us consider all terms in the normalization integral $\langle\psi|\psi\rangle$ that have P pairs excited. These number $N!/(N-2P)!P!2^P$; $P\leq N/2$. Their average coefficient is $\langle\sum_{p_1p_2}|\langle m_1m_2|A|p_1p_2|^2\rangle_{\text{Av}}{}^P$; where the average is over hole states. Calling this latter coefficient ξ/N, we have for the contribution of all terms with P pairs excited the value

$$\frac{N!}{(N-2P)!P!N^P}\left(\frac{\xi}{2}\right)^P\cong\frac{N^P}{P!}\left(\frac{\xi}{2}\right)^P \quad \text{for} \quad \frac{P}{N}\ll 1. \tag{B-1}$$

Let us assume (B-1) to be valid for all P. Then (B-1) defines a Poisson distribution and we have for the mean fraction of excited pairs the value

$$\langle P\rangle_{\text{Av}}/\tfrac{1}{2}N=\xi. \tag{B-2}$$

If $\xi\ll 1$, then (B-1) is a good approximation. Further the relative dispersion about the $\langle P\rangle_{\text{Av}}$ term is small like $O(1)/N$, i.e., $\langle P^2\rangle_{\text{Av}}/\langle P\rangle_{\text{Av}}{}^2=1/\langle P\rangle_{\text{Av}}$. In this respect $\langle\psi|\psi\rangle$ is quite like the cluster expansion of the partition function in statistical mechanics where the relative dispersion about the mean cluster populations is again $O(1)/N$.

An expression for ξ is found by observing that for an isolated pair the wave function of the particles in the pair is

$$\psi_{k_1k_2}=\mathcal{A}[\phi_{k_1}\phi_{k_2}+\textstyle\sum_q\langle k_1k_2|A|k_1+q;k_2-q\rangle \times\phi_{k_1+q}\phi_{k_2-q}, \tag{B-3}$$

so that

$$\xi=\frac{1}{v}\int|\langle|\psi_{\text{pair}}|^2\rangle_{\text{Av}}-1| \tag{B-4}$$

$$=\frac{\text{volume over which a pair is correlated}}{\text{volume per nucleon}}. \tag{B-5}$$

Bethe and Goldstone[4] have shown that for a hard-core interaction of the correlation range is of the order of the core radius. Further, they expect that an attractive well outside the core will not change this substantially. This would give an estimate for ξ from (B-5) of $\xi\approx 1/10$.

PHYSICAL REVIEW VOLUME 111, NUMBER 5 SEPTEMBER 1, 1958

Longitudinal Polarization of Bremsstrahlung and Pair Production*

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(Received May 8, 1958)

The Born approximation bremsstrahlung (and pair production) cross sections, valid at all energies and angles, are given for all possible states of longitudinal polarization of the particles involved.

When the photon-incoming electron angle (θ_0) and the photon-outgoing electron angle (θ) are both zero, a cancellation of Feynman diagrams causes *all* cross sections to vanish in Born approximation. Further, if both θ_0 and θ are small compared to m^2/E_0^2, the "spin-flip" cross sections are small (of order θ^2) relative to the "non-spin-flip" ones. When account is taken of the above cancellation, angular momentum conservation is sufficient to determine this small-angle behavior, but it explains neither the sign nor the magnitude of the bremsstrahlung circular polarization.

SINCE the recognition by Goldhaber *et al.*[1] that the circular polarization of bremsstrahlung can serve as a useful means of measuring longitudinal electron polarization, considerable interest has developed in the bremsstrahlung cross sections for specific polarization states of the incoming and outgoing particles. Although several cross-section calculations are now available,[2] little emphasis has been given to the differential cross sections for the most general combinations of longitudinal polarization, nor has an attempt been made to understand the physical origin of the polarization effects. We shall present these cross sections in detail, and call attention to their seemingly anomalous behavior at small angles. The explanation of this anomaly provides some physical insight into the details of the process, and brings to light the rather unexpected role played by orbital angular momentum in the polarization phenomena.

