and Schrieffer, Phys. Rev. **108**, 1175 (1957).
[2] M. J. Buckingham, Nuovo cimento **5**, 1763 (1957); M. R. Schafroth, Phys. Rev. **111**, 72 (1958).
[3] J. Bardeen, Nuovo cimento **5**, 1765 (1957).
[4] P. W. Anderson (to be published).
[5] N. N. Bogoliubov, Nuovo cimento **7**, 794 (1958). See also J. G. Valatin, Nuovo cimento **7**, 843 (1958).
[6] M. R. Schafroth, Helv. Phys. Acta **24**, 645 (1951).

eliminate H_{gA} in (3), let

$$H''=\exp(-K_{gA})H'\exp(K_{gA}),\quad [H_0,K_{gA}]=-H_{gA}, \tag{6}$$

$$H''=H_0+H_g+H_{ggA}+\text{terms}\propto \mathbf{A}^2,\quad H_{ggA}=[H_g,K_{gA}]. \tag{7}$$

Again, H_{ggA} is eliminated by

$$H'''=\exp(-K_{ggA})H''\exp(K_{ggA}),\quad [H_0,K_{ggA}]=-H_{ggA}, \tag{8}$$

$$H'''=H_0+H_g+\text{terms}\propto g^3\mathbf{A} \text{ and } \propto \mathbf{A}^2. \tag{9}$$

Thus, in the approximation we need, H''' is the field-free Hamiltonian [i.e., (1) with $\mathbf{A}=0$], making immediate application of Bogoliubov's method possible (Sec. 3). The current density operator becomes in this representation:

$$\mathbf{j}'''=\mathbf{j}_0+\mathbf{j}_A'+\mathbf{j}_{gA}+\mathbf{j}_{ggA}+\cdots,\quad \mathbf{j}_{gA}=[\mathbf{j}_0,K_{gA}],\quad \mathbf{j}_{ggA}=[\mathbf{j}_0,K_{ggA}]. \tag{10}$$

As to $\mathbf{j}_{ggA}$, it will be sufficient to compute its expectation value for the unperturbed Bogoliubov ground state, whereas in $\mathbf{j}_{gA}$ the admixtures of order g will contribute. It will be shown that, compared with ordinary perturbation theory, new terms appear (in the case of $\mathbf{j}_{ggA}$ boosted by small energy denominators) which have the nature of supercurrents satisfying the London equation.

Thus, we confirm Bardeen's contention that his model shows a Meissner effect, although, in quantitative detail, our results are necessarily different.†

2. FREE-PARTICLE REPRESENTATION

Following Bogoliubov, we adopt the well-known Bloch-Fröhlich Hamiltonian. Using the symbol a_{ks} for the electron absorption operator (momentum $\mathbf{k}$, spin state s) and b_p correspondingly for phonon absorption:

$$H_0=\sum_{ks}E(k)a_{ks}^*a_{ks}+\sum_p \omega(p)b_p^*b_p, \tag{11}$$

$$H_g=g(2V)^{-\frac{1}{2}}\sum_{pkk's}\delta(\mathbf{k}-\mathbf{k}'-\mathbf{p})\times[\omega(p)]^{\frac{1}{2}}(b_p+b_p^*)a_{ks}^*a_{k's}, \tag{12}$$

where $E(k)=k^2/2m$; $\delta(\mathbf{q})=0$ for $\mathbf{q}\neq 0$, $\delta(\mathbf{0})=1$.

