In the original model, R_4 is the probability that the nucleus evaporates three neutrons but still has an excitation energy greater than the neutron binding energy. To take fission into account, we must use the probability that the nucleus evaporates three neutrons but still has an excitation energy greater than the fission activation energy. Hence,

$$R_4 = G_{n1}G_{n2}G_{n3}I(\Delta_3{}^f,5),$$

where

$$\Delta_3{}^f = (E - B_1 - B_2 - B_3 - E_{\text{th}})/T,$$

and E_{th} is the activation energy for fission.[24] The probability for evaporation of exactly three neutrons is

$$P(E,3) = R_3 - R_4 = G_{n1}G_{n2}G_{n3}[I(\Delta_3,3) - I(\Delta_3{}^f,5)].$$

PHYSICAL REVIEW VOLUME 111, NUMBER 5 SEPTEMBER 1, 1958

Phenomenological Analysis of the Production of Pion Pairs*

RONALD F. PEIERLS
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
(Received April 17, 1958)

The angular distribution for the production of pairs of pions by photons or pions incident on nucleons is analyzed in terms of the various angular momentum states involved. A general expression is derived and then the effect of various assumptions about which states should be important is examined. It is found that an examination of the relative azimuth of the pions should give information about the nature of the process, and in particular about the existence of a resonant state of the nucleon, and its angular momentum.

I. INTRODUCTION

RECENTLY there have been a number of experimental investigations of the multiple production of pions from nucleons, both by pions[1] and also by photons.[2] Apart from any purely experimental difficulties, the examination of a process involving a three-body final state has the difficulty that the number of different parameters which can be examined is very large. There have been some attempts to calculate the expected cross section for such processes[3–6] but it is not all clear just which of the features of these predictions are sensitive to the assumptions made or the model used, and therefore it is hard to tell how to compare the experimental results with the theories.

Before very much was known about the single meson-nucleon interaction, it was found that many of the striking features of the experiments could be explained on quite general phenomenological grounds,[7] without assuming any detailed model. One might therefore hope that a similar analysis of the double production process, making use of the known single-meson information wherever possible, might give a qualitative insight into the nature of the process, and should at least enable one to pick out those aspects of the theoretical predictions which are sensitive to the model used.

In the present paper an analysis of this sort is attempted. Most of the formulas apply equally well for production by pions and by photons; the differences coming in the number and values of the various arbitrary constants which are produced. The main object is to make use of the known properties of angular momentum to investigate the angular distributions to be expected. We shall not say very much about the energy dependence of the cross section, nor about the isotopic spin, but restrict ourselves to trying to interpret the angular distributions.

II. GENERAL EXPRESSION FOR THE CROSS SECTION

In order to discuss the angular distribution we proceed to define the S matrix for the process in question.[8] We assume that when the particles concerned are sufficiently far apart they behave like free particles, and their wave functions may then be described by suitably normalized ingoing or outgoing spherical waves. The production process may be considered as the transition from one set of ingoing waves to another set of outgoing ones. The various possible states of the separated particles may be divided into a series of

* Work supported in part by the joint program of the Office of Naval Research and the U. S. Atomic Energy Commission.

1 M. Blau and M. Calton, Phys. Rev. **96**, 150 (1954).

2 M. Bloch and M. Sands, Phys. Rev. **108**, 1101 (1957); Sellen, Cocconi, Cocconi, and Hart, Bull. Am. Phys. Soc. Ser. II, **3**, 33 (1957).

3 R. D. Lawson, Phys. Rev. **92**, 1272 (1953).

4 S. Barshay, Phys. Rev. **103**, 1102 (1956); J. Franklin, Phys. Rev. **105**, 1101 (1957).

5 R. E. Cutkosky and F. Zachariasen, Phys. Rev. **103**, 1108 (1956).

6 A. Bincer, Phys. Rev. **105**, 1399 (1957).

7 See, for example, M. Gell-Mann and K. M. Watson, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Stanford, 1954), Vol. 4, p. 219.

8 See, for example, J. M. Blatt and L. C. Biedenharn, Revs. Modern Phys. **24**, 258 (1952). The discussion of this section is just an extension of the first part of their paper to cover three-body final states.

"channels" labeled by the indices α for the ingoing states and β for the outgoing ones. The indices α and β specify the number and nature of the particles, the momenta necessary to specify their motion and the eigenvalues of a suitable set of angular momentum operators. Suppose that we have an incident wave of unit amplitude in one channel α, say; the resulting outgoing wave will in general be a mixture of all channels β with amplitudes $S_{\alpha\beta}$. This set of quantities $S_{\alpha\beta}$ is the S matrix for the process in question, and determines the cross section.

