Majorana Formula

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The Majorana formula for the transition probability of a general spin is here rederived and brought into a more compact form. The method is based on the use of the projection operators which characterize the spin vector as definitely quantized along a specified unit vector. If the Hamiltonian contains only terms linear in the spin, it is possible to define a moving instantaneous axis along which the moving spin vector maintains its quantization. The Majorana problem is reduced to the calculation of the joint probability of quantization along two different axes.

The usual expression of the Majorana formula contains a factor which is the transition probability of a spin-$\frac{1}{2}$ vector. It is shown here how this factor is related to the angle through which the general spin is turned.

THERE is a formula, originally derived by Majorana, which gives the probability of a spin transition from a state of magnetic quantum number m to one of magnetic quantum number m'.[1] The spin is in a uniform magnetic field, whose direction is taken as the axis of quantization, and a perpendicular rf field is applied to cause transitions. The existing derivations of the formula are based on the consideration that an angular momentum of value s can be handled as the resultant of $2s$ angular momenta of value $\frac{1}{2}$. The spin-$\frac{1}{2}$ case is easily solved; the general case is solved by synthesis to yield the formula for the transition probability for state m to m':

$$P_{m,m'}=(s-m)!(s+m)!(s-m')!(s+m')!(\sin\tfrac{1}{2}\alpha)^{4s}\times\left[\sum_r\frac{(-1)^r(\cot\frac{1}{2}\alpha)^{m+m'+2r}}{(s-m-r)!(s-m'-r)!(m+m'+r)!r!}\right]^2. \quad (1)$$

The transition probability for a moment with the same gyromagnetic ratio and spin $\frac{1}{2}$ is

$$P_{\frac{1}{2},-\frac{1}{2}}=\sin^2(\tfrac{1}{2}\alpha). \quad (2)$$

This note offers another derivation of the Majorana formula. The derivation does not compound the general case out of the spin-$\frac{1}{2}$ case but directly follows the dynamics of the spin vector. The formula, as derived by this method, is much more concise than Eq. (1) and reveals an interesting physical interpretation of the angle α.

METHOD

First, the following problem is posed: given a spin definitely quantized along a unit vector $\mathbf{a}$ with component m, what is the probability that it is quantized along a unit vector $\mathbf{b}$ with component m'? The answer is[2]

$$\sum_{n=0}^{p-1}p_n(m)p_n(m')P_n(\mathbf{a}\cdot\mathbf{b}), \quad (3)$$

where $p=2s+1$, and $P_n(\mathbf{a}\cdot\mathbf{b})$ is the ordinary Legendre polynomial of $\mathbf{a}\cdot\mathbf{b}$, normalized so that $P_n(1)=1$. $p_n(m)$ is a certain polynomial in m, originally investigated by Tchebichef,[3] further described in the appendix. The important properties to be noted here are the following:

$$p_n(m)=0,\quad n\geq p \quad (4)$$

$$\sum_{n=0}^{p-1}p_n(m)p_n(m')=\delta_{m,m'}, \quad (5)$$

$$\sum_{m=-s}^{s}p_n(m)p_{n'}(m)=\delta_{n,n'}. \quad (6)$$

The first few polynomials are

$$p_0=(p)^{-\frac{1}{2}},$$
$$p_1=2(3)^{\frac{1}{2}}[p(p^2-1)]^{-\frac{1}{2}}m,$$
$$p_2=\tfrac{1}{2}(5)^{\frac{1}{2}}[p(p^2-1)(p^2-4)]^{-\frac{1}{2}}[12m^2-p^2+1],$$
$$p_3=(7)^{\frac{1}{2}}[p(p^2-1)(p^2-4)(p^2-9)]^{-\frac{1}{2}}[20m^2-3p^2+7]m.$$

Also,

$$p_n(s)=(p-1)!(2n+1)^{\frac{1}{2}}[(p+n)!(p-n-1)!]^{-\frac{1}{2}}.$$

Equation (3) was derived by the use of projection operators. If $\rho_m(\mathbf{a}\cdot\mathbf{S})$ is the projection operator for the m state along $\mathbf{a}$, and $\rho_{m'}(\mathbf{b}\cdot\mathbf{S})$ is the projection operator for the m' state along $\mathbf{b}$, then the evaluation of

$$\mathrm{Trace}\rho_m(\mathbf{a}\cdot\mathbf{S})\rho_{m'}(\mathbf{b}\cdot\mathbf{S})$$

yields Eq. (3). The trace expression is a direct transcription of the posed question.

