predicted. If the assumption of a slow energy variation of X and Y is accepted, the knowledge of the partial decay rates is sufficient to determine uniquely the electron spectrum and the two possible muon spectra (and corresponding the average polarizations), without need of observing the pion emitted. The general requirement $\tau(K_{e3}) \gtrsim \sim 0.5\tau(K_{\mu3})$ which follows from the theory, is apparently satisfied by the present data.

If the present discrepancy with the $\pi \rightarrow e+\nu$ to $\pi \rightarrow \mu+\nu$ ratio will still persist and eventually be confirmed by further tests of the kind examined here, then some of the following possibilities should be considered in detail:

The hypothesis of a universal interaction, in the sense of strict equality of coupling constants, is not true.

The hypothesis of universality is valid but the universal interaction, still local, has a more complicated form than the simple $A \pm V$ mixture—for instance, a small pseudoscalar term is present which almost exactly cancels the contribution from A to electron decay..

The universal interaction is nonlocal, such that the two leptons are emitted at different points [but always with the projection $\frac{1}{2}(1+\gamma_5)$]. Such a nonlocality must however extend up to very long wavelengths corresponding to a mass of 100 Mev or even less.

The universal interaction is nonlocal and furthermore the leptons are not required to interact only through the projection $\frac{1}{2}(1+\gamma_5)$—but such a requirement is only valid in the local limit. Such a form of the interaction has been proposed by Sirlin[10] and it offers a more redundant solution of the $\pi \rightarrow e+\nu$ problem than the simpler introduction of a small local pseudoscalar term. Moreover, as pointed out by Feynman and Gell-Mann,[11] the requirement that the rate of $\mu \rightarrow e+\gamma$ be slow imposes stringent conditions.

If one wants to insist on the universal $A \pm V$ form, one can speculate about a possible breakdown of present electrodynamics that may offer a possibility for an explanation.[12]

[10] A. Sirlin, Phys. Rev. **111**, 337 (1958).
[11] R. P. Feynman and M. Gell-Mann (private communication).
[12] R. Gatto and M. Ruderman, Nuovo cimento **8**, 775 (1958); **9**, 556 (1958).

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Vertex Function in Quantized Field Theories*

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An integral representation is given for the vertex function $F(k^2, p^2, (k-p)^2)$. This representation is obtained on the basis of local commutativity and the spectral conditions. It exhibits the set of points k^2, p^2, for which F is analytic in $(k-p)^2$ except for the physical cut. The limitations found in the representation are discussed on the basis of examples obtained from perturbation theory. These examples give some insight into the question of further analytic continuation in k^2 and p^2.

IN a previous article[1] we have obtained certain analyticity properties of the vertex function on the basis of the general axioms of field theory. However, the information contained in these axioms has not been completely exhausted, and especially the unitarity condition was not used at all. It is the purpose of the present note to give a representation for the vertex function which exhibits the analytic properties obtained so far, and to discuss the limitations which one encounters on the basis of examples obtained from perturbation theory.

1

Let us consider first the general vertex function

$$G(k,p)=\int d^4x \int d^4y\, e^{ik\cdot x-ip\cdot y}\tilde{G}(x,y), \qquad (1)$$

where $\tilde{G}(x,y)$ may be written in the form

$$\tilde{G}(x,y)=\theta(x)\{\theta(y-x)\langle 0|[[B(y),A(x)],C(0)]|0\rangle +\theta(y)\langle 0|[A(x),[B(y),C(0)]]|0\rangle\}. \qquad (2)$$

The quantities A, B, C can be considered as current operators which satisfy the spectral conditions $\langle 0|A|n\rangle=0$ unless $p_n^2 \geq a^2$, $\langle 0|B|n\rangle=0$ unless $p_n^2 \geq b^2$, and $\langle 0|C|n\rangle=0$ unless $p_n^2 \geq c^2$. Here p_n with $p_{n0} \geq 0$ is the four-vector describing the total energy and momentum of the state $|n\rangle$. Using the methods described earlier one can show that $G(k,p)$ is a boundary value of an analytic function $F(z_1z_2z_3)$ with $z_1=k^2$, $z_2=p^2$, $z_3=(k-p)^2$, $z_j=x_j+iy_j$; this function F is regular in the z_3 plane except for a cut $y_3=0$, $x_3 \geq c^2$, provided the variables z_1, z_2 are restricted to a certain domain D in the space of two complex variables. The domain D is most easily obtained if the methods of BOT are supplemented by the general representation for the causal commutator, which has been proven by Dyson.[2] Using