Suppose that we perform an experiment in which the incident beam is a mixture of channels α with amplitudes A_α. Then clearly the outgoing wave function will be

$$\Psi_{\text{out}}=\sum_{\alpha,\beta} A_\alpha S_{\alpha\beta}\psi_\beta, \tag{1}$$

where ψ_β is the wave function in channel β. Hence the cross section will be

$$d\sigma=d\tau\Big|\sum_{\alpha\beta} A_\alpha S_{\alpha\beta}\psi_\beta\Big|^2, \tag{2}$$

where $d\tau$ is the appropriate kinematical and phase space factor which will depend on the choice of normalization.

In investigating the angular distribution, we shall only be concerned with the angular-momentum part of the channel labels α, β, and we shall omit any explicit reference to other quantities such as energy, isotopic spin, etc. For convenience of notation the angular momenta $\mathbf{l}$, $\mathbf{l}'$, $\mathbf{l}_1$, $\cdots$, etc. will be associated with the quantum numbers l, l', l_1, $\cdots$, etc. for their magnitudes and μ, μ', μ_1, $\cdots$, etc. for their z components. Similarly $\mathbf{j}$, j, m, $\cdots$; $\mathbf{s}$, s, m_s, $\cdots$; $\mathbf{J}$, J, M, $\cdots$, etc.

The initial state involves the angular momentum of the incident beam, $\mathbf{l}$, and the initial spin of the nucleon, $\mathbf{s}_i$. These combine to give total angular momentum

$$\mathbf{J}=\mathbf{l}+\mathbf{s}_i. \tag{3}$$

To label the incident channels we take the quantum numbers J, M (which are both conserved), l, and s_i:

$$\alpha\equiv(JMls_i). \tag{4}$$

The outgoing channels are states of two pions and a nucleon, and are therefore described by *two* orbital momenta $\mathbf{l}_1$ and $\mathbf{l}_2$, and the final spin of the nucleon,

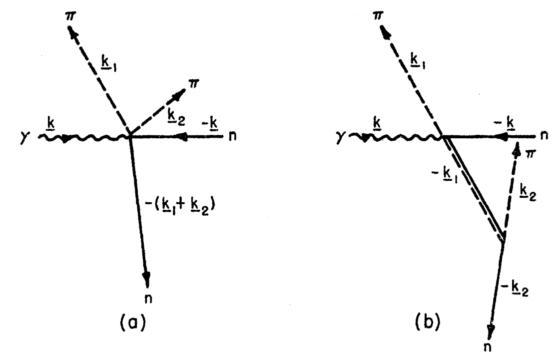

FIG. 1. Alternative choices of coordinate system for simultaneous production (a), and production via a "compound particle" (b).

$\mathbf{s}_f$, which combine to give:

$$\mathbf{J}=\mathbf{l}_1+\mathbf{l}_2+\mathbf{s}_f. \tag{5}$$

For the moment we do not specify precisely what we mean by $\mathbf{l}_1$ and $\mathbf{l}_2$; we shall return to this point later. Thus the outgoing channels are labeled in a similar way to the incident ones by J, M, l_1, l_2, and s_f. However, when three angular momenta are coupled to form an eigenstate of the total angular momentum, we must also specify the eigenvalue of *one* of the three sums:

$$\begin{aligned}\mathbf{l}_1+\mathbf{l}_2&=\mathbf{L},\\ \mathbf{l}_1+\mathbf{s}_f&=\mathbf{j}_1,\\ \mathbf{l}_2+\mathbf{s}_f&=\mathbf{j}_2.\end{aligned} \tag{6}$$

We choose to specify j_1, for reasons which we shall discuss later. Thus,

$$\beta\equiv(JMl_1l_2s_fj_1). \tag{7}$$

Since s_i and s_f are always $\frac{1}{2}$, we shall generally omit explicit reference to them.

It now remains to determine ψ_β. We know the angular dependence of an eigenstate of $(l_1l_2\mu_1\mu_2)$; it is given by the product of spherical harmonics:

$$Y_{l_1\mu_1}(\theta_1\phi_1)Y_{l_2\mu_2}(\theta_2\phi_2),$$

where $(\theta_1\phi_1)$, $(\theta_2\phi_2)$ are the directions associated with the angular momenta $\mathbf{l}_1$ and $\mathbf{l}_2$. An eigenstate of β can be expressed as a linear combination of such eigenstates, using the appropriate Clebsch-Gordan coefficients. When this is done, and the results inserted in (2), we obtain

$$d\sigma=\bar{\lambda}^2\,\mathrm{Re}\Big\{\sum_{\alpha\beta,\alpha'\beta'} A_\alpha^* A_{\alpha'} S_{\alpha\beta}^* S_{\alpha'\beta'}\Big[\sum_{L_1L_2M_1}\Theta(\alpha\beta\alpha'\beta';L_1L_2M_1)P_{L_1}^{M_1}(\theta_1)P_{L_2}^{-M_1}(\theta_2)\cos M_1\phi\Big]\Big\}, \tag{8}$$