† *Note added in proof.*—For the terms linear in $\mathbf{A}$, it makes no difference if the succession of unitary transformations (2), (6), (8), $\cdots$, is replaced by the single transformation

$$\bar{H}=\exp(-L)H\exp(L),$$

where

$$L=K_A+K_{gA}+K_{ggA}+\cdots$$

satisfying

$$[(H_0+H_g),L]=-H_A.$$

This choice of the transformation is, however, the appropriate one for the calculation of the terms quadratic in $\mathbf{A}$ of the Hamiltonian. Indeed, to this order,

$$\bar{H}=(H_0+H_g)+\tfrac{1}{2}[H_A,L]+H_{AA}+\cdots.$$

Then, using (16) and (19) together with (10), one obtains:

$$\bar{H}=(H_0+H_g)-\tfrac{1}{2}V\{(\mathbf{A}^*\cdot\mathbf{j}_A'+\mathbf{j}_{gA}+\mathbf{j}_{ggA}+\cdots)+\text{h.c.}\}.$$

This secures the proper relationship between the energy and the induced current.

For the vector potential, only one Fourier component will be considered which we write

$$\mathbf{A}\exp(i\mathbf{q}\cdot\mathbf{r})+\text{c.c.}$$

($\hbar=1$). Then, in the current density, only the corresponding Fourier component is relevant:

$$\mathbf{j}\exp(i\mathbf{q}\cdot\mathbf{r})+\text{H.c.},$$

where

$$\mathbf{j}=\mathbf{j}_0+\mathbf{j}_A, \tag{13}$$

$$\mathbf{j}_0=(e/2m)V^{-1}\sum_{kk's}\delta(\mathbf{k}-\mathbf{k}'+\mathbf{q})(\mathbf{k}+\mathbf{k}')a_{ks}^*a_{k's}, \tag{14}$$

$$\mathbf{j}_A=-(e^2/m)V^{-1}N\mathbf{A}+\cdots,\quad N=\sum_{ks}a_{ks}^*a_{ks}. \tag{15}$$

In $\mathbf{j}_A$ we have suppressed terms proportional to $A^*a_{ks}^*a_{k's}$, with $\mathbf{k}-\mathbf{k}'=-2\mathbf{q}$, which cannot contribute to the expectation value $\langle\mathbf{j}\rangle$ in our approximation. (The usual factor $1/c$ in $\mathbf{j}_A$ has been absorbed into $\mathbf{A}$.) We also ignore the (paramagnetic) spin-field interaction. In this notation,

$$H_A=-V(\mathbf{A}\cdot\mathbf{j}_0^*+\mathbf{A}^*\cdot\mathbf{j}_0). \tag{16}$$

The operator K_A in the transformation (2) is now readily constructed, using (11) and (16) with (14). Writing

$$E(k)-E(k')=(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{k}+\mathbf{k}')/2m$$

for the energy denominators, we have

$$K_A=e\sum_{kk's}\delta(\mathbf{k}-\mathbf{k}'-\mathbf{q})\times\left(\frac{\mathbf{A}\cdot(\mathbf{k}+\mathbf{k}')}{\mathbf{q}\cdot(\mathbf{k}+\mathbf{k}')}\right)a_{ks}^*a_{k's}-\text{H.c.} \tag{17}$$

In the transformed current operator (5), we meet with the commutator

$$[\mathbf{j}_0,K_A]=\frac{e^2}{2m}V^{-1}\sum_{kk's}\delta(\mathbf{k}-\mathbf{k}'-\mathbf{q})\left(\frac{\mathbf{A}\cdot(\mathbf{k}+\mathbf{k}')}{\mathbf{q}\cdot(\mathbf{k}+\mathbf{k}')}\right)\times(\mathbf{k}+\mathbf{k}')(a_{k's}^*a_{k's}-a_{ks}^*a_{ks})+\cdots,$$

the dots indicating terms proportional to A^* which can be omitted for the same reason as those in $\mathbf{j}_A$ (15). More conveniently, one writes

$$[\mathbf{j}_0,K_A]=\frac{e^2}{m}V^{-1}\sum_{ks}\left[\left(\frac{\mathbf{A}\cdot(\mathbf{k}+\frac{1}{2}\mathbf{q})}{\mathbf{q}\cdot(\mathbf{k}+\frac{1}{2}\mathbf{q})}\right)(\mathbf{k}+\tfrac{1}{2}\mathbf{q})-\left(\frac{\mathbf{A}\cdot(\mathbf{k}-\frac{1}{2}\mathbf{q})}{\mathbf{q}\cdot(\mathbf{k}-\frac{1}{2}\mathbf{q})}\right)(\mathbf{k}-\tfrac{1}{2}\mathbf{q})\right]a_{ks}^*a_{ks}+\cdots. \tag{18}$$