* This work has been performed while the author was at the Institute for Advanced Study, Princeton, New Jersey.
[1] Bremermann, Oehme, and Taylor, Phys. Rev. **109**, 2178 (1958). This paper will be referred to as BOT; it contains further references.
[2] F. J. Dyson, Phys. Rev. **110**, 1460 (1958).

these tools we obtain a representation for $F(z_1z_2z_3)$, which may be written in the form

$$F(z_1z_2z_3)=\int_{c^2}^{\infty}d\sigma^2\int_0^1 d\xi\int_{\xi-1}^{1-\xi}d\eta\int_{\kappa_0}^{\infty}d\kappa\frac{\chi(\kappa,\xi,\eta,\sigma^2)}{\sigma^2-z_3}\times\frac{1}{[2\kappa^2+\frac{1}{2}(1+\xi^2-\eta^2)\sigma^2-(z_1+z_2)+\eta(z_1-z_2)]^2-\xi^2\lambda(z_1z_2\sigma^2)},\quad(3)$$

where $\lambda(z_1z_2z_3)=z_1^2+z_2^2+z_3^2-2z_1z_2-2z_1z_3-2z_2z_3$. The lower limit of the κ integration is given by

$$\kappa_0=\max\{0,\ a-\tfrac{1}{2}\sigma[(1+\eta)^2-\xi^2]^{\frac{1}{2}},\ b-\tfrac{1}{2}\sigma[(1-\eta)^2-\xi^2]^{\frac{1}{2}}\}.\quad(4)$$

In general Eq. (3) must be supplemented by convergence factors and additive polynominals (subtractions), but these are unimportant for the purpose of this note. In addition to the unitarity condition, the information contained in the Jacobi identity for the double commutator has not been exhausted in the derivation of Eq. (3). It is questionable whether the latter property of the vertex function will enlarge the domain D, which is given by the set of points (z_1,z_2) for which

$$[2\kappa^2+\tfrac{1}{2}(1+\xi^2-\eta^2)\sigma^2-(z_1+z_2)+\eta(z_1-z_2)]^2-\xi^2\lambda(z_1z_2\sigma^2)\neq 0\quad(5)$$

for all $\sigma^2\geq c^2$, $0\leq\xi\leq 1$, $|\eta|\leq 1-\xi$, $\kappa\geq\kappa_0(\xi,\eta,\sigma)$. From the examples in Sec. 2 we will see that at least for certain, physically important parts of the boundary of D such an extension is not possible on the basis of the axioms (i.e., causality and spectrum, excluding unitarity) alone.

For reasons of brevity we consider here only the case $a=b$, in which we may take $\eta=0$ in Eq. (5). The real points $z_1=x_1$, $z_2=x_2$ are then contained in D provided $x_1<a^2$, $x_2<a^2$ and

$$x_1+x_2<\min_{\sigma\geq c}\{\sigma^2-2\sigma a+2a^2\},\quad(6)$$

which leads to

$$\begin{aligned}x_1+x_2&<a^2 &&\text{for } c\leq a\\ &<c^2-2ca+2a^2 &&\text{for } c\geq a.\end{aligned}\quad(6\text{a})$$

In the equal-mass case we have $a=c=2m$ and Eq. (6) leads to the condition $x_1+x_2<4m^2$. For the pion-nucleon vertex we obtain with $a=M+\mu$, $c=3\mu$ the restriction $x_1+x_2<(M+\mu)^2$. For $x_1=x_2=M^2$, this inequality gives the well known unphysical condition $\mu>(\sqrt{2}-1)M$.[1] It is important to note that the most serious restriction comes from intermediate states with total mass $\sigma=M+\mu$ and nucleon number zero.

