a spin of 9/2+, 7/2+, or 5/2+ may be assigned to the corresponding level. All that can be said about the characteristics of the principal gamma radiations is that they are very little converted and the measurements of the internal conversion coefficients by Deutsch and Hedgran[8] or Caird and Mitchell[9] are not accurate enough to draw any conclusions from them.

ACKNOWLEDGMENTS

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Variational Methods and the Nuclear Many-Body Problem*

R. BROUT
Department of Physics, Cornell University, Ithaca, New York
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The general form of the energy of the ground state of a many-fermion system is shown to be exactly of the form proposed by Brueckner and Bethe, without approximation. In a variational treatment, if the trial wave function is picked containing only pair correlations, together with all possible unlinked pairs, it is described by a two-body excitation matrix $\langle m_1 m_2 | A | p_1 p_2 \rangle$. Variation of this matrix in the Ritz-Rayleigh principle yields a set of integral equations of the scattering type for the matrix A. Hole-state energies are given self-consistently in terms of the matrix A, but particle-state energies are Hartree-Fock energies. This may be corrected for by widely enlarging the class of terms admitted into the wave function. If the approximation is then made of omitting a class of terms, defined as cross-linked clusters in $\langle \psi | H | \psi \rangle$, the particle-state energies are easily renormalized. Variation then leads to an infinite hierarchy of integral equations.

I. INTRODUCTION

In recent years considerable progress has been made on the fermion many-body problem by Brueckner and collaborators[1] and by Bethe.[2] The essential feature of this work is the remark that states may be classified in terms of single-particle wave functions, though the wave function itself be extremely complicated through interparticle correlation. Formally, the energy is expressed in terms similar to the Hartree-Fock theory, however the matrix elements of the potential of interaction, v, are now replaced by a scattering matrix G (in the notation of Bethe). It is the task of the theory to calculate G at least in some consistent approximate fashion, in terms of a single-particle complete orthogonal set. The single-particle functions are then chosen to minimize the energy via the construction of a central potential (in terms of the G matrix)—in exact analogy to the Hartree-Fock equations.

In this theory one is confronted with difficulties in formulating the scattering equation for the G matrix. These are two-fold: (1) Which intermediate states should be included and which excluded?—the Pauli principle dictates that excited intermediate states should lie outside the domain of occupied states; however it also allows for particle-hole and hole-hole scattering. (2) More difficult and more subtle is the question of the choice of the energies of the intermediate states, remembering that a particle is in interaction with the rest of the medium as well as with the "collision" partner in question. This latter point is discussed fully by Bethe where it is emphasized that a unique definition of energies above the Fermi level is not supplied by the theory. The effect of this arbitrariness is supposed to be small.

It is the purpose of this paper to attempt a derivation of the Brueckner-Bethe results variationally.[3] A search for a wave function that without approximation gives rise to the desired theory has failed. However, it is possible to construct a trial function which in an approximate sense, variationally gives rise to the Brueckner-Bethe integral equations. Thus the point of view taken in this paper is that the Brueckner-Bethe integral equations follow variationally from part of $\langle \psi | H | \psi \rangle / \langle \psi | \psi \rangle$ where ψ is a certain trial function. The neglected terms are explicitly shown and, in principle, calculable using the approximate G matrix. At present, numerical calculations of Bethe[2] and Bethe and Gold-

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[1] K. A. Brueckner and C. A. Levinson, Phys. Rev. **97**, 1344 (1955). For a complete list of Brueckner's contributions to the problem see reference 2.

[2] H. A. Bethe, Phys. Rev. **103**, 1353 (1956).

[3] There have been several excellent treatments of this problem in the framework of perturbation theory. The first of these was given by J. Goldstone (reference 4). This work has been amplified by W. Tobocman, Phys. Rev. **107**, 203 (1957), and N. M. Hugenholtz, Physica **23**, 481, 533 (1957). It might be well to point out here that Hugenholtz's high-density limit in the second of the above mentioned papers is the same as the high-density limit introduced by M. Gell-Mann and K. A. Brueckner [Phys. Rev. **106**, 364 (1957)] in their study of the correlation energy of an electron gas.

stone[4] indicate that the neglected terms are indeed small in the nuclear problem. In general, however, they need not be small and application of this theory to special cases must always be justified.

The integral equations derived in this paper include hole-particle and hole-hole scattering in principle. These terms correspond to three and four body clusters, respectively, and have been shown to be negligible in the nuclear problem.[2] Hence, it is sufficient in the nuclear problem to drop these terms. To avoid inessential complication, this will be done in the latter half of this paper, (Secs. IV and V).

Though an approximate wave function is used to calculate $\langle\psi|H|\psi\rangle$, the energy of the ground state is given by a simple expression which is formally valid for the exact eigenfunctions of the Hamiltonian. This expression is derived in Sec. II and states that when the exact solution is expanded in antisymmetrized products of a complete orthogonal set of one-particle wave functions, the energy is completely expressible in terms of the two-particle excitation coefficient only (interpreted as a scattering matrix). The expression for the energy then takes on a simple Hartree-Fock form with v replaced by G, as prescribed by Brueckner.