Note that $\mathbf{j}_A'$, as defined in (5), to first order in $\mathbf{A}$, is given by

$$\mathbf{j}_A'=\mathbf{j}_A+[\mathbf{j}_0,K_A]. \tag{19}$$

For a longitudinal vector potential, *viz.*, $\mathbf{A}=\alpha\mathbf{q}$, the two terms in (19) cancel precisely (including the $\cdots$ terms),

$\mathbf{j}_A'=0$. For a transverse $\mathbf{A}$, we shall choose

$$\mathbf{q}=(0,0,q),\quad \mathbf{A}=(A,0,0),\quad \mathbf{j}=(j,0,0); \tag{20}$$

$$j_A'=A\frac{e^2}{m}V^{-1}\left\{-N+\frac{1}{q}\sum_{ks}k_x^2 \times\left(\frac{1}{k_z+\frac{1}{2}q}-\frac{1}{k_z-\frac{1}{2}q}\right)a_{ks}^*a_{ks}\right\}+\cdots. \tag{21}$$

For the unperturbed ($g=0$) ground state, this must reduce to Landau's diamagnetic current. This is verified easily by using[7]

$$\int_{|\mathbf{k}|<k_F} d^3k\, k_x^2\left(\frac{1}{k_z+\frac{1}{2}q}-\frac{1}{k_z-\frac{1}{2}q}\right)$$
$$=\tfrac{1}{4}\pi\int_{-k_F}^{k_F}dk_z(k_F^2-k_z^2)^2\left(\frac{1}{k_z+\frac{1}{2}q}-\frac{1}{k_z-\frac{1}{2}q}\right)$$
$$=\tfrac{1}{2}\pi\left[(k_F^2-\tfrac{1}{4}q^2)^2\ln\left|\frac{k_F+\frac{1}{2}q}{k_F-\frac{1}{2}q}\right|+(5/3)k_F^3q-\tfrac{1}{4}k_Fq^3\right]$$
$$=\frac{4\pi}{3}k_F^3q\left(1-\frac{1}{4}\frac{q^2}{k_F^2}+\cdots\right)\quad\text{for}\quad |q|\ll k_F. \tag{22}$$

(Taking principal values in the k_z integrals is justified, here and later.)

Next, we compute H_{gA}, defined in (3) and (4), using (12) and (17):

$$H_{gA}=eg(2V)^{-\frac{1}{2}}\sum_{pkk's}\delta(\mathbf{k}-\mathbf{k}'-\mathbf{p}-\mathbf{q})[\omega(p)]^{\frac{1}{2}} \times\left(\frac{\mathbf{A}\cdot(\mathbf{k}'+\frac{1}{2}\mathbf{q})}{\mathbf{q}\cdot(\mathbf{k}'+\frac{1}{2}\mathbf{q})}-\frac{\mathbf{A}\cdot(\mathbf{k}-\frac{1}{2}\mathbf{q})}{\mathbf{q}\cdot(\mathbf{k}-\frac{1}{2}\mathbf{q})}\right) \times(b_p+b_{-p}^*)a_{ks}^*a_{k's}+\text{H.c.} \tag{23}$$

For a longitudinal $\mathbf{A}$, H_{gA} vanishes identically, as expected, and the same is true for the operators K_{gA}, H_{ggA}, K_{ggA}, defined in (6), (7), (8). Without loss of generality, we can from now on restrict ourselves to the transverse gauge (20).