2

We consider now the vertex function $V(z_1z_2z_3,m_1m_2m_3)$ in lowest order perturbation theory. The parameters m_1, m_2, and m_3 are the masses of the three internal lines opposite to the vertices associated with k, p, and $k-p$, respectively. In order to have a finite expression, it is convenient to use the combination

$$F_p(z_1z_2z_3,m_1m_2m_3)=\sum_{i=1}^{3}\frac{\partial}{\partial m_i^2}V(z_1z_2z_3,m_1m_2m_3)=\int_0^1\int_0^1\int_0^1\frac{d\alpha d\beta d\gamma\,\delta(1-\alpha-\beta-\gamma)}{[\alpha m_1^2+\beta m_2^2+\gamma m_3^2-\beta\gamma z_1-\alpha\gamma z_2-\alpha\beta z_3]^2},\quad(7)$$

which has the same singularities as V. The integration in Eq. (7) has been carried out by Källèn and Wightman.[3] For our present purpose, it is sufficient to take the special case $z_1=z_2=z$, $m_1=m_2=m$, for which we obtain, with $\gamma(z)=[z-(m_3-m)^2][z-(m_3+m)^2]$,

$$F_p(zzz_3,mmm_3)=\frac{1}{m_3^2z_3+\gamma(z)}\left\{-\frac{2(z+m_3^2-m^2)}{\{\gamma(z)\}^{\frac{1}{2}}}\times\ln\frac{1}{2m_3m}[m_3^2+m^2-z+\{\gamma(z)\}^{\frac{1}{2}}]+\frac{z_3-2(z-m_3^2+m^2)}{\{z_3(z_3-4m^2)\}^{\frac{1}{2}}}\times\ln\frac{1}{2m^2}[2m^2-z_3+\{z_3(z_3-4m^2)\}^{\frac{1}{2}}]\right\}.\quad(8)$$

For $y=0$, $|y_3|>\epsilon>0$, F_p has no singularities as a function of z_3. This property is a general feature of perturbation theory. From the example of Jost[4] we know that it does not follow from the axioms without unitarity, but at present one cannot exclude the possibility that the unitarity condition changes this situation. We see from Eq. (8) that F_p has always the "static" branch lines $y=0$, $x\geq(m_3+m)^2$ and $y_3=0$, $x_3\geq 4m^2$. For $y=0$, $x<m_3^2+m^2$ we have analyticity in the z_3 plane except for the static cut $y_3=0$, $x_3\geq 4m^2$. [If the restriction $x_1=x_2$ is relaxed, one finds the condition $x_1+x_2<2(m_3^2+m^2)$.] But if $y=0$, $m_3^2+m^2<x<(m_3+m)^2$ we can produce singularities for $y_3=0$, $x_3\geq -m_3^{-2}\gamma(x)$ in addition to the static cut.[5] It is instructive to consider some special cases: (a) equal-mass case: taking $m_3=m$, we have only the static cut $y_3=0$, $x_3\geq 4m^2$ provided $x<2m^2$ [or $x_1+x_2<4m^2$ if $x_1\neq x_2$]. *This is exactly the same condition as the one obtained from Eq (6)* in Sec. 1. For $2m^2<x<4m^2$, we can have singularities for $y_3=0$, $x_3\geq x(4-xm^{-2})$. We conclude that in the case of completely symmetric spectral conditions the limitation $x_1+x_2<4m^2$ is already

[3] G. Källèn and A. Wightman (private communication). We would like to thank Professor Wightman for many interesting discussions concerning the vertex function.
[4] R. Jost (to be published), and reference 1, footnote 18.
[5] These conditions have been obtained independently by Y. Nambu (to be published) and by Karplus, Sommerfield, and Wichmann (to be published).

