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Structure of a Shock Wave in Fully Ionized Hydrogen*

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The Fokker-Planck equations are used to examine the structure of a shock wave in fully ionized hydrogen. This is done by assuming a bimodal Maxwellian distribution for the protons in the interior of the shock and noting that the electrons are in thermal equilibrium with themselves but not necessarily with the protons. The method is essentially an extension of that used by Mott-Smith in his analysis of the Boltzmann equation for a shock wave in a gas of neutral atoms.

1. INTRODUCTION

A CONSIDERABLE amount of work has been done over the past few years by various authors[1–4] making use of the Fokker-Planck equation for the study of plasmas. This equation provides an extension of the Boltzmann equation and determines the way in which the distribution function of the ions and electrons in the plasma changes. The long-range Coulomb interaction between these particles gives rise to a diffusion contribution on the scattering side of the Boltzmann equation.[5] These diffusion terms account for that part of the change in the distribution function due to the large number of distant encounters between particles.

The purpose of this paper is to make an analysis of the structure of a shock wave in a plasma using the Fokker-Planck equation. This is done by extending the method used by Mott-Smith[6] to analyze the structure of shock waves in ordinary gases. We consider the simple case of fully ionized hydrogen in the absence of a magnetic field, i.e., the shock is established by collisions between the particles. The results thus have applicability to shocks produced in the laboratory or to cases in astrophysics of shocks propagated along magnetic field lines. It may also be possible to extend this type of treatment to the more complicated case in which a magnetic field is present in the plasma.

The advantage of the Mott-Smith method is that, since its starting point is the distribution function for particles in the medium, it gives a description of shocks of high Mach number. In this region the continuum equations of ordinary hydrodynamics or of plasma dynamics which are based on the assumption of small deviations from equilibrium distributions no longer apply.

2. DISTRIBUTION FUNCTIONS

The Fokker-Planck equations[4] for the proton and electron distribution functions F and f become, when simplified for two-body Coulomb interactions,

$$\frac{\partial F}{\partial t}+\mathbf{c}\cdot\frac{\partial F}{\partial \mathbf{r}}+\frac{e\mathbf{E}}{M}\cdot\frac{\partial F}{\partial \mathbf{c}}=\left(\frac{\partial F}{\partial t}\right)_c, \tag{2.1}$$

$$\frac{\partial f}{\partial t}+\mathbf{c}\cdot\frac{\partial f}{\partial \mathbf{r}}-\frac{e\mathbf{E}}{m}\cdot\frac{\partial f}{\partial \mathbf{c}}=\left(\frac{\partial f}{\partial t}\right)_c, \tag{2.2}$$

where $\partial F/\partial\mathbf{a}$ denotes the vector $(\partial F/\partial a_x, \partial F/\partial a_y, \partial F/\partial a_z)$ and where the collision terms are given by

$$\frac{1}{\Gamma}\left(\frac{\partial F}{\partial t}\right)_c=-\frac{\partial}{\partial c_i}\left(F\frac{\partial H}{\partial c_i}\right)+\frac{1}{2}\frac{\partial}{\partial c_i}\frac{\partial}{\partial c_j}\left(F\frac{\partial}{\partial c_i}\frac{\partial}{\partial c_j}G\right),$$
$$\frac{1}{\gamma}\left(\frac{\partial f}{\partial t}\right)_c=-\frac{\partial}{\partial c_i}\left(f\frac{\partial h}{\partial c_i}\right)+\frac{1}{2}\frac{\partial}{\partial c_i}\frac{\partial}{\partial c_j}\left(f\frac{\partial}{\partial c_i}\frac{\partial}{\partial c_j}g\right), \tag{2.3}$$

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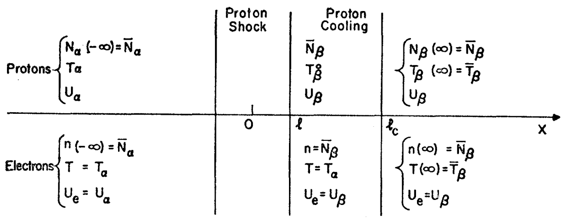

FIG. 1. The density, temperature, and stream velocity in various regions of the shock.

and

$$h=2\int d\mathbf{c}_1\frac{f(\mathbf{c}_1)}{|\mathbf{c}-\mathbf{c}_1|}+\left(\frac{m+M}{M}\right)\int d\mathbf{c}_1\frac{F(\mathbf{c}_1)}{|\mathbf{c}-\mathbf{c}_1|},\tag{2.4}$$

$$g=G=\int d\mathbf{c}_1(f+F)|\mathbf{c}-\mathbf{c}_1|,\tag{2.5}$$

$$H=\left(\frac{M+m}{m}\right)\int d\mathbf{c}_1\frac{f(\mathbf{c}_1)}{|\mathbf{c}-\mathbf{c}_1|}+2\int d\mathbf{c}_1\frac{F(\mathbf{c}_1)}{|\mathbf{c}-\mathbf{c}_1|}.\tag{2.6}$$

The slowly varying quantity Γ is

$$\Gamma=\frac{4\pi e^4}{M^2}\ln\left[\frac{3}{4(\pi n)^{\frac{1}{2}}}\left(\frac{kT}{e^2}\right)^{\frac{3}{2}}\right],\tag{2.7}$$

and γ is obtained by replacing M with m in (2.7). The notation used is the same as that used by Rosenbluth, MacDonald, and Judd.[4]

The terms on the right-hand side of Eqs. (2.1) and (2.2) account for the change in the distribution function due to the large number of distant collisions a particle (proton or electron) suffers with other particles in the medium. These collisions occur with particles inside a distance equal to the Debye radius about the "test particle." Outside the Debye radius the Coulomb field of a particle drops off exponentially due to local polarization of the medium. Thus charge fluctuations affecting the test particle involve a large number of particles outside this radius and therefore long characteristic periods. The medium outside the Debye radius can therefore be considered as a continuous charge distribution producing a field $\mathbf{E}$ at the test particle which is included on the left side of Eqs. (2.1) and (2.2). In a violent shock wave, we might expect to displace the electrons from the protons in the plasma in such a way as to give rise to a charge distribution across the shock. In fact this displacement is extremely small as will be seen later.