Dividing each term in (23) by the appropriate energy denominator, we find

$$K_{gA}=eAg(2V)^{-\frac{1}{2}}\sum_{pkk's}\delta(\mathbf{k}-\mathbf{k}'-\mathbf{p}-\mathbf{q})[\omega(p)]^{\frac{1}{2}} \times(X_{pkk'}b_p+Y_{pkk'}b_{-p}^*)a_{ks}^*a_{k's}-\text{H.c.}, \tag{24}$$

$$X_{pkk'}=\frac{1}{q}\left(\frac{k_x'}{k_z'+\frac{1}{2}q}-\frac{k_x}{k_z-\frac{1}{2}q}\right)\frac{1}{\omega(p)+E(k')-E(k)},$$
$$Y_{pkk'}=\frac{1}{q}\left(\frac{k_x'}{k_z'+\frac{1}{2}q}-\frac{k_x}{k_z-\frac{1}{2}q}\right)\frac{1}{-\omega(p)+E(k')-E(k)}. \tag{25}$$

[7] For a slightly different derivation, see G. Wentzel, Phys. Rev. **108**, 1593 (1957), Sec. 4.

With this result, we can now compute H_{ggA} and K_{ggA}, according to (7) and (8). K_{ggA} will be used only in application to "no phonon" states. Therefore only the terms involving products $b_pb_p^*$ are of interest, and, in these, $b_pb_p^*$ can be replaced by **1**:

$$K_{ggA}=eAg^2(2V)^{-1}\sum_{pkk'sll's'}\frac{\delta(\mathbf{l}-\mathbf{l}'+\mathbf{p})\delta(\mathbf{k}-\mathbf{k}'-\mathbf{p}-\mathbf{q})}{E(l')+E(k')-E(l)-E(k)} \times\omega(p)[Y_{pkk'}a_{ls'}^*a_{l's'}a_{ks}^*a_{k's} -X_{pkk'}a_{ks}^*a_{k's}a_{ls'}^*a_{l's'}]+\cdots. \tag{26}$$

Finally, inserting (14), (24), and (26) into (10), we obtain

$$j_{gA}=(e^2/m)Ag(2V)^{-\frac{1}{2}}V^{-1} \times\sum_{pkk's}\delta(\mathbf{k}-\mathbf{k}'-\mathbf{p}-\mathbf{q})[\omega(p)]^{\frac{1}{2}} \times(X_{pkk'}b_p+Y_{pkk'}b_{-p}^*) \times(k_xa_{k-qs}^*a_{k's}-k_x'a_{ks}^*a_{k'+qs})+\cdots, \tag{27}$$

$$j_{ggA}=\frac{e^2}{2m}Ag^2V^{-2}$$
$$\times\sum_{pkk'sll's'}\frac{\delta(\mathbf{l}-\mathbf{l}'+\mathbf{p})\delta(\mathbf{k}-\mathbf{k}'-\mathbf{p}-\mathbf{q})}{E(l')+E(k')-E(l)-E(k)}\omega(p)$$
$$\times[Y_{pkk'}(l_xa_{l-qs'}^*a_{l's'}-l_x'a_{ls'}^*a_{l'+qs'})a_{ks}^*a_{k's}$$
$$+Y_{pkk'}a_{ls'}^*a_{l's'}(k_xa_{k-qs}^*a_{k's}-k_x'a_{ks}^*a_{k'+qs})$$
$$-X_{pkk'}(k_xa_{k-qs}^*a_{k's}-k_x'a_{ks}^*a_{k'+qs})a_{ls'}^*a_{l's'}$$
$$-X_{pkk'}a_{ks}^*a_{k's}(l_xa_{l-qs'}^*a_{l's'}-l_x'a_{ls'}^*a_{l'+qs'})]$$
$$+\cdots. \tag{28}$$

3. BOGOLIUBOV'S TRANSFORMATION

Let

$$\alpha_{k0}=u_ka_{k,\frac{1}{2}}-v_ka_{-k,-\frac{1}{2}}^*,$$
$$\alpha_{k1}=u_ka_{-k,-\frac{1}{2}}+v_ka_{k,\frac{1}{2}}^*, \tag{29}$$