contained in a perturbation theory with the most simple interaction. It may be of interest to note that in the present case the Jost example,[4]

$$F_j=[2(4m^2-x)^{\frac{1}{2}}+(4m^2-z_3)^{\frac{1}{2}}-(B+iC)]^{-1},\qquad (B+C)<2m;\quad B,\quad C\geq 0,$$

becomes applicable for $x\geq 3m^2$, and for $x>3m^2+\epsilon$, $\epsilon>0$, it gives also singularities off the real x_3 axis. (b) Pion-nucleon vertex: for $x=M^2$ (on the mass shell) perturbation theory with the *usual* interaction leads always to the static cut $y_3=0$, $x_3\geq(3\mu)^2$ only.[6] But we may use more general "interactions" which are compatible with the spectral conditions, although they may make no physical sense. The only question we ask is whether or not we can expect that the axioms alone, taken as mathematical conditions, already guarantee the validity of the usual dispersion relation. Taking for instance $m=\frac{3}{2}\mu$, $m_3=M-\frac{1}{2}\mu$ and $x=M^2$, we obtain singularities for $x_3\geq(4\mu)^2(2M+\mu)(M-\mu)(2M-\mu)^{-2}<(3\mu)^2$, $y_3=0$ in addition to the static cut. Only for $x<M^2-\frac{1}{2}\mu(2M-5\mu)<M^2$ these additional singularities would disappear. We conclude that the causality and spectral conditions do not suffice to guarantee the validity of the "normal" dispersion relation for the pion-nucleon vertex $\langle N|\pi|N\rangle$.

The case $m_3=m=\frac{1}{2}(M+\mu)$ is also of some interest. For $x<\frac{1}{2}(M+\mu)^2$, $y=0$ the function F_p is analytic in z_3 except for the cut $y_3=0$, $x_3\geq(M+\mu)^2$, but if $x=M^2$ we can have in addition singularities on the real axis for $x_3\geq 4M^2\mu(2M+\mu)(M+\mu)^{-2}$, $y_3=0$. Here the lower limit is between $(3\mu)^2$ and $(M+\mu)^2$. The condition $x<\frac{1}{2}(M+\mu)^2$ is identical to the one we have obtained in BOT and Sec. 1 for the analyticity of $F(z_1z_2z_3)$ in the cut z_3 plane. In the present example, as well as in the general case, the restriction is due to states with total mass $\sigma=M+\mu$. Note that the example does not introduce new points of singularity, because we have already the cut $x_3\geq(3\mu)^2$, $y_3=0$. However, it shows that for $x=M^2$, the causality and spectral conditions alone cannot guarantee that states with $\sigma=M+\mu$ lead only to singularities for $x_3\geq(M+\mu)^2$.

[6] Y. Nambu, Nuovo cimento **6**, 1064 (1957).

A situation similar to either the first or the second example described above prevails for all states with

$$(M+\mu)-[(M-\mu)^2-2\mu^2]^{\frac{1}{2}}<\sigma<(M+\mu)+[(M-\mu)^2-2\mu^2]^{\frac{1}{2}}.$$

Note that for $a=m+m_3$, $\sigma=2m$ the condition (6) becomes $x_1+x_2<2(m_3^2+m^2)$, which is exactly what we obtained from Eq. (7). In the most general case we find from the representation (3) the condition

$$x_1(\sigma-a+b)+x_2(\sigma+a-b)<\sigma^2(\sigma-a-b)+2\sigma ab$$

for all $\sigma\geq c$. For $a=m_2+m_3$, $b=m_1+m_3$, and $\sigma=m_1+m_2$, this yields the restriction

$$m_1(x_1-m_2^2-m_3^2)+m_2(x_2-m_1^2-m_3^2)<0,$$

which is the same as in perturbation theory.

There are many other cases for which we find a close connection between the limitations obtained from Eq. (5) and the singularities of corresponding examples from perturbation theory. We mention here only the deuteron-photon vertex,[7] for which we may take $m_3=m=M$. With $x=M_D^2=(2M-\epsilon)^2>2M^2$, one has singularities for $y_3=0$, $x_3\geq x_{30}=4M_D^2(1-M_D^2/4M^2)\approx 16M\epsilon$ in addition to the static cut $y_3=0$, $x_3\geq 4\mu^2$. Here these additional singularities are physically reasonable. The general representation (3) would give the static cut alone only under the unphysical condition $M_D<\sqrt{2}M$, which also leads to $x_{30}=4M^2$ for $M_D=\sqrt{2}M$.

We hope that the connections discussed in the present note may give some indication about the assumptions which have to be added to the present axioms in order to guarantee the validity of dispersion relations in some physically interesting cases. They suggest that more information about the properties of certain intermediate states should be used.

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[7] Y. Nambu, reference 5.