The comparatively rare close encounters between particles should in general be accounted for in (2.1) and (2.2) by adding the single scattering terms which occur in the ordinary Boltzmann equation. However, these make a less important contribution to the collision side of the equation and it is usual to neglect them.

The general solution of these equations for the case of an infinite plane shock wave is of course difficult to obtain. We adopt a procedure similar to that of Mott-Smith,[6] which is to choose a distribution function of a particularly simple form depending on our physical picture of the conditions in the interior of the shock and in its neighborhood. In order to construct such a picture somewhat in advance, we need to consider the various relaxation times of the system.

Consider a frame in which the shock is at rest (Fig. 1). Then in front of and behind the shock (at $x<0$ and $x>0$) we have plasma flowing along the positive x axis. At sufficiently large distances from the shock on either side, we have equilibrium distributions corresponding to the appropriate stream velocities and temperatures in these two regions. The proton and electron distributions will be both in equilibrium with themselves and with each other. Across the shock itself, however, we have a nonequilibrium region in which the collision terms of Eqs. (2.1) and (2.2) can be thought of as describing the way in which particles migrate from the one equilibrium state to the other in traversing the shock.

Using the suffix α for conditions ahead of the shock and β for behind the shock, we take for the proton distribution function,

$$\begin{aligned}F=&N_\alpha(x)\left(\frac{M}{2\pi kT_\alpha}\right)^{\frac{3}{2}}\exp\left[-\frac{M}{2kT_\alpha}(\mathbf{c}-\mathbf{i}U_\alpha)^2\right]\\&+N_\beta(x)\left(\frac{M}{2\pi kT_\beta}\right)^{\frac{3}{2}}\exp\left[-\frac{M}{2kT_\beta}(\mathbf{c}-\mathbf{i}U_\beta)^2\right]\\=&F_\alpha+F_\beta,\end{aligned}\tag{2.8}$$

where N_α and N_β are densities, T_α and T_β temperatures, and U_α and U_β stream velocities along the x direction $\mathbf{i}$. The quantities N_α and N_β are functions of x together with $T_\beta(x)$ for reasons discussed below. Thus the proton distribution inside the shock is represented by two interacting Maxwellian distributions. In this zero-order picture, the collision terms of the Fokker-Planck equation describe the way in which protons jump back and forth between the α and β distributions in crossing the shock.

Now consider the coupling between the electron gas and the proton gas in the plasma. This derives in our case from electron-proton collisions. As we shall see, over a large range of Mach numbers the proton shock thickness l is too small to allow the electrons to reach thermal equilibrium with the protons in this region, and in fact the proton shock can be considered approximately to involve both a momentum and an energy transfer from one proton stream to another which takes place as if the electrons were absent. Only somewhat farther behind the shock does the temperature T of the electron stream rise to that of the protons. This in turn cools the β proton stream behind the shock over a region of characteristic length $l_c>l$. This cooling takes place in such a way that the sum of the electron and proton pressures behind the shock remains constant; i.e., there are no further pressure gradients to change the stream velocity U_β in this region. It should be pointed out that for weak shocks, i.e. $K\simeq 1$, the proton shock thickness l becomes very large and the electrons are thus able to reach thermal equilibrium with the protons inside the shock, in which case there is no further cooling of the β stream and $T_\beta(x)=T_\beta(\infty)$.

Next consider the effect the protons have on the electrons. Due to the higher electron thermal velocity, the electrons reach equilibrium with themselves faster than the protons do with themselves,[3] and considerably faster than the electrons with the protons. These three times are in the ratio $1:(M/m)^{\frac{1}{2}}:(M/m)$. Thus we take for the electron distribution one of self-equilibrium, in which the interaction with the protons gives rise to the slow passage of the electron gas through a series of equilibrium states, i.e.,

$$f(x)=n(x)\left(\frac{m}{2\pi kT(x)}\right)^{\frac{3}{2}}\exp\left[-\frac{m}{2kT}(\mathbf{c}-\mathbf{i}U_e)^2\right], \quad (2.9)$$

where n, T, and U_e are all functions of x. It should be pointed out however that in making this assumption we are excluding effects of *thermal conduction* in the electron gas which are of course important for describing the variation of the electron temperature through the shock. Thus in the approximation of this paper we shall not obtain the rise in electron temperature extending *ahead* of the proton shock as was obtained in the work of Jukes using the Navier-Stokes equations.[7] Inclusion of this effect in the "strong shock" approach is an obvious next step.

Using the distribution functions (2.8) and (2.9), we take moments of the Fokker-Planck equation and apply the appropriate boundary conditions shown in Fig. 1. Quantities which depend on x have values at $x=\pm\infty$ which are written with a bar, e.g., $N_\alpha(-\infty)=\bar{N}_\alpha$, $N_\alpha(+\infty)=0$, etc.

Basically there are three important lengths to consider in this problem, the proton shock thickness l, the length l_c in which the electrons reach thermal equilibrium with the protons, and the characteristic length l_v over which any stream velocity differences between the protons and electrons are removed. These three lengths are plotted as a function of Mach number in Fig. 2.

We expect that $l_v \ll l_c$ or l which is the case since, although electron-proton collisions do not involve appreciable energy transfers, they do involve large-angle scattering of the electrons which destroys their stream velocity relative to that of the protons.