(u_k,v_k real, $u_k^2+v_k^2=1$). The α_{ki}, α_{ki}^* obey the same (Fermion) commutation relations as the a_{ks}, a_{ks}^*, and we can go over to a representation where the "occupation numbers" $\alpha_{ki}^*\alpha_{ki}$ (eigenvalues 0 and 1) are diagonal. For instance (assuming $u_{-k}=u_k$, $v_{-k}=v_k$):

$$\sum_sa_{ks}^*a_{k's}=2v_k^2\delta(\mathbf{k}-\mathbf{k}')+u_ku_{k'}(\alpha_{k0}^*\alpha_{k'0}+\alpha_{-k1}^*\alpha_{-k'1})$$
$$-v_kv_{k'}(\alpha_{k'1}^*\alpha_{k1}+\alpha_{-k'0}^*\alpha_{-k0})$$
$$+u_kv_{k'}(\alpha_{k0}^*\alpha_{k'1}^*+\alpha_{-k'0}^*\alpha_{-k1}^*)$$
$$+v_ku_{k'}(\alpha_{k1}\alpha_{k'0}+\alpha_{-k'1}\alpha_{-k0}). \tag{30}$$

The coefficients u_k,v_k, are determined by the condition

$$(k|H_0-E_FN|0) +\sum_n(k|H_g|n)(E_0^0-E_n^0)^{-1}(n|H_g|0)=0, \tag{31}$$

where $|0)$ = unperturbed ground state $[\alpha_{ki}|0)=0]$;

$|k)=$ state containing one "pair" of total momentum 0 and no phonon $[|k)=\alpha_{k0}{}^*\alpha_{k1}{}^*|0)]$; $|n)=$ intermediate states containing two "particles" and one phonon. Bogoliubov's justification for imposing the condition (31) is that, if the matrix element describing pair creation from the unperturbed ground state did not vanish, higher order perturbations would become dangerously large, owing to small energy denominators [the "dangerous" region is the vicinity of the Fermi surface: $E(k)=E_F$]. Writing out Eq. (31), by inserting (30) in H_0, N, and H_g, one obtains an integral equation for u_k, v_k, which in the case of weak coupling (g small, see below) can be readily solved in closed form, with the result

$$u_k^2=\tfrac{1}{2}[1+\xi(c^2+\xi^2)^{-\frac{1}{2}}],\quad v_k^2=\tfrac{1}{2}[1-\xi(c^2+\xi^2)^{-\frac{1}{2}}], \tag{32}$$

where

$$\begin{aligned}\xi&=E(k)-E_F,\quad c=\bar{\omega}e^{-1/\rho},\\ \rho&=g^2(dn/dE)\equiv g^2(mk_F/2\pi^2).\end{aligned} \tag{33}$$

(v_k^2 is identical with the function called h_k by BCS.[1]) $\bar{\omega}$ is a certain mean phonon energy, and the condition for "weak coupling" is $\rho\ll 1$, implying that $c/\bar{\omega}$ is extremely small. Bogoliubov goes on to show that single "particle" excitations require an energy $\geqslant c$. Comparison with the BCS theory indicates that this "energy gap" c must be identified, in order of magnitude, with the critical temperature T_c (times the Boltzmann constant). For actual superconductors, this leads to values of the order $\frac{1}{4}$ for the coupling parameter ρ, small enough to expect valid results from the weak-coupling approximation (32), (33).