where

$$\Theta\equiv\frac{i^{(l_1+l_2-l_1'-l_2')}}{4\pi}\left\{\frac{(2l_1+1)(2l_2+1)(2l_1'+1)(2l_2'+1)(2j_1+1)(2j_2'+1)}{(2L_1+1)(2L_2+1)}\right\}^{\frac{1}{2}}$$
$$\times C(l_1l_1'L_1;000)C(l_2l_2'L_2;000)W(l_1j_1l_1'j_1';\tfrac{1}{2}L_1)$$
$$\times\sum_{\mu_2}\{(-1)^{\frac{1}{2}+M}C(l_2j_1J;\mu_2m_1M)C(l_2'j_1'J';\mu_2'm_1'M)C(l_2l_2'L_2;-\mu_2\mu_2'-M_1)C(j_1j_1'L_1;m_1m_1'M_1)\},$$

$C(\cdots)$ are the Clebsch-Gordan coefficients, $W(\cdots)$ is the Racah coefficient, P_L^M are the associated Legendre functions, $\phi=\phi_1-\phi_2$, $\lambda\!\!\!^{-}$ is the de Broglie wavelength in the incident channel,[9] and the summations are taken between the limits

$$\begin{array}{ll} 0\leq l\leq\infty, & 0\leq l'\leq\infty, \\ |l-\tfrac{1}{2}|\leq J\leq l+\tfrac{1}{2}, & |l'-\tfrac{1}{2}|\leq J'\leq l'+\tfrac{1}{2}, \\ 0\leq l_1\leq\infty, & 0\leq l_1'\leq\infty, \\ |l_1-\tfrac{1}{2}|\leq j_1\leq l_1+\tfrac{1}{2}, & |l_1'-\tfrac{1}{2}|\leq j_1'\leq l_1'+\tfrac{1}{2}, \\ |J-j_1|\leq l_2\leq J+j_1, & |J'-j_1'|\leq l_2'\leq J'+j_1', \\ |l_1-l_1'|\leq L_1\leq l_1+l_1', & |l_2-l_2'|\leq L_2\leq l_2+l_2', \end{array}$$

$$-J,\ -J'\leq M\leq J,\ J',$$
$$-L_1,\ -L_2\leq M_1\leq L_1,\ L_2,$$
$$-l_2\leq \mu_2\leq l_2,$$

the remaining quantum numbers being given by

$$\mu_2'=\mu_2-M_1,$$
$$m_1=M-\mu_2,$$
$$m_1'=M-\mu_2'.$$

This result is derived in the appendix. The general term results from interference between the states labeled by the unprimed quantum numbers and those labeled by the primed ones. L_1, L_2, and M_1 are just indices describing the expansion of the angular distribution as a sum of products of spherical harmonics.

III. CHOICE OF REPRESENTATION

So far, we have not properly defined the angular momenta $\mathbf{l}_1$ and $\mathbf{l}_2$. If we work in some definite frame of reference, then momentum conservation leaves two arbitrary independent momenta to be specified in the final state. Let us denote these by $\mathbf{k}_1$ and $\mathbf{k}_2$. Then the statement that the final state is in an eigenstate of l_1, μ_1 and l_2, μ_2 corresponding to $\mathbf{k}_1$ and $\mathbf{k}_2$ is merely the statement that the angular dependence of its wave function is given by

$$Y_{l_1\mu_1}(\hat{k}_1)Y_{l_2\mu_2}(\hat{k}_2),$$

where $\hat{k}_1$ and $\hat{k}_2$ are the directions of $\mathbf{k}_1$ and $\mathbf{k}_2$. If we were really to take all states of the system into account, i.e., if we knew all of the quantities $S_{\alpha\beta}$, then the choice of β would not matter. Similarly, we have seen that we must choose whether to label the outgoing states by the eigenvalues of $\mathbf{j}_1$, $\mathbf{j}_2$, or $\mathbf{L}$; here again the choice would not matter if we knew all the $S_{\alpha\beta}$.

However, since we are trying to make a phenomenological analysis without assuming a detailed theory, we do not know any of the $S_{\alpha\beta}$, but try to find qualitative reasons for neglecting all but a few of them, so as to leave our result with as few arbitrary parameters as possible. Hence in choosing our definition of the β, we try to select them so that they differ in ways which correspond to qualitative physical differences. First of all, let us discuss the choice of $\mathbf{l}_1$ and $\mathbf{l}_2$. In terms of the angles θ_1, θ_2, ϕ which we have used above:

$$\cos\theta_1=\frac{\mathbf{k}'\mathbf{k}_1}{|\mathbf{k}||\mathbf{k}_1|},\quad \cos\theta_2=\frac{\mathbf{k}\cdot\mathbf{k}_2}{|\mathbf{k}||\mathbf{k}_2|},$$
$$\cos\phi=\frac{(\mathbf{k}\times\mathbf{k}_1)\cdot(\mathbf{k}\times\mathbf{k}_2)}{|\mathbf{k}\times\mathbf{k}_1||\mathbf{k}\times\mathbf{k}_2|},\quad (9)$$

where $\mathbf{k}$ is the momentum of the incident beam, whose direction is taken as the quantization axis.