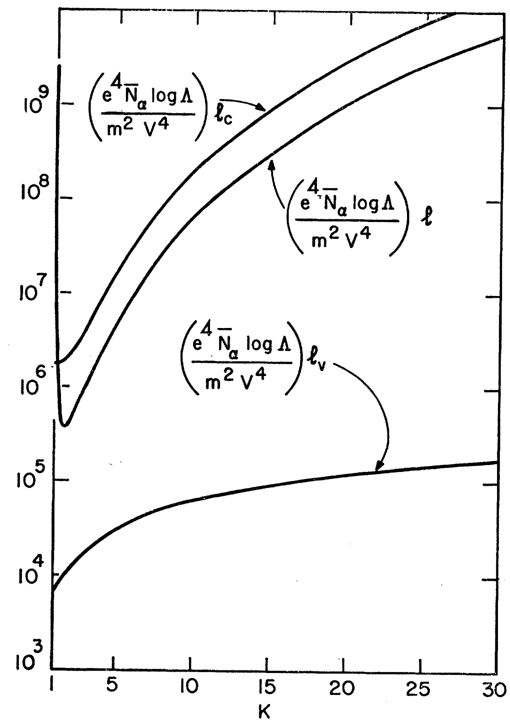

FIG. 2. Proton shock thickness l and cooling length l_c as a function of Mach number K.

3. MOMENTS OF THE FOKKER-PLANCK EQUATION

In the usual way, the first three moments obtained by multiplying (2.1) by 1, Mu, $\frac{1}{2}Mc^2$ and (2.2) by 1, mu, and $\frac{1}{2}mc^2$ and integrating over $\mathbf{c}$ yield the following equations for the flow of mass momentum and energy in the proton and electron streams, respectively:

$$\frac{\partial}{\partial x}[N_\alpha U_\alpha+N_\beta U_\beta]=0, \quad (3.1)$$

$$\frac{\partial}{\partial x}[N_\alpha(kT_\alpha+MU_\alpha^2)+N_\beta(kT_\beta+MU_\beta^2)] -e(N_\alpha+N_\beta)E(x)=\left(\frac{\partial P_i}{\partial t}\right)_c, \quad (3.2)$$

$$\frac{\partial}{\partial x}\left[MN_\alpha U_\alpha\left(U_\alpha^2+\frac{5kT_\alpha}{M}\right)+MN_\beta U_\beta\left(U_\beta^2+\frac{5kT_\beta}{M}\right)\right] -2e(N_\alpha U_\alpha+N_\beta U_\beta)E=\left(\frac{\partial Q_i}{\partial t}\right)_c, \quad (3.3)$$

$$\frac{\partial}{\partial x}[nU_e]=0, \quad (3.4)$$

$$\frac{\partial}{\partial x}[n(kT+mU_e^2)]+enE=\left(\frac{\partial P_e}{\partial t}\right)_c, \quad (3.5)$$

$$\frac{\partial}{\partial x}\left[mnU_e\left(U_e^2+\frac{5kT}{m}\right)\right]+2enU_eE=\left(\frac{\partial Q_e}{\partial t}\right)_c, \quad (3.6)$$

where

$$\left(\frac{\partial P_i}{\partial t}\right)_c=\int Mu\left(\frac{\partial F}{\partial t}\right)_c d\mathbf{c},$$

$$\left(\frac{\partial Q_i}{\partial t}\right)_c=\frac{1}{2}\int Mc^2\left(\frac{\partial F}{\partial t}\right)_c d\mathbf{c}.$$

[7] J. D. Jukes, J. Fluid Mech. 3, 275 (1957).

The conservation of total momentum and energy are then expressed by

$$\left(\frac{\partial P_i}{\partial t}\right)_c+\left(\frac{\partial P_e}{\partial t}\right)_c=0, \quad \left(\frac{\partial Q_i}{\partial t}\right)_c+\left(\frac{\partial Q_e}{\partial t}\right)_c=0. \tag{3.7}$$

The mass flow of the electrons and protons in the x direction is conserved independently. Thus using

$$\partial E/\partial x=4\pi e(N_\alpha+N_\beta-n), \tag{3.8}$$

together with the condition that there are no currents flowing along the x axis at $\pm\infty$ we find for the conservation equations,

$$N_\alpha U_\alpha+N_\beta U_\beta=nU_e, \tag{3.9}$$

$$N_\alpha(kT_\alpha+MU_\alpha^2)+N_\beta(kT_\beta+MU_\beta^2)+n(kT+mU_e^2)-(E^2/8\pi)=P, \tag{3.10}$$

$$MN_\alpha U_\alpha\left(U_\alpha^2+\frac{5kT_\alpha}{M}\right)+MN_\beta U_\beta\left(U_\beta^2+\frac{5kT_\beta}{M}\right)+mnU_e\left(U_e^2+\frac{5kT}{m}\right)=Q. \tag{3.11}$$

These equations enable us to obtain relations between the α and β quantities at $x=\pm\infty$ under our conditions of equilibrium $T(\infty)=T_\beta(\infty)$ etc., at these points, i.e., the Rankine-Hugoniot relations. They also apply for all x, i.e., in the interior of the shock. However in order to discuss the shock region where the various quantities depend on x we require further moment equations in which the collision terms appear explicitly. We shall use the transport equations for v^2 of the protons and electrons $[\mathbf{c}=(u,v,w)]$.