Returning to our specific problem, we first investigate the term j_{ggA} (28) in the current density operator (10) (x component). As remarked earlier, it is sufficient in our g^2A approximation to compute its expectation value for the *unperturbed* Bogoliubov ground state, *viz.*, $(0|j_{ggA}|0)$. From (30) one finds easily, assuming $\mathbf{k}\neq\mathbf{k}'$ and $\mathbf{l}\neq\mathbf{l}'$ [this applies to all not trivially vanishing terms in (28)]:

$$\begin{aligned}(0|\textstyle\sum_{s'} a_{ls'}{}^*a_{l's'}\sum_s a_{ks}{}^*a_{k's}|0)&=2u_k^2v_{k'}^2\delta(\mathbf{l}'-\mathbf{k})\delta(\mathbf{l}-\mathbf{k}')\\ &\quad+2u_kv_ku_{k'}v_{k'}\delta(\mathbf{l}+\mathbf{k})\delta(\mathbf{l}'+\mathbf{k}').\end{aligned} \tag{34}$$

At this point, it is helpful to make a comparison with "ordinary perturbation theory", which would result by interpreting $|0)$ in (34) as the ground state of the *free* ($g=0$) Fermi gas. This is equivalent to letting $c\to 0$ in (34), namely, accordingly to (32):

$$\begin{aligned}v_k^2\to N_k^0&\equiv 1\quad\text{for}\quad \xi<0,\quad\text{or}\quad |\mathbf{k}|<k_F;\\ &\equiv 0\quad\text{for}\quad \xi>0,\quad\text{or}\quad |\mathbf{k}|>k_F;\\ u_k^2v_{k'}^2&\to(1-N_k^0)N_{k'}^0;\ u_kv_ku_{k'}v_{k'}\to 0.\end{aligned}$$

In particular, the terms $\propto\delta(\mathbf{l}+\mathbf{k})\delta(\mathbf{l}'+\mathbf{k}')$ in (34), which originate in an "exchange" among coherent intermediate pair states $[\alpha_{k0}{}^*\alpha_{k'1}{}^*|0)$ and $\alpha_{-k'0}{}^*\alpha_{-k1}{}^*|0)$, see (30)], vanish in the limit $c\to 0$, that is to say, they are absent in the ordinary perturbation theory. We claim that just these "exchange terms", when inserted into $(0|j_{ggA}|0)$, owing to the small energy denominators in (28), give rise to a supercurrent.

To prove this, we write

$$(0|j_{ggA}|0)=j^{\text{ord}}+j^{\text{exch}}, \tag{35}$$

the two terms corresponding to the two terms in (34). One sees easily that j^{ord} is practically the same whether c is strictly zero or only small compared to $\bar{\omega}$; the relative change is of the order $(c/\bar{\omega})^2=e^{-2/\rho}$, and terms of this order will be neglected. On the other hand,

$$\begin{aligned}j^{\text{exch}}=&(e^2/m)Ag^2V^{-2}\textstyle\sum_{pll'}\delta(\mathbf{l}-\mathbf{l}'+\mathbf{p})\omega(p)\\ &\times\Bigg[\frac{l_x(Y-X)_{p,-l+q,-l'}}{2E(l')-E(l)-E(l-q)}(uv)_{l'}\{(uv)_{l-q}-(uv)_l\}\\ &-\frac{l_x'(Y-X)_{p,-l,-l'-q}}{E(l')+E(l'+q)-2E(l)}(uv)_l\{(uv)_{l'+q}-(uv)_{l'}\}\Bigg].\end{aligned} \tag{36}$$

($l-q$, of course, refers to the vector difference, $\mathbf{l}-\mathbf{q}$, etc.) We note

$$\begin{aligned}\omega(p)(Y-X)_{p,-l,-l'}&\\ &=\frac{2}{q}\left(\frac{l_x}{l_z+\frac{1}{2}q}-\frac{l_x'}{l_z'-\frac{1}{2}q}\right)\frac{\omega(p)^2}{\omega(p)^2-(\xi-\xi')^2},\end{aligned} \tag{37}$$

$$(uv)_l=\tfrac{1}{2}c(c^2+\xi^2)^{-\frac{1}{2}}\equiv w_\xi, \tag{38}$$

where

$$\xi=E(l)-E_F,\quad \xi'=E(l')-E_F.$$

According to Schafroth's criterion[6] we have to examine (36) in the limit $q\to 0$. Expanding in the curly brackets, e.g.,