Now, given the motion of a spin under the influence of any Hamiltonian, the probability of observing the state m along $\mathbf{a}$ is given by

$$\mathrm{Trace}\rho_m(\mathbf{a}\cdot\mathbf{S})\rho,$$

where ρ is the projection operator for the particular state of motion. For a general Hamiltonian, it may not be possible to characterize the states by spin component quantization, but in the case to be considered here, it is

[1] The history of the formula and a derivation are given in N. F. Ramsey, *Molecular Beams* (Oxford University Press, Oxford, 1956).
[2] A. Meckler (unpublished).
[3] G. Szegö, *Orthogonal Polynomials* (American Mathematical Society, New York, 1939), p. 32.

possible to define an instantaneous axis, a moving axis of quantization relative to which the moving spin vector maintains its alignment. Once the equation of motion of the instantaneous axis is known, Eq. (3) can be used to give the probability of m along $\mathbf{a}$ at any time. Here, $\mathbf{b}$ will be the moving instantaneous axis. If the initial condition is that $\mathbf{b}$ at time $t=0$ is along $\mathbf{a}$, then Eq. (3) must contain the Majorana formula.

DERIVATION

The Hamiltonian is

$$\mathcal{H}=-\omega_0\mathbf{a}\cdot\mathbf{S}-\lambda\mathbf{h}\cdot\mathbf{S}, \tag{7}$$

where h_μ is a unit vector:

$$h_\mu=(2)^{-\frac{1}{2}}(a_\mu{}^+e^{-i\omega t}+a_\mu{}^-e^{i\omega t}). \tag{8}$$

$a_\mu{}^+$ and $a_\mu{}^-$ are complex vectors perpendicular to a_μ such that

$$\begin{aligned} a_\mu{}^+a_\mu&=a_\mu{}^-a_\mu=0,\\ a_\mu{}^+a_\mu{}^-&=1,\\ \epsilon_{\sigma\mu\nu}a_\mu{}^+a_\nu&=ia_\sigma{}^+,\\ \epsilon_{\sigma\mu\nu}a_\mu{}^-a_\nu&=-ia_\sigma{}^-,\\ \epsilon_{\sigma\mu\nu}a_\mu{}^+a_\nu{}^-&=-ia_\sigma. \end{aligned}$$

Here, the summation convention is used, and $\epsilon_{\sigma\mu\nu}$ is the antisymmetrical unit pseudotensor.[4] It is antisymmetric in all its indices and $\epsilon_{123}=1$.

The instantaneous axis, b_μ, is defined by the condition

$$(d/dt)(\mathbf{b}\cdot\mathbf{S})=0, \tag{9}$$

or

$$\dot{b}_\sigma S_\sigma=-b_\sigma\dot{S}_\sigma. \tag{10}$$

The quantum rule is[5]

$$\dot{S}_\sigma=i(\mathcal{H},S_\sigma).$$

Application of the spin commutation rule allows the solution of Eq. (10) as

$$\dot{b}_\sigma=-\epsilon_{\sigma\mu\nu}b_\mu(\omega_0a_\nu+\lambda h_\nu). \tag{11}$$

It is desirable to know $\mathbf{b}$ in terms of $\mathbf{a}$, $\mathbf{a}^+$, and $\mathbf{a}^-$. Set

$$b_\sigma=Z(t)a_\sigma+Z^+(t)e^{i\omega t}a_\sigma{}^-+Z^-(t)e^{-i\omega t}a_\sigma{}^+, \tag{12}$$

where normalization and reality demand that

$$Z^2+2Z^+Z^-=1,\quad Z^+=(Z^-)^*.$$

The equations for the Z's are

$$\begin{aligned} \dot{Z}&=-(i\lambda/\sqrt{2})(Z^+-Z^-),\\ \dot{Z}^+&=i(\omega_0-\omega)Z^+-(i\lambda/\sqrt{2})Z,\\ \dot{Z}^-&=-i(\omega_0-\omega)Z^-+(i\lambda/\sqrt{2})Z. \end{aligned} \tag{13}$$

[4] J. L. Synge and A. Schild, *Tensor Calculus* (University of Toronto Press, Toronto, 1952).
[5] $\hbar$ is set equal to unity.

These equations can be made to look like a matrical Schrödinger equation by the introduction of the column vector

$$\Psi=\begin{bmatrix} Z\\ Z^+\\ Z^-\end{bmatrix}, \tag{14}$$

which obeys the equation

$$\mathbf{H}\Psi=i\partial\Psi/\partial t, \tag{15}$$

where the matrix $\mathbf{H}$ is

$$\mathbf{H}=\begin{bmatrix} 0 & \lambda/\sqrt{2} & -\lambda/\sqrt{2}\\ \lambda/\sqrt{2} & (\omega-\omega_0) & 0\\ -\lambda/\sqrt{2} & 0 & -(\omega-\omega_0)\end{bmatrix}. \tag{16}$$

$\mathbf{H}$ is Hermitian and so the normalization of Ψ is preserved in time. A particular solution of Eq. (15) is

$$\Psi=\psi e^{-iWt}, \tag{17}$$

with

$$\mathbf{H}\psi=W\psi. \tag{18}$$

The possible values of W are the eigenvalues of $\mathbf{H}$, which are

$$W=0,\ [\lambda^2+(\omega-\omega_0)^2]^{\frac{1}{2}},\ -[\lambda^2+(\omega-\omega_0)^2]^{\frac{1}{2}}.$$