The most convenient frame of reference is the center-of-mass frame defined by the incident beam and the struck nucleon. The simplest choices of $\mathbf{k}_1$ and $\mathbf{k}_2$ are then the momenta of the two outgoing pions in this frame of reference. This treats the two pions in a symmetrical fashion and also has one particular advantage from the point of view of experiment. In this case the angle ϕ, which is the angle between the production planes defined by the incident beam and each of the pions, is invariant under Lorentz transformation along the beam direction. This means that it is also the angle between the production planes in the laboratory system, which can be directly measured even in the case of photoproduction, when the center-of-mass frame varies.

However, it might be that the two pions are not emitted symmetrically but instead are emitted successively, with an intermediate "compound particle" existing in between. In this case the most sensible choice of $\mathbf{k}_2$ is the momentum of the second pion relative to the center of mass of the outgoing nucleon and second pion. Note that the direction of $\mathbf{k}_1$ in this case is the same as that in the previous case and is also the direction in which the "compound particle" moves in the total center-of-mass system. These two choices are illustrated in Fig. 1. They are not, of course, the only possible choices; which of the two is more reasonable should be made clear by the kinematics of the observed process.

Next we must consider the choice of coupling scheme. Here the question is whether we expect the production process to be more sensitive to the total angular momentum of the two pions, or to the total angular momentum of a pion-nucleon pair. If the former is the case, then this means that the production should depend strongly on the direct interaction between the two mesons. If we make the assumption that the direct pion-pion interaction is small, or even the much less severe assumption that it is of much shorter range than the pion-nucleon interaction, then it is convenient to make the analysis in terms of eigenstates of $\mathbf{j}_1$. This is also convenient because of the known strong dependence of the single meson-nucleon interaction on the total angular momentum.

[9] We have used the definition and normalization of reference 8 for our wave functions; S is a unitary matrix.

IV. EVALUATION OF THE ANGULAR DISTRIBUTION

Before trying to work out the coefficients for particular cases, let us examine some of the general features of the expression (8) for the cross section. This expression is of the form

$$d\sigma=\sum_{L_1L_2M_1} A_{L_1L_2M_1}P_{L_1}^{M_1}(\cos\theta_1)\times P_{L_2}^{-M_1}(\cos\theta_2)\cos M_1\phi, \quad (10)$$

where the $A_{L_1L_2M_1}$ are some complicated functions of the energies and momenta involved. We shall assume, where necessary, that we are discussing the first choice of the definition of $\mathbf{k}_1$ and $\mathbf{k}_2$ discussed in the last section. As we should expect, the cross section depends only on the *difference* in azimuth of the two pions, and is therefore invariant under rotation of the coordinate system about the z-direction (direction of the incident beam). The fact that it depends only on the cosine of multiples of ϕ is related to parity conservation.

Next we may examine which angular momentum states can contribute to each of the $A_{L_1L_2M_1}$. First of all, the Clebsch-Gordan coefficients $C(l_1l_1L_1;000)$ and $C(l_2l_2'L_2;000)$ vanish unless $(l_1+l_1'+L_1)$ or $(l_2+l_2'+L_2)$ are even, respectively. This greatly reduces the number of terms to be considered in any particular case. The physical interpretation of this rule is simply that interference terms between orbital angular momenta of opposite parities give rise to angular distributions involving spherical harmonics of odd order, and vice versa.

After this we observe that the coefficient of $P_{L_1}^{M_1}$, say, comes only from channels β, β' such that

$$l_1+l_1'\geq L_1, \quad (11)$$

and a similar rule holds for $P_{L_2}^{-M_1}$. If the observed cross section in a given energy region is observed to involve only low values of L_1, L_2, then we can assume that in this region high values of l_1, l_2, l_1', l_2' are not important, and ignore their contributions in all terms One way of examining this is by looking at the cross section as a function of the angle ϕ, which, as pointed out above, should be particularly easy experimentally, since this angle can be measured directly in the laboratory system. If this distribution is analyzed into its Fourier components, then the highest value of L_1 which is important (L_{1m}) and the highest value of L_2 (L_{2m}) must both be greater than or equal to the highest Fourier component which occurs (M_m):

$$L_{1m}\geq M_m; \quad L_{2m}\geq M_m.$$

In the absence of accidental cancellations, in fact, we should expect M_m to be equal to the smaller of L_{1m}, L_{2m}. If one looks for production of pairs of mesons having roughly the same energy, then clearly we expect

$$L_{1m}=L_{2m}=M_m.$$

In other words, making use of (11), the highest value of M observed in the angular distribution as a function of ϕ should be twice the maximum pion orbital angular momentum which contributes appreciably to the production at the energies involved.