$$(uv)_{l-q}-(uv)_l=-[q(l_z-\tfrac{1}{2}q)/m]dw_\xi/d\xi+O(q^3), \tag{39}$$

where $q(l_z-\frac{1}{2}q)$ cancels with the denominator in $(Y-X)$ [the terms $\propto l_xl_x'$ vanish for symmetry reasons, whereas $l_x^2\to\frac{1}{3}k_F^2$], we can let $q\to 0$ in all remaining factors. Since only the vicinity of the Fermi surface will contribute $[(\xi-\xi')^2\lesssim c^2\ll\bar{\omega}^2]$, the last factor in (37) can be replaced by 1. Then,

$$\lim_{q=0}j^{\text{exch}}=-(e/m)^2Ag^2(\tfrac{1}{3}k_F^2)(mk_F/2\pi)^2I, \tag{40}$$

$$I=\int d\xi\int d\xi'(\xi'-\xi)^{-1}[w_{\xi'}dw_\xi/d\xi-w_\xi dw_{\xi'}/d\xi']. \tag{41}$$

With (38), the "small denominator" $(\xi'-\xi)$ cancels out:

$$\begin{aligned}I&=\frac{1}{4}\int d\xi\int d\xi'\,c^2(c^2-\xi\xi')(c^2+\xi^2)^{-\frac{3}{2}}(c^2+\xi'^2)^{-\frac{3}{2}}\\ &=1,\end{aligned}$$

(whereas, in "ordinary perturbation theory", $I\equiv 0$). Introducing Bogoliubov's parameter ρ [see (33)], we obtain finally

$$\lim_{q=0} j^{\text{exch}} = -(e^2/m)A\rho k_F^3/6\pi^2 \equiv \tfrac{1}{2}\rho j_A, \qquad (42)$$

where j_A [compare (15)] is the London, or BCS, value for the supercurrent. As will be shown, this term is not compensated by any other term in (10). Our theory, therefore, predicts a penetration depth for the magnetic fields in a superconductor which is larger by a factor $(2/\rho)^{\frac{1}{2}}$ than that predicted by the BCS theory.

The expansion (39) is valid only if

$$qv_F \ll c \ (v_F = k_F/m).$$

For larger q values, *viz.*,

$$c \ll qv_F \ll \bar{\omega},$$

we have estimated j^{exch} [it is then convenient to change variables such that $(uv)_l(uv)_{l'}$ appears as a common factor] and found that the value (42) is reduced by a factor $\sim c(v_Fq)^{-1}$. This $1/q$ dependence will determine the deviation from the London equation which amounts to an integral, or nonlocal, relationship between $\mathbf{j}(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$, as first proposed by Pippard.[8] (Discussion of this problem is left to a later paper.)

It remains to examine the terms j_{gA} and j_A' in (10). In j_{gA}, (27), the admixtures of order g in the Bogoliubov ground state must be taken into account; in other words, an expression of the form $(0|[j_{gA},K_g]|0)$ must be calculated, where $|0)$ refers again to the unperturbed ground state. Similar to (35), this can be split in "ordinary" and "exchange" terms. But, now, K_g carries an energy denominator $[\omega(p)+\cdots]$ which takes the place of $(\xi'-\xi)$ in (41), among other changes. One finds that the exchange terms resulting from $[j_{gA},K_g]$ are by a factor of the order $(e^{-1/\rho}/\rho)^2$ smaller than (42), and therefore of no interest. The "ordinary" term is practically the same as for $c=0$, and in this approximation just cancels the term j^{ord} in (35), as was to be expected from Schafroth's perturbation result, quoted earlier.[6] (The value of j^{ord}, for $q\rightarrow 0$, is $2\rho j_A$.)