In both bremsstrahlung and pair production, the total number of incoming and outgoing particles (apart from the static nucleus) is three, and since each particle has two states of longitudinal polarization, there are eight possible cross sections. However, by Lenard's theorem,[3] two cross sections which differ from each other only in having all three spin directions reversed are equal in Born approximation, so to this approximation there are only four distinct cross sections. If only one of the outgoing particles is to be

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] Goldhaber, Grodzins, and Sunyar, Phys. Rev. **106**, 826 (1957).

[2] A definitive calculation of those integrated cross sections which are of most immediate experimental interest has recently been supplied by C. Fronsdal and H. Überall [Phys. Rev. **111**, 580 (1958)], which contains references to previous calculations.

[3] A. Lenard, Phys. Rev. **107**, 1712 (1957).

observed, then we are interested in the cross section which is summed over the two spin states of the other outgoing particle, and it is these (two) cross sections which are given in reference 2. However, if one wishes to use these processes as sources of longitudinally polarized outgoing particles, or if the longitudinal polarization is to be followed in the particles of a developing electromagnetic cascade,[4] then the four cross sections, in which the polarization states of all three particles are specified, are of interest. The relativistic limits of these cross sections were published some time ago[5]; the expressions given below are valid for all energies and angles to which the Born approximation is applicable. Screening has been neglected in these expressions, but can be included merely by modifying the q^{-4} factor; it becomes important only for relativistic particles, where its effects are those given in reference 2 and reference 5.

I. BREMSSTRAHLUNG

We shall specify the spin states of the particles involved in bremsstrahlung by the helicities ε_0, ε, and δ,[6] for the incoming electron, outgoing electron, and photon, respectively. They take on only the values ± 1, $+1$ referring in each case to a "forward-spin" particle, i.e., one whose spin and momentum define a right-handed screw. We use the notation of Bethe and Heitler,[7] in which $(\mathbf{p}_0,E_0)$, $(\mathbf{k},k)$, and $(\mathbf{p},E)$ are the momentum and energy of the incident electron, photon, and outgoing electron, respectively, and $c=1$. Also θ_0 is the $(\mathbf{p}_0,\mathbf{k})$ angle, θ the $(\mathbf{p},\mathbf{k})$ angle, α the $(\mathbf{p}_0,\mathbf{p})$ angle, and ϕ the angle between the $(\mathbf{p}_0\mathbf{k})$ and $(\mathbf{p},\mathbf{k})$ planes. $E_0=E+k$ and $\mathbf{q}=\mathbf{p}_0-\mathbf{p}-\mathbf{k}$. The four cross sections can be written in the following unified fashion:

$$d\sigma(\varepsilon_0,\delta,\varepsilon)=\frac{Z^2e^4}{137}\frac{p}{p_0}\frac{dk}{k}\frac{d\Omega_0 d\Omega}{(2\pi)^2}\frac{1}{q^4}X(\varepsilon_0,\delta,\varepsilon),$$