At energies not too far above threshold, one would expect the most important part of the cross section to be due to s- and p-state pions. If no higher states are involved, then, by the above argument the highest value of M which should be observed is 2, and the values of L_1 and L_2 which will occur are 0, 1, and 2. In Table I the various possible values of l_1, l_2, l_1', l_2' which can contribute to each of the $A_{L_1L_2M_1}$ are tabulated for this case.

Up to this point, we have made no real approximations except to neglect high values of l_1, l_2, l_1', l_2', which, as we have seen, is an assumption which can be checked experimentally. However, if we consider all the possible values of J and j_1 going with the above values of l_1, l_2, we find that the complete expression involves 10 complex amplitudes $S_{\alpha\beta}$, or 19 real parameters. Even if we were to examine only the coefficient of $\cos2\phi$, there are still 5 amplitudes, or 9 parameters. (These numbers apply for the case of production by pions; in the case of photoproduction, where both parities are present in the initial state, there are even more.) Hence it is clear that we must make some further approximation.

The assumption that we shall make is that the mesons produced in p states interact with the nucleon only through the resonant state ($j=\frac{3}{2}$). This also involves ignoring the direct pion-pion interaction in comparison with that between pion and nucleon (or taking it to be of much shorter range: see above). This is a fairly severe assumption which should be carefully considered. In the region of the resonance energy, i.e., when the energy of the pion "relative to the nucleon" (measured in the center-of-mass system of the pion and nucleon) is near the energy of the pion-nucleon scattering resonance, then the approximation should be quite good. However, we must remember that this is now a statement about the angular momentum measured in the center-of-mass system of the pion and nucleon, whereas the rest of our calculations and observations refer to the center-of-mass system of all three particles. This

TABLE I. Values of (l_1,l_2,l_1',l_2') which can contribute to each of the coefficients $A_{L_1L_2M_1}$.

M_1	0			1			2		
$L_1 \backslash L_2$	0	1	2	0	1	2	0	1	2
0	(0000) (0101) (1010) (1111)	(0001) (0100) (1011) (1110)	(0101) (1111)	...	...	...	...	...	...
1	(0010) (1000) (0111) (1101)	(0011) (1100) (0110) (1001)	(0111) (1101)	...	(0110) (1001)	(0111) (1101)	...	...	...
2	(1010) (1111)	(1011) (1110)	(1111)	...	(1011) (1110)	(1111)	...	...	(1111)

should not have very much effect in practice for the case of resonance pions. For in this case the maximum velocity of the Lorentz transformation between these two frames of reference is about $c/10$, whereas a pion at resonance is fairly strongly relativistic. In other words, the angular dependence of the pion wave function will be amost the same in the two systems, and a pure $j=\frac{3}{2}$ state in one system will be mainly $j=\frac{3}{2}$ in the other system also. The approximation does break down at energies far from the resonance; however these regions should give relatively small contributions.

At this point we may compare some points of this discussion with the specific calculations of Cutkosky and Zachariasen.[5] They make a calculation using the Chew-Low theory for the specific case when one s- and one p-state meson are produced, and ignore all other cases. This seems a reasonable approximation if one pion is produced with very low energy and the other near resonance. However, to neglect the production with both mesons in p states implies that the amplitude for production of the pion with the lower energy in a p state is small, and hence the amplitude for such production with the near-resonance pion in an s state should be at least as small as that for two p-state pions (and probably much smaller). Hence, in Eq. (17) of reference 5 only one of the three terms should be appreciable, if the approximations are valid; it will be either the first or the second according as the near-resonance pion is positive or negative. In our notation this corresponds to saying that we must only allow

$$L_1=0; \quad L_2=0,\ 2,$$

(2 denoting the higher energy pion) but that the interference term with $l_1=1$, $l_2=0$, $l_1'=0$, $l_2'=1$ should be as small as the other neglected terms. The angular distribution as a function of θ_2 can be calculated either by substitution into the general expression (8) or, more simply, by noting that the only effect of the emission of the s-state pion is to change the parity of the system with no effect on the angular distribution. In other words, the distribution will be the same as that for the production of a single p-state scalar pion. For photoproduction this can be obtained from the usual expression for single production by interchanging electric and magnetic amplitudes, giving

$$|E_1|^2(2+3\sin^2\theta_2)+|M_2|^2(3+3\cos^2\theta_2) +2\sqrt{3}\,\mathrm{Re}(E_1{}^*M_2)(3\cos^2\theta_2-1), \quad (12)$$

where E_2 and M_2 are the electric dipole and magnetic quadrupole amplitudes, respectively, (assuming only $J=\frac{3}{2}$ occurs). For production by pions the angular distribution is of the form

$$1+3\cos^2\theta_2. \quad (12a)$$

The predictions of reference 5, which are based on a static, cut-off model, only include the electric dipole term. In general one would expect the magnetic quadrupole term to be much smaller (it should be much less important than the electric quadrupole term for single production). Since in this case the interference term would be the important one, and if the phases of the two amplitudes are suitably related it has the same sort of angular dependence as the electric dipole term, it would be difficult to detect anything but a very large magnetic quadrupole contribution.