Multiplying (2.1) by v^2 and integrating the collision terms (2.3)–(2.6) over $\mathbf{c}$ by parts using $F(c=\pm\infty)=f(c=\pm\infty)=0$, we find

$$U_\alpha\left(\frac{kT_\alpha}{M}\right)\frac{\partial N_\alpha}{\partial x}+U_\beta\left(\frac{kT_\beta}{M}\right)\frac{\partial N_\beta}{\partial x}+\frac{U_\beta N_\beta k}{M}\frac{\partial T_\beta}{\partial x}$$
$$=2\left(\frac{M+m}{m}\right)\Gamma\int d\mathbf{c}\, vF(\mathbf{c})\int d\mathbf{c}_1\, f(\mathbf{c}_1)\frac{\partial}{\partial v}|\mathbf{c}-\mathbf{c}_1|^{-1}$$
$$+\Gamma\int d\mathbf{c}\, F(\mathbf{c})\int d\mathbf{c}_1\, F(\mathbf{c}_1)\frac{\partial_2}{\partial v^2}|\mathbf{c}-\mathbf{c}_1|$$
$$+4\Gamma\int d\mathbf{c}\, vF(\mathbf{c})\int d\mathbf{c}_1\, F(\mathbf{c}_1)\frac{\partial}{\partial v}|\mathbf{c}-\mathbf{c}_1|^{-1}$$
$$+\Gamma\int d\mathbf{c}\, f(\mathbf{c})\int d\mathbf{c}_1\, F(\mathbf{c}_1)\frac{\partial_2}{\partial v^2}|\mathbf{c}-\mathbf{c}_1|, \tag{3.12}$$

for the proton v^2 equation.

Similarly, from (2.2) one obtains for the electrons,

$$\frac{\partial}{\partial x}\left[nU_e\left(\frac{kT}{m}\right)\right]$$
$$=2\left(\frac{m+M}{M}\right)\gamma\int d\mathbf{c}\, vf(\mathbf{c})\int d\mathbf{c}_1\, F(\mathbf{c}_1)\frac{\partial}{\partial v}|\mathbf{c}-\mathbf{c}_1|^{-1}$$
$$+\gamma\int d\mathbf{c}\, f(\mathbf{c})\int d\mathbf{c}_1\, F(\mathbf{c}_1)\frac{\partial_2}{\partial v^2}|\mathbf{c}-\mathbf{c}_1|. \tag{3.13}$$

It should be noted that in all of these equations we have taken the distributions to be time independent; i.e., neglected an explicit consideration of how rapidly the shock is dissipating energy and thus altering shape.

4. PROTON DISTRIBUTION FUNCTION

In this section we consider Eq. (3.12) for the variation of the densities $N_\alpha(x)$ and $N_\beta(x)$ through the shock wave.

Using (2.8) with (3.12), we find

$$U_\alpha\left(\frac{kT_\alpha}{M}\right)\frac{\partial N_\alpha}{\partial x}+U_\beta\left(\frac{kT_\beta}{M}\right)\frac{\partial N_\beta}{\partial x}+\frac{U_\beta N_\beta k}{M}\frac{\partial T_\beta}{\partial x}$$
$$=2\Gamma\int d\mathbf{c}\, F_\alpha(\mathbf{c})\int d\mathbf{c}_1\, F_\beta(\mathbf{c}_1)\times\left[|\mathbf{c}-\mathbf{c}_1|^{-1}-\frac{3(v-v_1)^2}{|\mathbf{c}-\mathbf{c}_1|^3}\right]. \tag{4.1}$$

The α-α and β-β terms vanish, which is to be expected for a Maxwellian distribution interacting with itself. Further, the terms deriving from electron-proton scattering are calculated explicitly later in connection with the electron v^2 transport equation (they determine the cooling length l_c), and are small compared with the proton terms above where they have been neglected. This electron-proton interaction is responsible for the comparatively slow variation of $T_\beta(x)$ through the shock (for $l<l_c$) compared with the variation we find for $N_\alpha(x)$ and $N_\beta(x)$. Thus T_β will be held constant as far as the proton shock is concerned, so that (4.1) becomes

$$U_\alpha\left(\frac{kT_\alpha}{M}\right)\frac{\partial N_\alpha}{\partial x}+U_\beta\left(\frac{kT_\beta}{M}\right)\frac{\partial N_\beta}{\partial x}=\frac{2\Gamma N_\alpha N_\beta}{\pi^3}\left(\frac{M}{2kT_\alpha}\right)^{\frac{1}{2}}\Psi, \tag{4.2}$$

where

$$\Psi=\left(\frac{2kT_\alpha}{M}\right)^{\frac{1}{2}}\int d\mathbf{c}d\mathbf{c}_1\exp[-(c^2+c_1^2)]\times\left\{\left|\left(\frac{2kT_\alpha}{M}\right)^{\frac{1}{2}}\mathbf{c}-\left(\frac{2kT_\beta}{M}\right)^{\frac{1}{2}}\mathbf{c}_1+\mathbf{i}(U_\alpha-U_\beta)\right|^{-1}-\frac{3[(2kT_\alpha/M)^{\frac{1}{2}}v-(2kT_\beta/M)^{\frac{1}{2}}v_1]^2}{|(2kT_\alpha/M)^{\frac{1}{2}}\mathbf{c}-(2kT_\beta/M)^{\frac{1}{2}}\mathbf{c}_1+\mathbf{i}(U_\alpha-U_\beta)|^3}\right\}. \tag{4.3}$$