$$\begin{aligned}
X(\varepsilon_0,\delta,\varepsilon)=&\frac{1}{(E_0-p_0\cos\theta_0)^2}\{p_0^2\sin^2\theta_0(EE_0+m^2+\varepsilon\varepsilon_0 pp_0)(1+\varepsilon\varepsilon_0\cos\alpha)\\
&+k^2(E_0-\delta\varepsilon_0 p_0)(E-\delta\varepsilon p)(1-\delta\varepsilon\cos\theta)(1+\delta\varepsilon_0\cos\theta_0)\\
&+\delta\varepsilon p_0 k[(E_0-\delta\varepsilon_0 p_0)(E-\delta\varepsilon p)+m^2][(1+\delta\varepsilon_0\cos\theta_0)\sin\theta\sin\theta_0\cos\varphi+\varepsilon\varepsilon_0(1-\delta\varepsilon\cos\theta)\sin^2\theta_0]\}\\
&+\frac{1}{(E-p\cos\theta)^2}\{p^2\sin^2\theta(EE_0+m^2+\varepsilon\varepsilon_0 pp_0)(1+\varepsilon\varepsilon_0\cos\alpha)\\
&+k^2(E_0+\delta\varepsilon_0 p_0)(E+\delta\varepsilon p)(1-\delta\varepsilon\cos\theta)(1+\delta\varepsilon_0\cos\theta_0)\\
&+\delta\varepsilon_0 pk[(E_0+\delta\varepsilon_0 p_0)(E+\delta\varepsilon p)+m^2][(1-\delta\varepsilon\cos\theta)\sin\theta\sin\theta_0\cos\varphi+\varepsilon\varepsilon_0(1+\delta\varepsilon_0\cos\theta_0)\sin^2\theta]\}\\
&-\frac{1}{(E_0-p_0\cos\theta_0)(E-p\cos\theta)}\{2(EE_0+m^2+\varepsilon\varepsilon_0 pp_0)(1+\varepsilon\varepsilon_0\cos\alpha)pp_0\sin\theta\sin\theta_0\cos\varphi\\
&+2k^2m^2(1-\delta\varepsilon\cos\theta)(1+\delta\varepsilon_0\cos\theta_0)\\
&+\delta\varepsilon p_0 k[(E_0+\delta\varepsilon_0 p_0)(E+\delta\varepsilon p)+m^2][(1+\delta\varepsilon_0\cos\theta_0)\sin\theta\sin\theta_0\cos\varphi+\varepsilon\varepsilon_0(1-\delta\varepsilon\cos\theta)\sin^2\theta_0]\\
&+\delta\varepsilon_0 pk[(E_0-\delta\varepsilon_0 p_0)(E-\delta\varepsilon p)+m^2][(1-\delta\varepsilon\cos\theta)\sin\theta\sin\theta_0\cos\varphi+\varepsilon\varepsilon_0(1+\delta\varepsilon_0\cos\theta_0)\sin^2\theta]\}.
\end{aligned}\tag{1}$$

The validity of Lenard's theorem[3] is evident from the fact that the helicity coefficients appear only bilinearly, as $\varepsilon_0\varepsilon$, $\varepsilon_0\delta$, or $\varepsilon\delta$.

As for comparisons with previously published cross sections, the sum of all four cross sections is just the Bethe-Heitler expression, as it should be. In addition, if only the spin-states of the outgoing electron are summed over, the resulting expression agrees with Eq. (13) of reference 2, for the special case of $\mathbf{s}$ parallel to $\mathbf{p}_1$ in that equation.

[4] F. J. Dyson, Ann. Phys. (to be published).
[5] K. W. McVoy and F. J. Dyson, Phys. Rev. **106**, 1360 (1957).

II. PAIR PRODUCTION

The pair production cross sections are obtainable from those for bremsstrahlung, of course, by the substitution law,[8] but since spin states are involved, we shall describe the substitution in detail. If we take as energy and momenta $(k',\mathbf{k}')$, $(E_-,\mathbf{p}_-)$, and $(E_+,\mathbf{p}_+)$ for the photon, electron, and positron, respectively, the

[6] For more explicit definitions, see, e.g., K. W. McVoy, Phys. Rev. **106**, 828 (1957).
[7] See, e.g., W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, Oxford, 1947), second edition, p. 164.
[8] See, e.g., J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Press, Cambridge, 1955), p. 161.

simplest substitution seems to be

$$\mathbf{k}'=\mathbf{k},\ k'=k;\quad \mathbf{p}_-=\mathbf{p}_0,\ E_-=E_0;\quad \mathbf{p}_+=-\mathbf{p},\ E_+=-E;$$
$$\delta'=\delta,\ \varepsilon_-=\varepsilon_0,\text{ and }\varepsilon_+=-\varepsilon, \tag{2}$$

where ε_+ is the helicity of the physical positron (not of the "hole"). In addition, Eq. (1) must be multiplied by p^2dE_+/k^2dk.