V. EFFECT OF THE "SECOND RESONANCE"

When we consider the production of two p-state pions, we are led to a very complicated distribution, for in general there are three possible values of J, all of which may interfere. However, it has been suggested[10] that nearly all the photoproduction phenomena in the region of 700-Mev photon energy, and also the pion-nucleon scattering at corresponding energies, can be fitted on the assumption that they all take place through a resonant state of the nucleon, lying above the known $J=\frac{3}{2}$ level and below the rather higher (probably $J=\frac{5}{2}$) level observed in the scattering.[11] Amongst the phenomena which can be observed in this region is the production of two p-state pions, and it seems worthwhile to examine the consequences. This introduces a great simplification into the calculation, for it means that we only allow one value of J, though this may be any of the three values.

If the analysis is done for the coefficient of $\cos 2\phi$ in the distribution (again assuming only $j_1=\frac{3}{2}$ occurs), then we find the following result for the coefficient of $\sin^2\theta_1\sin^2\theta_2\cos 2\phi$ (for photoproduction):

$$\begin{aligned} J=\tfrac{1}{2}, &\quad (3/8)|M_1|^2, \\ =\tfrac{3}{2}, &\quad -(1/5)(3|M_1|^2-4/3\,\mathrm{Re}(M_1{}^*E_2)+9|E_2|^2), \\ =\tfrac{5}{2}, &\quad (9/20)(|E_2|^2+(4\sqrt{2}/3)\,\mathrm{Re}(M_3{}^*E_2)+2|M_3|^2), \end{aligned} \quad (13)$$

where M_1, E_2, M_3 are the amplitudes for magnetic dipole, electric quadrupole, and magnetic octupole transitions, respectively. For production by pions the form is similar but with only the "magnetic" terms. The significant feature of (13) is that for $J=\frac{1}{2},\frac{5}{2}$, this coefficient is positive, while for $J=\frac{3}{2}$ it is negative, whatever the relative phases of E_2, M_1, M_3. Furthermore, for $J=\frac{3}{2},\frac{5}{2}$, production of two s-state pions cannot occur, while for $J=\frac{5}{2}$ production of one s- and one p-state pion cannot occur. For the s- and p-production in the case when $J=\frac{3}{2}$, we now make no further assumption in assuming $j_1=\frac{3}{2}$, and therefore the angular distributions should be strictly as given in (12). In other words, if there is a $J=\frac{3}{2}$ resonance then near the resonance, if both pions are produced with energies in the region of 150 Mev relative to the nucleon, the distribution should have a term in $\cos 2\phi$ whose coefficient should be negative.

[10] R. R. Wilson (private communication).
[11] Cool, Piccioni, and Clark, Phys. Rev. **103**, 1082 (1956).

Hence, if we assume that the resonance exists, it should be quite possible to determine its angular momentum, provided its parity is even so that $2p$ production is possible.

VI. SUMMARY

We have seen that it is quite possible to derive a general expression (8) for the production cross section for pion pairs in terms of unknown complex amplitudes or matrix elements. By making various approximations and assumptions it was then possible to make some definite predictions without any detailed model of the production process. These conclusions may be summarized as follows:

For the production of two mesons of similar energies, if l is the maximum orbital angular momentum (of either one) which contributes appreciably to the production under the given conditions, then the angular distribution, as a function of their relative azimuth ϕ should have no Fourier components higher than $\cos 2l\phi$. This is quite generally true, whatever the choice of coordinate system, though in any particular case the value of l thus obtained will depend on the choice of reference system. Hence, if we assume the pion-pion interaction to have negligible effect, a measurement of this sort can give information about the angular momentum dependence of the *single* pion-nucleon system.

If only s and p states are important, which, as seen above, is an assumption which can be checked, then:

(a) If we assume that for a p-state pion, whose energy relative to the nucleon is near the resonance, only the $j=\frac{3}{2}$ state is important, then for the production of one low-energy pion and one near resonance the angular distribution should be given by (12).

(b) If we assume in addition that production takes place through a definite resonant state of the nucleon, then for the production of two pions with energies near resonance (both energies measured relative to the nucleon) the angular distribution should show an azimuthal dependence in which the coefficient of $\cos 2\phi$ is positive if $J=\frac{1}{2}$ or $J=\frac{5}{2}$, and is negative if $J=\frac{3}{2}$. If the latter is the case, then for the case mentioned in (a) the angular distributions given in (12) should be independent of any separate assumption that $j_1=\frac{3}{2}$.

Some estimate of the validity of assuming $j_1=\frac{3}{2}$ can be made by looking at the coefficient of $\cos\phi$. If this is negligible except at energies where there is also an appreciable term in $\cos 2\phi$, then the assumption is probably valid.