This integral can be evaluated by using the Fourier integrals

$$|\mathbf{c}-\mathbf{c}_1|=-\frac{1}{\pi^2}\int\frac{d\mathbf{k}}{k^4}\exp[i\mathbf{k}\cdot(\mathbf{c}-\mathbf{c}_1)],$$
$$|\mathbf{c}-\mathbf{c}_1|^{-1}=\frac{1}{2\pi^2}\int\frac{d\mathbf{k}}{k^2}\exp[i\mathbf{k}\cdot(\mathbf{c}-\mathbf{c}_1)], \tag{4.4}$$

and noting that

$$\Psi=\left(\frac{2kT_\alpha}{M}\right)^{\frac{1}{2}}\left(I-\frac{3}{2}\frac{\partial^2 I_1}{\partial U_\alpha^2}\right), \tag{4.5}$$

where

$$I=\int d\mathbf{c}d\mathbf{c}_1\exp[-(c^2+c_1^2)]\times\left|\left(\frac{2kT_\alpha}{M}\right)^{\frac{1}{2}}\mathbf{c}-\left(\frac{2kT_\beta}{M}\right)^{\frac{1}{2}}\mathbf{c}_1+\mathbf{i}(U_\alpha-U_\beta)\right|^{-1}, \tag{4.6}$$

$$I_1=\int d\mathbf{c}d\mathbf{c}_1\exp[-(c^2+c_1^2)]\times\left|\left(\frac{2kT_\alpha}{M}\right)^{\frac{1}{2}}\mathbf{c}-\left(\frac{2kT_\beta}{M}\right)^{\frac{1}{2}}\mathbf{c}_1+\mathbf{i}(U_\alpha-U_\beta)\right|. \tag{4.7}$$

This gives the result

$$\Psi=\frac{3\pi^2\sqrt{\pi}}{a^2}(1+b^2)^{\frac{1}{2}}\exp\left[-\frac{a^2}{(1+b^2)}\right]+\frac{2\pi^2\sqrt{\pi}}{a^3}[a^2-\tfrac{3}{2}(1+b^2)]\int_0^{a/(1+b^2)^{\frac{1}{2}}}\exp(-c^2)dc, \tag{4.8}$$

where

$$a=(M/2kT_\alpha)^{\frac{1}{2}}(U_\alpha-U_\beta),\quad b=(T_\beta/T_\alpha)^{\frac{1}{2}}. \tag{4.9}$$

We next make use of the conservation of mass equation (3.9) for the protons; i.e.,

$$N_\alpha U_\alpha+N_\beta U_\beta=\bar{N}_\alpha U_\alpha,\quad \frac{\partial N_\beta}{\partial x}=-\frac{U_\alpha}{U_\beta}\frac{\partial N_\alpha}{\partial x}, \tag{4.10}$$

so that (4.2) becomes

$$\frac{\partial N_\alpha}{\partial x}\left[\frac{1}{N_\alpha}+\frac{1}{\bar{N}_\alpha-N_\alpha}\right]=\frac{2\Gamma\bar{N}_\alpha}{\pi^3U_\beta}\left(\frac{M}{2kT_\alpha}\right)^{\frac{1}{2}}\frac{M\Psi}{k(T_\alpha-T_\beta)}. \tag{4.11}$$

If one chooses the origin $x=0$ at the point where $N_\alpha(x)$ drops to half its value $\bar{N}_\alpha$ at $-\infty$, the solution of (4.11) is

$$N_\alpha=\frac{\bar{N}_\alpha e^{-x/l}}{(1+e^{-x/l})}, \tag{4.12}$$

$$N_\beta=\frac{\bar{N}_\beta}{(1+e^{-x/l})}, \tag{4.13}$$

where

$$l=\frac{\pi^3U_\beta k(T_\beta-T_\alpha)}{2\Gamma\bar{N}_\alpha M\Psi}\left(\frac{2kT_\alpha}{M}\right)^{\frac{1}{2}}, \tag{4.14}$$

with Ψ given by (4.8). The shock region is then represented by a simple exponential penetration of the α stream into the β stream as is to be expected.

We might now ask, since we neglected the dependence of T_β on x, what the appropriate value for T_β should be in the above formula. This will depend on the cooling length l_c for the β protons which is calculated in Sec. 7. For $l_c>l$, we take the value of T_β at $x\simeq l$ which is the same as that obtained by applying the conservation laws of Sec. 3 across the shock as if the electron degrees of freedom did not enter. If, on the other hand, $l_c\cong l$, we use $\bar{T}_\beta=T_\beta(\infty)$ obtained by using the conservation laws as if we had complete electron-proton equilibrium. This latter case covers only a small range of Mach numbers.

5. RELATIONS BETWEEN THE α AND β VARIABLES

Using the conservation equations (3.9) to (3.11) at $x=-\infty$, $x\simeq l$, and $x=+\infty$, we have for $l<l_c$,

$$\bar{N}_\alpha U_\alpha=\bar{N}_\beta U_\beta, \tag{5.1}$$

$$\bar{N}_\alpha(kT_\alpha+MU_\alpha^2)+\bar{N}_\alpha kT_\alpha=\bar{N}_\beta(kT_\beta^0+MU_\beta^2)+\bar{N}_\beta kT_\alpha=\bar{N}_\beta(k\bar{T}_\beta+MU_\beta^2)+\bar{N}_\beta k\bar{T}_\beta, \tag{5.2}$$

$$M\left(U_\alpha^2+\frac{5kT_\alpha}{M}\right)+5kT_\alpha=M\left(U_\beta^2+\frac{5kT_\beta^0}{M}\right)+5kT_\alpha=M\left(U_\beta^2+\frac{5k\bar{T}_\beta}{M}\right)+5k\bar{T}_\beta, \tag{5.3}$$

where T_β^0 is the proton temperature immediately behind the shock at $x\simeq l$. In the above equations we have used the result of Sec. 8 that the electron stream velocity U_e approaches the mean stream velocity of the protons rapidly in a characteristic length $l_v\ll l$ or l_c owing to the angular scattering of electrons by protons. Thus we deduce from the conservation equation (3.9) that since $U_e(x)\simeq\bar{U}=(N_\alpha U_\alpha+N_\beta U_\beta)N^{-1}$, then $n\simeq N_\alpha+N_\beta=N$; i.e., the displacement of the electrons from the protons is small, and in fact we would expect the resulting $E^2/8\pi$ in Eq. (3.10) to be small. Thus we have neglected it to the order in which we require to relate the proton β variables to the α variables over this range. Equations (5.1) to (5.3) then give