As for the generality of these expressions, they are, of course, not restricted to beams of completely longitudinally polarized particles, but are applicable to any beam of incoming particles described by a spin density matrix which is diagonal (with respect to eigenstates of definite helicity). Physically, for an electron beam this is one in which all electron spins make the same angle with the momentum direction, but are distributed randomly over azimuthal angle. This is indeed just the type of beam which occurs most commonly in β decay. The above cross sections can than be employed in the following way. Let σ_R and σ_L be the bremsstrahlung cross sections given above for incoming right and left electrons, both leading to the same final states. Then the corresponding cross section for a beam of incoming electrons of the above type in which $|a_R|^2$ and $|a_L|^2$ are the diagonal elements of the spin density matrix $(|a_R|^2+|a_L|^2=1)$ is,

$$\sigma=|a_R|^2\sigma_R+|a_L|^2\sigma_L.$$

Since the measurement of the spin states of both outgoing particles implies a measurement of their production-angles as well, only the differential cross sections are of interest in this case, and there seems to be no point in integrating them over the directions of the outgoing particles.[9]

III. THE "EXPLANATION" OF THE BREMSSTRAHLUNG POLARIZATION IN TERMS OF ANGULAR MOMENTUM CONSERVATION

The "naive" explanation of the high degree of circular polarization observed in bremsstrahlung from relativistic, longitudinal electrons goes as follows. Let us label the four cross sections by helicity suffixes, σ_{ABC}, denoting the helicities of the incident electron, photon and outgoing electron, reading from left to right. Each suffix can be either R or L, "R" denoting a "right particle" or "spin-forward" particle. Now the bremstrahlung from relativistic electrons is emitted mostly at small angles, so let us consider the special case in which both the photon and outgoing electron momenta are parallel to that of the incident electron. If we quantize angular momentum along this axis, none of the particles have components of orbital momentum along the axis, so their spin components along it must be conserved. For an initial R electron, $J_z=S_z=+\frac{1}{2}$ along this axis. This can be achieved in the final state only by adding $+1$ from the photon to $-\frac{1}{2}$ from the electron, so all cross sections except σ_{RRL} must be zero in this case, and the bremsstrahlung helicity should be the same as that of the incident electron.

This argument, however, is invalid, for σ_{RRL} is *also* zero, and in Born approximation there is no radiation at all, when $\mathbf{p}_0$, $\mathbf{p}$, and $\mathbf{k}$ are parallel. If the electron had no spin, this result *would* be a consequence of angular momentum conservation. The initial and final "electrons" would have $J_z=0$, and could only radiate a photon with $J_z=S_z=0$; but $S_z=\pm1$ for the electromagnetic radiation field, so the process would be forbidden in all orders. Dirac electrons, however, can conserve angular momentum in the process by spin-flip, and there seems to be no conservation law which forbids the radiation process in this case. Its nonoccurrence appears to be a peculiarity of the Born approximation, and can be traced to a cancellation between the two Feynman diagrams which contribute in this order. It can readily be seen (e.g., from the expressions given in reference 6) that both diagrams are zero in each of the cases (*RRR*), (*RLR*), and (*RLL*) if $\mathbf{p}_0$, $\mathbf{p}$, and $\mathbf{k}$ are parallel. For the case (*RRL*), however, which is allowed by angular momentum conservation, neither diagram is zero when all momenta are parallel, but the two cancel exactly in this case.[10]

Furthermore, the above "naive argument," invalid for $\theta_0=\theta=0$, does not even retain approximate validity for small θ_0 and θ. This is because the entire matrix element is infinitesimal in this limit, so that the $L_z\neq0$ contributions, while infinitesimal, are not negligible, and orbital angular momentum actually plays an important role as we shall show explicitly below.

However, angular momentum conservation, when applied carefully, does explain certain important properties of the cross sections of Eq. (1). Specifically, the angular dependence of the cross sections in the small-angle limit can be explained in this way, but the polarization of the bremsstrahlung *cannot*. Before presenting the argument, the exact limiting results we wish to explain are the following. Consider the limit $\theta_0\to0$, $\theta\to0$ of Eq. (1), and keep only the leading terms in ϵ, where we shall let ϵ stand for any of the small quantities θ_0, θ, and α. None of the cross sections remain finite in this limit (which is clear from the fact that the Bethe-Heitler cross section is proportional to ϵ^2 in this

[9] For very relativistic electrons, the outgoing particles are very much forward, and the average cross section for a given polarization state may be of use. The integrations for this limiting case were given in reference 5, and an example of the their application is reference 4.