Finally, we go back to Eq. (21) for j_A', where $\sum_s a_{ks}{}^*a_{ks}$ is given by (30), with $\mathbf{k}=\mathbf{k}'$, and admixtures up to order g^2 must be included. For $q\rightarrow 0$, however, this term can be shown to vanish to arbitrary order in g. It is only necessary to use the trivial fact that the ground-state expectation value $\langle\sum_s a_{ks}{}^*a_{ks}\rangle$ is isotropic, say $f(k^2)$. According to (21) and (15), a spherical shell in $\mathbf{k}$-space, δk, contributes to j_A' an amount proportional to

$$f(k^2)\int_{\delta k} d^3k\left\{-1+\frac{1}{q}k_x^2\left(\frac{1}{k_z+\frac{1}{2}q}-\frac{1}{k_z-\frac{1}{2}q}\right)\right\}.$$

But this vanishes as $q\rightarrow 0$; indeed, this follows from Eq. (22) by differentiation with respect to k_F.

Hence, j^{exch} (42) is the only supercurrent of any size, as long as the weak coupling approximation is valid. Extension of the perturbation theory to higher orders in g will lead to terms $\sim\rho^2 j_A$, and so on, but none of these can cancel the leading term (42).

At this point, we want to emphasize once more that a *longitudinal* vector potential cannot contribute to any of the j terms discussed. For instance, in Eq. (37), the numerators l_x and l_x' would be replaced by $l_z+\frac{1}{2}q$ and $l_z'-\frac{1}{2}q$, respectively, making the whole expression "manifestly" zero.‡

4. CONCLUDING REMARKS

The foregoing mathematical study reveals which features of the BCS-Bogoliubov model are essential for the appearance of a Meissner effect. Of pre-eminent importance is the dissolution of the sharp Fermi surface into a zone of width $\sim c/v_F$ in $\mathbf{k}$ space where neither u_k nor v_k is very small [$w_\xi\sim\frac{1}{2}$, see (38)] such that according to (29), in the ground state, electrons (or holes) having momenta $+\mathbf{k}$ and $-\mathbf{k}$ are strongly correlated. That such pair correlations must play a crucial role in the phenomena of superconductivity has been anticipated by several authors. (We quote the extensive discussion by Schafroth, Butler, and Blatt[9]; see also their list of references.) Of course, the specific nature of these correlations, as expressed, e.g., in the functions u_k, v_k (32), is also of importance. It depends on such details that the value of the factor I in (40) is 1, and not, say, $e^{-2/\rho}$.

The theory, as presented here, calls for generalization in various directions. No such attempt has been made so far. Needless to say, the oversimplification inherent in the BCS model will always leave some doubt whether a quantitative comparison with experimental data is meaningful. But irrespective of success or failure in numerical details, it is gratifying that, at last, a reasonable model and an adequate mathematical treatment have been found which seem to provide a qualitative understanding of the Meissner effect.

‡ *Note added in proof*.—It should also be noted that the momenta $\mathbf{l}$ or $\mathbf{k}$, e.g., as arguments of the functions u_k, v_k, have a gauge-invariant meaning in our final representation. This has enabled us to define the ground state in a gauge-invariant manner. The situation is very different if one introduces the Bogoljubov transformation (29) already in the original Hamiltonian [(1) with (11), (12), and (16)]. Then, it is the vector $\mathbf{k}-e\mathbf{A}=m\mathbf{v}$ which has the gauge-invariant meaning; a change of gauge alters the meaning of $\mathbf{k}$, and of u_k, v_k. In this approach, the main problem would be the gauge-invariant definition of the ground state. Simply choosing a particular gauge, like $\nabla\cdot\mathbf{A}=0$ [G. Rickayzen, Phys. Rev. (to be published)], does not solve the problem and, indeed, leads to a result at variance with ours.

[8] A. B. Pippard, Proc. Roy. Soc. (London) **A216**, 547 (1953).

[9] Schafroth, Butler, and Blatt, Helv. Phys. Acta **30**, 93 (1957).