$$U_\beta=\frac{10kT_\alpha+MU_\alpha^2}{4MU_\alpha},\tag{5.4}$$

$$\bar{N}_\beta=\frac{4M\bar{N}_\alpha U_\alpha^2}{10kT_\alpha+MU_\alpha^2},\tag{5.5}$$

$$k\bar{T}_\beta=\frac{M}{8}\left(\frac{3U_\alpha^2}{4}-\frac{5k^2T_\alpha^2}{M^2U_\alpha^2}+\frac{7kT_\alpha}{M}\right),\tag{5.6}$$

$$kT_\beta^0=\frac{M}{4}\left(\frac{3U_\alpha^2}{4}-\frac{5k^2T_\alpha^2}{M^2U_\alpha^2}+\frac{3kT_\alpha}{M}\right).\tag{5.7}$$

It is now useful to express these quantities in terms of the Mach number K for the stream ahead of the shock; i.e.,

$$K=(U_\alpha/V),\tag{5.8}$$

where V is the velocity of sound in the plasma. This can be obtained in the usual way by linearizing the conservation equations (3.9) to (3.11) after including the time-dependent contributions from (2.1) and (2.2), and is

$$V=(10kT_\alpha/3M)^{\frac{1}{2}}.\tag{5.9}$$

Thus Eqs. (5.4) to (5.7) can also be written as

$$U_\beta=\left(\frac{3+K^2}{4K^2}\right)U_\alpha,\tag{5.10}$$

$$\bar{N}_\beta=\frac{4K^2\bar{N}_\alpha}{(3+K^2)},\tag{5.11}$$

$$k\bar{T}_\beta=\frac{MU_\alpha^2}{8}\left(\frac{3}{4}-\frac{9}{20K^4}+\frac{21}{10K^2}\right),\tag{5.12}$$

$$kT_\beta^0=\frac{MU_\alpha^2}{4}\left(\frac{3}{4}-\frac{9}{20K^4}+\frac{9}{10K^2}\right).\tag{5.13}$$

We see that at $K=1$, $U_\beta=U_\alpha$, $\bar{N}_\beta=\bar{N}_\alpha$, etc., in which case we have no shock. It should also be noted that if we have a proton cooling region in which $T_\beta(x)$ approaches $\bar{T}_\beta$, then the pressure $(\bar{N}_\beta kT+\bar{N}_\beta kT_\beta)$ of the electrons and protons in this region is constant and we have no pressure gradients to give rise to a change in U_β. There is no further interchange between the stream kinetic energy and the thermal energy of the plasma in this region. This interchange occurs entirely in the proton shock $|x|\lesssim l$.

6. DISCUSSION OF PROTON SHOCK THICKNESS l

If we express the thickness l given by Eq. (4.14) in terms of the Mach number K, we have for $l_c>l$

$$l\left(\frac{\bar{N}_\alpha\ln\Lambda}{V^4}\right)=\frac{3\pi^2}{128}\left(\frac{3}{5}\right)^{\frac{1}{2}}\frac{M^2K(3+K^2)}{e^4\Psi}\times\left(\frac{1}{4}-\frac{3}{20K^4}-\frac{1}{10K^2}\right),\tag{6.1}$$

where $\Psi(a,b)$ is given by (4.8) with

$$a=\frac{3}{4}\left(\frac{5}{3}\right)^{\frac{1}{2}}K\left(1-\frac{1}{K^2}\right),$$
$$b=\left(\frac{T_\beta^0}{T_\alpha}\right)^{\frac{1}{2}}=K\left(\frac{5}{2}\right)^{\frac{1}{2}}\left(\frac{1}{4}-\frac{3}{20K^4}+\frac{3}{10K^2}\right)^{\frac{1}{2}},\tag{6.2}$$

$$\Lambda=\frac{3}{4(\pi\bar{N}_\alpha)^{\frac{1}{2}}}\left(\frac{kT_\alpha}{e^2}\right)^{\frac{3}{2}}.\tag{6.3}$$

For the weaker shock $l_c\cong l$, we have approximately

$$l\left(\frac{\bar{N}_\alpha\ln\Lambda}{V^4}\right)=\frac{3\pi^2}{256}\left(\frac{3}{5}\right)^{\frac{1}{2}}\frac{M^2K(3+K^2)}{e^4\Psi}\times\left(\frac{1}{4}-\frac{3}{20K^4}-\frac{1}{10K^2}\right),\tag{6.4}$$

where now

$$b=\left(\frac{\bar{T}_\beta}{T_\alpha}\right)^{\frac{1}{2}}=\frac{K\sqrt{5}}{2}\left(\frac{1}{4}-\frac{3}{20K^4}+\frac{7}{10K^2}\right)^{\frac{1}{2}}.\tag{6.5}$$

Now consider how this behaves at high and low values of K. From (4.8), we have

$$\Psi\rightarrow 0.309\pi^4/a \quad\text{as}\quad K\rightarrow\infty,$$
$$\Psi\rightarrow 2\sqrt{2}\pi^2\pi^{\frac{1}{2}}a^2/15 \quad\text{as}\quad K\rightarrow 1,$$

so that

$$l\rightarrow\frac{29.1K^4V^4}{512\pi\bar{N}_\alpha\Gamma}\quad\text{as}\quad K\rightarrow\infty,\tag{6.6}$$

and

$$l\rightarrow\frac{3\sqrt{\pi}}{10\sqrt{2}}\left(\frac{3}{5}\right)^{\frac{1}{2}}\frac{V^4}{\bar{N}_\alpha\Gamma(K-1)}\quad\text{as}\quad K\rightarrow 1.\tag{6.7}$$

Hence, at the onset of the shock at $K\simeq 1$, the thickness is very large and at first decreases with increasing K as in the case for ordinary gases. However, as the Mach number K increases, the shock thickness l begins to increase again and finally varies as K^4 for large K. This is due to the fact that the mean free path for transfer of a given energy from an α proton to a β proton becomes very large due to the high relative velocities of these two particles (both thermal and stream velocities). In order to cross the energy gap from the α to the β stream, a proton requires a mean free path proportional to $(U_\alpha-U_\beta)^4$ in this region due to the Coulomb interaction.