[10] This cancellation will occur for bremsstrahlung in *any* central field, providing the recoil energy of the nucleus is neglected. Peculiarly enough, this effect persists in bremsstrahlung even when the electrons are described by Sommerfeld-Maue-Furry wave functions, although it does not persist in pair production calculated in this way. Since the electron-positron interaction is not included in this calculation, the same dynamical conservation laws should apply to both the bremsstrahlung and pair production processes in this approximation. The fact that they behave differently when $\mathbf{p}_0$, $\mathbf{p}$, and $\mathbf{k}$ are parallel would thus seem to indicate that the behavior cannot be due to a conservation principle. For these calculations see H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954).

limit), and the individual cross sections have as leading terms,

$$\sigma_{RRR}\sim\epsilon^2, \quad \sigma_{RRL}\sim\epsilon^4, \qquad \sigma_{RLR}\sim\epsilon^2, \quad \sigma_{RLL}\sim\epsilon^4. \tag{3}$$

In other words, the cross sections involving no electron spin-flip are larger, at small angles, than those with spin-flip. This is true at all energies, and disagrees with what one would expect by extending the above "naive" argument from zero angle to small angles, for σ_{RRL} is actually one of the "small" cross sections. (However, the forward radiation from relativistic electrons *does* have predominantly the same helicity as the incident electron. This is because σ_{RRR} is the largest of the four cross sections, and is not a consequence of angular momentum conservation, as will be seen below.)

These small-angle results can be obtained very simply in the nonrelativistic case, which is worth a few moments' consideration. If we suppose for the purpose of illustration that Z is so small that $Ze^2/\hbar v\ll 1$ even for a nonrelativistic electron, we can use the Born approximation, in which case the angular dependence of the matrix element (for arbitrary θ_0 and θ) has the well-known form

$$q^{-4}[(\mathbf{p}_0-\mathbf{p})\cdot\mathbf{e}](u_0,u),$$

where $u_0(\mathbf{p}_0)$ and $u(\mathbf{p})$ are the Pauli (2-dimensional) spinors for states of definite helicity, and $\mathbf{e}=(\mathbf{e}_x+i\delta\mathbf{e}_y)/\sqrt{2}$ for a photon travelling along the z axis ($\delta=\pm1$ is the photon helicity). From this we see immediately that, if $\mathbf{p}_0$, $\mathbf{p}$ and $\mathbf{k}$ are parallel, then $(\mathbf{p}_0-\mathbf{p})\cdot\mathbf{e}=0$, and the process cannot occur at all, just as a dipole cannot radiate (transverse) photons along its own axis. Secondly, if we note that

$$u_R(\mathbf{p}_0)^\dagger u_R(\mathbf{p})\sim1, \quad u_R(\mathbf{p}_0)^\dagger u_L(\mathbf{p})\sim\alpha, \qquad \mathbf{p}_0\cdot\mathbf{e}\sim\theta_0, \quad \mathbf{p}\cdot\mathbf{e}\sim\theta, \tag{4}$$

as $\epsilon\to0$, we immediately get the results (3).[11]

To see that the small-angle behavior (at all energies) does indeed follow from angular momentum conservation,[12] let us consider the bremsstrahlung matrix element as an element of the S matrix,

$$\mathfrak{M}=\langle f|S|i\rangle. \tag{5}$$

The only property of the S matrix we need is $[S,J_z]=0$, i.e., $J_z=L_z+s_z$ is conserved. Since the nucleus is treated in the static approximation and exerts only a central force on the electron, it can absorb no angular momentum, so in (f) only the J_z of the photon and outgoing electron need be considered.

For simplicity consider the special (but entirely representative) case in which $\mathbf{k}$, taken along the z axis, is parallel to $\mathbf{p}_0$ but not to $\mathbf{p}$ (Fig. 1). The argument then proceeds exactly as before, except that $\theta\neq0$, so the outgoing electron carries both orbital and spin angular momentum components along the z axis. If the initial electron and the photon helicities are ε_0 and δ, the outgoing electron must carry total angular momentum $J_z=(\varepsilon_0/2-\delta)$ along the z axis, so if its wave function is expanded into J_z eigenstates, only that part with $J_z=(\varepsilon_0/2-\delta)$ can contribute to $\mathfrak{M}$.