In conclusion, it is interesting to note that although the three-body system is far more complicated than the two-body one, and in general much more difficult to analyze, with a few assumptions it is possible to use the extra complication of azimuthal dependence to sort out different angular momentum states, which is much more difficult to do in the simpler two-body case.

ACKNOWLEDGMENTS

The author wishes to express his thanks to Dr. E. L. Lomon for having suggested the investigation and for helpful discussions, and also to Dr. H. A. Bethe and Dr. E. E. Salpeter for helpful comments and suggestions.

APPENDIX. DERIVATION OF THE GENERAL EXPRESSION

In the notation of the rest of the paper, we must find the wave functions ψ_β to insert in (2). These are the asymptotic wave functions of a nucleon and two pions in eigenstates of

$$J, M, l_1, l_2, s_f, j_1.$$

Following the normalization of reference 8, we have the asymptotic form of the eigenstates of $l_1 l_2 s_f \mu_1 \mu_2 m_{sf}$:

$$\Psi \underset{r_1,r_2\to\infty}{\sim} Y_{l_1\mu_1}(\theta_1,\phi_1)Y_{l_2\mu_2}(\theta_2,\phi_2)\chi_{s_f m_{sf}} \times[r_1 r_2(v_1 v_2)^{\frac{1}{2}}]^{-1}e^{i(k_1 r_1-\frac{1}{2}l_1\pi)}e^{i(k_2 r_2-\frac{1}{2}l_2\pi)},$$

where v_1 and v_2 are the velocities of the two outgoing particles, $Y_{l\mu}$ are spherical harmonics and χ is the spin wave function of the nucleon. We may write this as

$$\Psi\sim i^{l_1+l_2}Y_{l_1\mu_1}(\theta_1,\phi_1)Y_{l_2\mu_2}(\theta_2,\phi_2)F, \quad \text{(A1)}$$

where now F is independent of angle or orbital angular momentum. We now expand the states in which we are interested in terms of these states:

$$\Psi(l_1 l_2 J j_1 M)=\sum_{m_1\mu_1\mu_2} C(l_1 s_f j_1;\mu_1 m_{sf} m_1)C(l_2 j_1 J;\mu_2 m_1 M) \times i^{l_1+l_2}Y_{l_1\mu_1}(\theta_1,\phi_1)Y_{l_2\mu_2}(\theta_2,\phi_2)F, \quad \text{(A2)}$$

where $C(jj'j'';mm'm'')$ is the Clebsch-Gordan coefficient. In this way we find that, after substituting into (1), Eq. (2) becomes

$$d\sigma_{Mm_{sf}}=d\tau\sum_{\alpha\beta\alpha'\beta'}\sum_{m_1'\mu_1'\mu_2'}\sum_{m_1\mu_1\mu_2} A_\alpha^* A_{\alpha'} S_{\alpha\beta}^* S_{\alpha'\beta'} \times C(l_1 s_f j_1;\mu_1 m_{sf} m_1)C(l_2 j_1 J;\mu_2 m_1 M) \times C(l_1' s_f j_1';\mu_1' m_{sf} m_1')C(l_2' j_1' J';\mu_2' m_1' M) \times i^{(l_1'+l_2'-l_1-l_2)}Y_{l_1\mu_1}^*(\theta_1,\phi_1)Y_{l_2\mu_2}^*(\theta_2,\phi_2) \times Y_{l_1'\mu_1'}(\theta_1,\phi_1)Y_{l_2'\mu_2'}(\theta_2,\phi_2). \quad \text{(A3)}$$

Note that although we have written this as a sum over channels we do not sum over values of M. Owing to the invariance of the problem under change of coordinate system, the S-matrix elements cannot depend on the magnetic quantum numbers. Since the spin orientation of the initial or final nucleon, or the polarization of the incident beam could, in principle, be measured without interfering with the experiment, there can be no interference between states in which these differ. If these are not measured, then we must sum this expression over values of m_{sf} and average over values of M. Note also that the fact that the normalization of the wave functions included v_1 and v_2 means that the term $d\tau$ is simply $\lambda^2 d\Omega_1 d\Omega_2$.