The quantity $l(e^4\bar{N}_\alpha\ln\Lambda/m^2V^4)$, which is a function of K only, is plotted in Fig. 2, from which one easily obtains l for a particular density and temperature of the plasma.

7. ELECTRON DISTRIBUTION FUNCTION

We now require the electron distribution function for the region behind the shock where $F_\alpha=0$. For this, we use the v^2 Eq. (3.13) which has the advantage that E does not appear explicitly. Upon integrating by parts and using

$$\frac{\partial F_\beta}{\partial v_1}=-\left(\frac{Mv_1}{kT_\beta}\right)F_\beta,\qquad \frac{\partial f}{\partial v}=-\left(\frac{mv}{kT}\right)f, \tag{7.1}$$

this becomes

$$\frac{\partial}{\partial x}\left[nU_e\left(\frac{kT}{m}\right)\right]=\frac{2\gamma m}{k}\left(\frac{1}{T}-\frac{1}{T_\beta}\right)\int d\mathbf{c}d\mathbf{c}_1\, fF_\beta vv_1|\mathbf{c}-\mathbf{c}_1|^{-1}+\gamma\int d\mathbf{c}d\mathbf{c}_1 f(\mathbf{c})F_\beta(\mathbf{c}_1)\left[|\mathbf{c}-\mathbf{c}_1|^{-1}-\frac{3(v-v_1)^2}{|\mathbf{c}-\mathbf{c}_1|^3}\right]. \tag{7.2}$$

The right-hand side of (7.2) tends to zero if the stream velocities U_e and U_β together with the temperatures T and T_β are equal. This is the equilibrium case. Further if $T=T_\beta$ and $U_e\neq U_\beta$, only the second term contributes; and if $T\neq T_\beta$ and $U_e=U_\beta$, only the first contributes. Thus the first term is primarily responsible for removing electron-proton temperature differences, and the second for removing stream velocity differences.

We first calculate an expression for the variation of the electron temperature T in the proton cooling region under the assumption $l_c>l$; i.e., with the boundary conditions that at $x\simeq l$ the electron temperature is T_α and proton temperature T_β^0. That the derivation under this assumption is self-consistent becomes obvious when we see that indeed l_c is greater than l for a large range of K and the result applies over this range. Further, since in Sec. 8 we show $U_e\simeq\bar{U}$, we use the distribution functions,

$$F_\beta=\bar{N}_\beta\left(\frac{M}{2\pi kT_\beta}\right)^{\frac{3}{2}}\exp\left[-\frac{M}{2kT_\beta}(\mathbf{c}-\mathbf{i}U_\beta)^2\right],\qquad f=\bar{N}_\beta\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\exp\left[-\frac{m}{2kT}(\mathbf{c}-\mathbf{i}U_\beta)^2\right], \tag{7.3}$$

in (7.2) which gives

$$\frac{\partial T}{\partial x}=\frac{2\gamma m^2\bar{N}_\beta^2}{3\bar{N}_\alpha U_\alpha\pi^3k^2}\left(\frac{2kT}{m}\right)^{\frac{1}{2}}\left(\frac{1}{T}-\frac{1}{T_\beta}\right)\int d\mathbf{c}d\mathbf{c}_1\times\exp[-(c^2+c_1^2)]\mathbf{c}\cdot\mathbf{c}_1\left|\mathbf{c}-\left(\frac{MT}{mT_\beta}\right)^{\frac{1}{2}}\mathbf{c}_1\right|^{-1}. \tag{7.4}$$

Upon making use of the conservation law (3.10); i.e., $2\bar{T}_\beta=T_\beta+T$, this becomes

$$\frac{T^{\frac{3}{2}}dT}{(\bar{T}_\beta-T)}=\frac{4\gamma m^2\bar{N}_\beta^2}{3\bar{N}_\alpha U_\alpha\pi^{\frac{1}{2}}k^2}\left(\frac{2k}{m}\right)^{\frac{1}{2}}\left(\frac{m}{M}\right)\times\left[1+\frac{mT_\beta}{MT}\right]^{-\frac{3}{2}}dx. \tag{7.5}$$

If we now note the fact that over the range $K<(8M/5m)^{\frac{1}{2}}$, the maximum value of $(T_\beta m/TM)<1$, then (7.5) is simply integrated to give

$$\bar{T}_\beta^{\frac{3}{2}}\ln\left[\left(\frac{\bar{T}_\beta^{\frac{1}{2}}+T^{\frac{1}{2}}}{\bar{T}_\beta^{\frac{1}{2}}-T^{\frac{1}{2}}}\right)\left(\frac{\bar{T}_\beta^{\frac{1}{2}}-\bar{T}_\alpha^{\frac{1}{2}}}{\bar{T}_\beta^{\frac{1}{2}}+\bar{T}_\alpha^{\frac{1}{2}}}\right)\right]-2T^{\frac{1}{2}}\left(\frac{T}{3}+\bar{T}_\beta\right)+2\bar{T}_\alpha^{\frac{1}{2}}\left(\frac{\bar{T}_\alpha}{3}+\bar{T}_\beta\right)=\frac{4\gamma m^2\bar{N}_\beta^2}{3\bar{N}_\alpha U_\alpha k^2\sqrt{\pi}}\left(\frac{2k}{m}\right)^{\frac{1}{2}}\left(\frac{m}{M}\right)x. \tag{7.6}$$