In order to accomplish this expansion, we can use the following spinor transformation. If $u_R(\mathbf{p}), u_L(\mathbf{p})$ are the eigenstates of $\boldsymbol{\sigma}\cdot\mathbf{p}$, and u_+, u_- the eigenstates of σ_z, then

$$u_R(\hat{p})=A_+\cos\tfrac{1}{2}\theta u_+ + A_-\sin\tfrac{1}{2}\theta e^{i\phi}u_-, \qquad u_L(\hat{p})=-A_-\sin\tfrac{1}{2}\theta e^{-i\phi}u_+ + A_+\cos\tfrac{1}{2}\theta u_-, \tag{6}$$

where $A_\pm$ are real numbers, independent of θ and ϕ, the polar angles of $\mathbf{p}$.

The desired expansion of the outgoing electron state is then

$$u(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{r}}=4\pi\sum_{lm} i^l j_l(pr)Y_{lm}^*(\hat{p})Y_{lm}(\hat{r})[\alpha u_+ + \beta u_-], \tag{7}$$

where $\alpha(\hat{p}), \beta(\hat{p})$ are the expansion coefficients of Eq. (6). Since $Y_{lm}(\hat{r})u_+$ and $Y_{lm}(\hat{r})u_-$ are J_z eigenstates with $M=m+\frac{1}{2}$ and $M=m-\frac{1}{2}$, respectively, we can immediately read off the terms which will contribute to $\mathfrak{M}$ for each of the four cross sections. This in turn gives the dependence of $\mathfrak{M}$ on the direction of $\mathbf{p}$ through the factors $Y_{lm}(\hat{p})$, $\alpha(\hat{p})$ and $\beta(\hat{p})$. It is convenient to define

$$\psi_m(\hat{p})=4\pi\sum_l i^l j_l(pr)Y_{lm}^*(\hat{p})Y_{lm}(\hat{r})\to\theta^m e^{-im\phi}, \tag{8}$$

the limiting expression giving its behavior as $\theta\to0$.

When (7) is substituted into (5), the conservation of J_z then gives the following nonzero contributions to $\mathfrak{M}$, for the four polarization states:

(ABC)	J_z eigenvalues $(\varepsilon_0/2-\delta)$	J_z eigenfunctions $[Y_{lm}(\hat{r})u_\pm]$	Angular dependence of $\mathfrak{M}(\mathbf{p})$ (General θ)	$(\theta\to0)$
RRR	$-\frac{1}{2}$	$Y_{l,0}u_-;\ Y_{l,-1}u_+$	$\psi_0\sin(\frac{1}{2}\theta)e^{i\varphi};\ \psi_{-1}\cos(\frac{1}{2}\theta)$	$\theta e^{i\varphi}$
RRL			$\psi_0\cos(\frac{1}{2}\theta);\ \psi_{-1}\sin(\frac{1}{2}\theta)e^{-i\varphi}$	$1;\ \theta^2$
RLR	$-\frac{3}{2}$	$Y_{l,1}u_+;\ Y_{l,2}u_-$	$\psi_1\cos(\frac{1}{2}\theta);\ \psi_2\sin(\frac{1}{2}\theta)e^{i\varphi}$	$\theta e^{-i\varphi}$
RLL			$\psi_1\sin(\frac{1}{2}\theta)e^{-i\varphi};\ \psi_2\cos(\frac{1}{2}\theta)$	$\theta^2 e^{-2i\varphi}$.

(9)

[11] Incidentally, we might note that $(\mathbf{p}_0-\mathbf{p})\cdot\mathbf{e}=\mathbf{q}\cdot\mathbf{e}=q_x+i\delta q_y$. Consequently $|\mathbf{q}\cdot\mathbf{e}|^2=q_x^2+q_y^2$ and is independent of δ, the photon's helicity, so the bremsstrahlung from nonrelativistic electrons has *no* circular polarization, now matter hat its angle of emission.