Henceforth, let $u=[\lambda^2+(\omega-\omega_0)^2]^{\frac{1}{2}}$. The general solution of Eq. (15) is

$$\Psi=\begin{bmatrix} -A(\omega-\omega_0)+2B\cos ut\\ \dfrac{A\lambda}{\sqrt{2}}+\dfrac{\sqrt{2}B}{\lambda}[(\omega-\omega_0)\cos ut-iu\sin ut]\\ \dfrac{A\lambda}{\sqrt{2}}+\dfrac{\sqrt{2}B}{\lambda}[(\omega-\omega_0)\cos ut+iu\sin ut]\end{bmatrix}. \tag{19}$$

A and B are to be determined by normalization and the initial condition. At $t=0$,

$$\Psi(0)=\begin{bmatrix} -A(\omega-\omega_0)+2B\\ \dfrac{A\lambda}{\sqrt{2}}+\dfrac{\sqrt{2}}{\lambda}(\omega-\omega_0)B\\ \dfrac{A\lambda}{\sqrt{2}}+\dfrac{\sqrt{2}}{\lambda}(\omega-\omega_0)B\end{bmatrix}. \tag{20}$$

The initial condition to be taken here is that

$$Z(0)=1,\quad Z^+(0)=Z^-(0)=0.$$

The equations for A and B are, therefore,

$$\begin{aligned} -A(\omega-\omega_0)+2B&=1,\\ A\lambda/\sqrt{2}+[\sqrt{2}(\omega-\omega_0)/\lambda]B&=0, \end{aligned}$$

so that

$$\begin{aligned} A&=-(\omega-\omega_0)/u^2,\\ B&=\lambda^2/(2u^2). \end{aligned} \tag{21}$$

b_μ is now completely determined as a function of time. The transition probability of flipping from state m along $\mathbf{a}$ at $t=0$ to state m' along $\mathbf{a}$ at t is, by Eq. (3):

$$P_{m,m'}=\sum_{n=0}^{p-1} p_n(m)p_n(m')P_n(\mathbf{a}\cdot\mathbf{b}),$$

or

$$P_{m,m'}=\sum_{n=0}^{p-1} p_n(m)p_n(m')P_n(Z),$$

where

$$Z=1-\frac{\lambda^2}{u^2}(1-\cos ut). \tag{22}$$

For the case of a spin-$\frac{1}{2}$ vector,

$$\begin{aligned}P_{\frac{1}{2},-\frac{1}{2}}&=p_0(\tfrac{1}{2})p_0(-\tfrac{1}{2})+p_1(\tfrac{1}{2})p_1(-\tfrac{1}{2})P_1(Z)\\&=\tfrac{1}{2}-\tfrac{1}{2}Z\\&=\tfrac{1}{2}(1-Z).\end{aligned}$$

In the Ramsey-Majorana notation, $P_{\frac{1}{2},-\frac{1}{2}}=\sin^2(\frac{1}{2}\alpha)$. The connection must be

$$Z=\cos\alpha. \tag{23}$$

The final, concise form of the Majorana formula can now be written as

$$P_{m,m'}=\sum_{n=0}^{p-1} p_n(m)p_n(m')P_n(\cos\alpha). \tag{24}$$

With this derivation, a physical and classical interpretation can be given to the angle α. It is the angle of inclination of the instantaneous axis to the direction of the uniform magnetic field.

APPENDIX

A recursion relation for the Tchebichef polynomials is

$$p_n(m)=mF_np_{n-1}-K_np_{n-2},$$

where

$$F_n{}^2=\frac{4(4n^2-1)}{n^2(p^2-n^2)}$$

and

$$K_n=F_n/F_{n-1}.$$

Set $m=-s+q$. Then

$$p_n(q)=n!\left[\frac{(2n+1)(p-n-1)!}{(p+n)!}\right]^{\frac{1}{2}}\Delta^n\binom{q}{n}\binom{q-p}{n},$$

where

$$\binom{q}{n}=\frac{q!}{n!(q-n)!},$$

$$\Delta f(x)=f(x+1)-f(x).$$

Subsequent to the preparation of this manuscript, Professor G. F. Koster expressed the suspicion that the Tchebichef polynomials, $p_n(s;m)$, are somehow equivalent to certain Clebsch-Gordan coefficients. Because of his insistence, it was proved, in fact, that

$$p_n(s;m)=(-1)^{s-m}C(s,s,n;m,-m),$$

where the C definition is taken from Rose.[6]

Added note.—The author has been made aware of a paper by Salwen[7] in which still another derivation and expression of the Majorana formula is given. The main concern of that paper is the case of unequally spaced magnetic levels, the anomalous Zeeman case, and an approximate solution to the time-dependent Schrödinger equation is presented. The normal Zeeman case is solved exactly, but the expression of the Majorana formula [Eq. (41) of Salwen], though more neat and convenient than the original one, is not the same as that of this paper. The rotating frame of Salwen is not that associated with an instantaneous axis.

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[6] M. E. Rose, *Multipole Fields* (John Wiley and Sons, Inc., New York, 1955), p. 13.
[7] H. Salwen, Phys. Rev. **99**, 1274 (1955).