This expression describes the way in which the electron temperature T rises behind the proton shock and also therefore the drop in the proton temperature $T_\beta=2\bar{T}_\beta-T$. The above expression would however be considerably modified in an improved approximation taking account of the thermal conductivity of the electron gas as discussed in Sec. 2. As T approaches $\bar{T}_\beta$ the logarithmic term dominates, and we have a simple exponential law with a characteristic length

$$l_c=\frac{3\bar{N}_\alpha U_\alpha(\pi)^{\frac{1}{2}}k^2T_\beta^{\frac{3}{2}}}{4\gamma m^2\bar{N}_\beta^2}\left(\frac{m}{2k}\right)^{\frac{1}{2}}\left(\frac{M}{m}\right). \tag{7.7}$$

Using the relations (5.10)–(5.12), we can now derive as before an expression which is a function of Mach number only, namely

$$l_c\left(\frac{e^4\bar{N}_\alpha\ln\Lambda}{m^2V^4}\right)=\frac{3}{8192\sqrt{\pi}}\left(\frac{M}{m}\right)^{\frac{5}{2}}(3+K^2)^2\times\left[\frac{3}{4}-\frac{9}{20K^4}+\frac{21}{10K^2}\right]^{\frac{3}{2}}. \tag{7.8}$$

This quantity is plotted in Fig. 2, from which we see that $l_c>l$ except for very small values of K where the shock becomes very thick and the electrons have time to reach thermal equilibrium with the protons. In this region the continuum equation based on the assumption of small deviations from equilibrium distributions probably give a more accurate result.

8. VELOCITY-MATCHING LENGTH l_v

In order to determine the characteristic length l_v in which the electrons adjust their stream velocity U_e to that of the background protons through scattering, we first consider the time-dependent equation for v^2 of the electrons,

$$\frac{\partial}{\partial t}\left[\frac{knT}{m}\right]+\frac{\partial}{\partial x}\left[\frac{nU_e kT}{m}\right]=\int\left(\frac{\partial f}{\partial t}\right)_c v^2 d\mathbf{c}. \tag{8.1}$$

Consider an infinite medium of protons at rest and electrons streaming in the x direction with velocity U_e. Then, since all quantities are independent of x, we first calculate the time, τ_v, taken to slow the electrons down by using the distribution functions,

$$F=N\left(\frac{M}{2\pi kT_\alpha}\right)^{\frac{3}{2}}\exp\left(-\frac{Mc^2}{2kT_\alpha}\right), \tag{8.2}$$

$$f=N\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\exp\left[-\frac{m}{2kT}(\mathbf{c}-\mathbf{i}U_e)^2\right]. \tag{8.3}$$

With (8.1) these give

$$\frac{\partial}{\partial t}\left(\frac{kNT}{m}\right)=\frac{\gamma N^2}{\pi^3}\int d\mathbf{c}d\mathbf{c}_1$$
$$\times\exp[-(c^2+c_1^2)]\left\{\left|\left(\frac{2kT}{m}\right)^{\frac{1}{2}}\mathbf{c}-\left(\frac{2kT_\alpha}{M}\right)^{\frac{1}{2}}\mathbf{c}_1+\mathbf{i}U_e\right|^{-1}\right.$$
$$\left.-\frac{3[(2kT/m)^{\frac{1}{2}}v-(2kT_\alpha/M)^{\frac{1}{2}}v_1]^2}{|(2kT/m)^{\frac{1}{2}}\mathbf{c}-(2kT_\alpha/M)^{\frac{1}{2}}\mathbf{c}_1+\mathbf{i}U_e|^3}\right\}. \tag{8.4}$$

We now use the result in advance that τ_v is small compared with the time required for the electrons to reach thermal equilibrium with the protons. This means that over the time taken for the electrons to adjust their stream velocity to that of the protons, we have energy interchange only between the stream kinetic and thermal energies of the electrons. Thus the time part corresponding to the energy conservation equation (3.11) gives

$$mN\frac{\partial}{\partial t}\left(U_e^2+\frac{3kT}{m}\right)=0. \tag{8.5}$$

The integral (8.4) is similar to (4.3) and can be obtained by writing

$$a=\left(\frac{m}{2kT}\right)^{\frac{1}{2}}U_e\ll 1,$$
$$b=\left(\frac{mT_\alpha}{MT}\right)^{\frac{1}{2}}\ll 1, \tag{8.6}$$

in the expression (4.8) for Ψ. Together with (8.5), this yields

$$U_e=U_e(0)e^{-t/\tau_v}, \tag{8.7}$$

where

$$\tau_v=\frac{5\sqrt{\pi}}{4\gamma N}\left(\frac{2kT}{m}\right)^{\frac{3}{2}}. \tag{8.8}$$

Thus, returning to our shock wave, if the electron stream velocity gets out of step with that of the protons, it adjusts itself through angular scattering in a characteristic length

$$l_v=\frac{5\sqrt{\pi}}{4\gamma\bar{N}_\alpha}\left(\frac{2kT_\alpha}{m}\right)^{\frac{3}{2}}U_\alpha, \tag{8.9}$$

in the frame in which the shock is at rest.

Expressed as a function of Mach number, l_v is given by

$$l_v\left(\frac{e^4\bar{N}_\alpha\ln\Lambda}{m^2V^4}\right)=\frac{3}{4}\left(\frac{3\pi}{5}\right)^{\frac{1}{2}}\frac{1}{4\pi}\left(\frac{M}{m}\right)^{\frac{3}{2}}K, \tag{8.10}$$

which is also shown in Fig. 2. We see that for all K, $l_v\ll l$ or l_c which justifies our writing $U_e=\bar{U}$ in the conservation equations, and indicates that the displacement of the electrons from the protons is negligible.