Now we use the relation

$$Y_{l_1\mu_1}^*(\theta_1,\phi_1)Y_{l_1'\mu_1'}(\theta_1,\phi_1)$$
$$=(-1)^{\mu_1}\sum_{L_1=|l_1-l_1'|}^{l_1+l_1'}\sum_{M_1=-L_1}^{L_1}\left\{\frac{(2l_1+1)(2l_1'+1)}{4\pi(2L_1+1)}\right\}^{\frac{1}{2}}$$
$$\times C(l_1l_1'L_1;000)C(l_1l_1'L_1;-\mu_1\mu_1'M_1)$$
$$\times Y_{L_1M_1}(\theta_1,\phi_1), \quad \text{(A4)}$$

and hence obtain

$$\frac{d^2\sigma}{d\Omega_1 d\Omega_2}=\lambda\!\!\!^{-2}\sum_{\alpha\alpha'\beta\beta'}\sum_{L_1L_2}A_\alpha^*A_{\alpha'}S_{\alpha\beta}^*S_{\alpha'\beta'}i^{(l_1'+l_2'-l_1-l_2)}$$
$$\times(4\pi)^{-1}\left\{\frac{(2l_1+1)(2l_2+1)(2l_1'+1)(2l_2'+1)}{(2L_1+1)(2L_2+1)}\right\}^{\frac{1}{2}}$$
$$\times C(l_1l_1'L_1;000)C(l_2l_2'L_2;000)$$
$$\times\sum_{m_1m_1'\mu_1'\mu_2'}\sum_{M_1M_2\mu_1\mu_2}\{(-1)^{\mu_1+\mu_2}C(l_1s_fj_1;\mu_1m_{sf}m_1)$$
$$\times C(l_2j_1J;\mu_2m_1M)C(l_1's_fj_1';\mu_1'm_{sf}m_1')$$
$$\times C(l_2'j_1'J';\mu_2'm_1'M)C(l_1l_1'L_1;-\mu_1\mu_1'M_1)$$
$$\times C(l_2l_2'L_2;-\mu_2\mu_2'M_2)Y_{L_1M_1}(\theta_1\phi_1)Y_{L_2M_2}(\theta_2\phi_2)\}. \quad \text{(A5)}$$

This can be simplified as follows. First we use the sum rule for the product of three Clebsch-Gordan coefficients,[12] which for this particular case becomes:

$$\sum_{\mu_1\mu_1'm_{sf}'}(-1)^{\mu_1}C(l_1's_fj_1';\mu_1'm_{sf}m_1')$$
$$\times C(l_1s_fj_1;\mu_1m_{sf}m_1)C(l_1l_1'L_1;-\mu_1\mu_1'M_1)$$
$$=(-1)^{\frac{1}{2}+L_1-l_1'-l_1+2j_1-m_1}(2j_1+1)^{\frac{1}{2}}(2j_1'+1)^{\frac{1}{2}}$$
$$\times W(l_1j_1l_1'j_1';s_fL_1)C(j_1j_1'L_1;-m_1m_1'M_1), \quad \text{(A6)}$$

where $W(\cdots)$ is the Racah coefficient. Then we note that $C(l_1l_1'L_1;000)$ vanishes unless $L_1-l_1-l_1'$ is an even integer; also we write $2j_1-m_1=2(j_1-m_1)+m_1$ and note that $2(j_1-m_1)$ must be an even integer. Next we use the fact that $\mu_2+m_1=M=\mu_2'+m_1'$, or

$$\mu_2'-\mu_2=m_1-m_1'. \quad \text{(A7)}$$

Since the expression for $d\sigma/d\Omega_1d\Omega_2$ contains the two terms $C(j_1j_1'L_1;-m_1m_1'M_1)$ and $C(l_2l_2'L_2;-\mu_2\mu_2'M_2)$, it follows from (A7) that

$$M_2=-M_1.$$

Making use of all these results we obtain, finally:

$$\frac{d^2\sigma}{d\Omega_1 d\Omega_2}=\lambda\!\!\!^{-2}\sum_{\alpha\alpha'\beta\beta'}\sum_{L_1L_2M_1}A_\alpha^*A_{\alpha'}S_{\alpha\beta}^*S_{\alpha'\beta'}i^{(l_1'+l_2'-l_1-l_2)}(4\pi)^{-1}\left\{\frac{(2l_1+1)(2l_1'+1)(2l_2+1)(2l_2'+1)(2j_1+1)(2j_1'+1)}{(2L_1+1)(2L_2+1)}\right\}^{\frac{1}{2}}$$
$$\times C(l_1l_1'L_1;000)C(l_2l_2'L_2;000)W(l_1j_1l_1'j_1';s_fL_1)$$
$$\times\sum_{\mu_2}\{(-1)^{\frac{1}{2}+M}C(l_2j_1J;\mu_2m_1M)C(l_2'j_1'J';\mu_2'm_1'M)C(l_2l_2'L_2;-\mu_2\mu_2'-M_1)$$
$$\times C(j_1j_1'L_1;-m_1m_1'M_1)\}P_{L_1}^{M_1}(\theta_1)P_{L_2}^{-M_2}(\theta_2)e^{iM_1(\phi_1-\phi_2)}. \quad \text{(A8)}$$

Symmetry considerations show that only terms in $\cos M_1\phi$ can survive in the sum, and hence we obtain the expression reproduced in (8). The origin of the various limits on the summations are all straightforward.

[12] See, for example, M. Rose, *Elementary Theory of Angular Momentum* (John Wiley and Sons, Inc., New York, 1957), p. 110.