[12] My thanks to Dr. J. Weneser, who suggested this form of the argument.

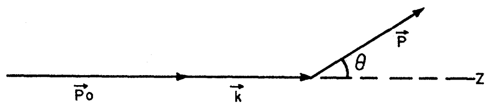

FIG. 1. Momentum relations for the special case under consideration.

In the small-angle limit, (RRL) would appear to remain finite and thus be the largest of the cross sections, as the "naive" argument suggested. However, we know by explicit calculation that *all* matrix elements are zero when $\theta=0$, so the contribution to the (RRL) matrix element from the $m=0$ term of Eq. (7) must in fact be identically zero in Born approximation. This is another way of describing the unexpected cancellation of Feynman diagrams discussed above, and does not seem to have a physical interpretation. Once we recognize the cancellation, though, and discard the "1" term, we see that $(RRL)\sim\theta^2$, and the angular momentum argument does reproduce the results (3) obtained in Born approximation. For the range of validity of Eq. (9) see Appendix I.

In summary, if we keep only terms through ϵ^2 in the small-angle limit, the expansions of the bremsstrahlung cross sections are[5]

nonrelativistic case:

$$[d\sigma_{RRR},d\sigma_{RRL},d\sigma_{RLR},d\sigma_{RLL}]\sim[1,0,1,0]\epsilon^2 d\Omega_0 d\Omega;$$

$$(10)$$

extreme relativistic case:

$$[d\sigma_{RRR},d\sigma_{RRL},d\sigma_{RLR},d\sigma_{RLL}]\sim[E_0^2,0,E^2,0]\epsilon^2 d\Omega_0 d\Omega.$$

Although this limiting angular dependence is a consequence of angular momentum conservation, the sign and magnitude of the bremsstrahlung's circular polarization arise from the fact that $\sigma_{RRR}\geq\sigma_{RLR}$, which is not obtainable from an angular momentum argument.

APPENDIX I. RANGE OF VALIDITY OF THE SMALL-ANGLE EXPANSIONS

It should be noted that the range of applicability of the small-angle expressions (3) and (9) is actually quite small, for the neglected terms in these expansions become significant even at very small angles. This is of most interest for very relativistic electrons, in which case the expansions are valid, as we shall see, only in the angular range given approximately by

$$\epsilon<(m^2/E_0^2). \qquad \text{(A-1)}$$

At these energies the determining factor is q^{-4}, and the Taylor expansion of q^2 begins with

$$q^2\approx(m^2k/2EE_0)^2+p_0^2\theta_0^2+p^2\theta^2-2pp_0\theta\theta_0\cos\varphi. \qquad \text{(A-2)}$$

If we consider E_0, E and k to be of the same order of magnitude, the ϵ^2 terms clearly are negligible only if $\epsilon\ll(m^2/E^2)$, and since the ϵ^2 terms *were* neglected in (3), this expansion cannot be used at larger angles. This is made somewhat clearer if we recognize that the maximum of the cross section comes at $\epsilon\sim m^2/E_0^2$, for it is apparent that the simple behavior of (3) cannot hold beyond this maximum. Since the cross section remains significant out to $\epsilon\sim m/E_0$, we see that the expansion (3) holds over only a very small fraction of the significant angular range.

From another point of view, the angular momentum argument, which was employed to derive the small-angle approximation of Eqs. (9), itself implies a severe restriction on the range of validity of this approximation. The limiting behavior was obtained from a small-angle expansion of $Y_{lm}(\theta,\varphi)$. For $m<l/2$, which is true in the cases under consideration, the required expansion has the general form

$$P_l^m(\cos\theta)\sim\theta^m[1+l^2\theta^2+l^4\theta^4+\cdots], \qquad \text{(A-3)}$$

so that the range of validity of the approximation $P_l^m\sim\theta^m$, which we used in deriving (9), is $\theta<1/l$. Since in fact we employed this approximation in the sum (8), this implies the strong restriction $\theta<1/L$, if the major contributions to (8) came from $l<L$.