In Sec. III, we explore the consequences of the most simple imaginable wave function containing pair correlations. It is shown that, in the approximation that the fraction of the total number of virtually excited particles is small, one recovers the Brueckner-Bethe theory with two modifications. (1) Particle-hole and hole-hole scattering is included. (2) The energy denominators in the scattering-matrix integral equation contain Hartree-Fock energies. This last defect is, of course, very serious in the case of a singular potential of interaction, in which case the whole procedure becomes meaningless.

In Sec. IV, it is shown how the hole-state energies are renormalized to finite self-consistent values with the simple wave function used in Sec. III. However, the energies of the particle states are still Hartree-Fock in the simple approximation. In Sec. V, the wave function is completely modified in such fashion as to insure finite energy of particle states. However, the modified wave function introduces complicated terms into the expression for $\langle\psi|H|\psi\rangle$ which completely destroy the simplicity of the form of the theory. If, at this stage, one introduces the approximation of neglecting "cross-linked" diagrams (to be explained in the text), then the theory is again simple and a hierarchy of integral equations follows by variation.

Throughout this paper we shall use the notation of Feynman diagrams as adapted for use in this problem by Goldstone.[5] In an appendix the notation is summarized for the use of the reader. For simplicity, all the work of this paper is carried out for an infinite medium where only momentum-conserving transitions need be considered. It is hoped that these methods will also be applicable to finite systems, though certain nontrivial features arise in this case.

II. GENERAL FORM OF ENERGY

Our basis of representation will be the one-particle complete orthogonal set $\phi_k(r)$. Throughout this paper we shall for simplicity deal with extended nuclear matter so that the $\phi_k(r)$ are plane waves and only momentum-conserving transitions need be considered. Let Φ_0 be an antisymmetrized product of $\phi_k(r)$. This will serve as the "model wave function." ψ, the true ground state function may be expanded in antisymmetrized products of $\phi_k(r)$ where the components are characterized by two, three etc. particles "excited" from the model ground state into excited states. These latter will be referred to as particle states and will be designated with the symbol p_i. The particle states occupied in the model wave function are said to be hole states since they usually enter into the theory as deficiencies. These states will be labeled m_i.

The wave function may then be expanded in the form

$$\psi=\Phi_0+\sum_{m_1m_2}{}'\sum_{p_1p_2}\langle m_1m_2|A|p_1p_2\rangle\mathfrak{A}(\phi_{p_1}(r_1)\phi_{p_2}(r_2)\phi_{m_3}(r_3)\cdots\phi_{m_N}(r_N))$$
$$+\sum_{m_1,m_2,m_3}{}'\sum_{p_1p_2p_3}\langle m_1m_2m_3|A|p_1p_2p_3\rangle\mathfrak{A}(\phi_{p_1}(r_1)\phi_{p_2}(r_2)\phi_{p_3}(r_3)\phi_{m_4}(r_4)\cdots\phi_{m_N}(r_N))+\cdots. \quad (2.1)$$

In (2.1), the operator $\mathfrak{A}$ is an "antisymmetrizer" and designates the operation of taking the antisymmetrized sum over products. The primes over summations means sums consistent with the Pauli principle. Let us write for (2.1) $\psi=\Phi_0+\chi$ where $\langle\Phi_0|\chi\rangle=0$. Then the energy E is given by the expression

$$E=\langle\Phi_0|H|\Phi_0+\chi\rangle=\langle\Phi_0|H|\Phi_0\rangle+\langle\Phi_0|H|\chi\rangle, \quad (2.2)$$

where $H=\sum T_i+\sum v_{ij}$; v_{ij} is the potential of interaction.

The first term on the right-hand side of Eq. (2.2) is the Hartree-Fock energy of the state Φ_0. It is

$$\sum_{m_1}\langle m_1|T|m_1\rangle+\tfrac{1}{2}\sum_{m_1m_2}{}'\langle m_1m_2|v|m_1m_2\rangle\equiv\langle T\rangle+\langle v\rangle. \quad (2.3)$$

The exchange term in the matrix element of v is always included, i.e.,

$$\langle m_1m_2|v|m_1m_2\rangle\equiv\langle m_1m_2|v|m_1m_2\rangle_D-\langle m_1m_2|v|m_2m_1\rangle. \quad (2.4)$$

The second term on the right-hand side of (2.2) has zero contribution from T, since T is a sum of one-body operators and χ differs from Φ_0 by at least two-body

[4] H. A. Bethe and J. Goldstone, Proc. Roy. Soc. (London) **A238**, 551 (1957).

[5] J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957).

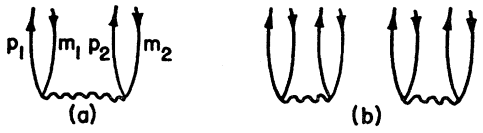

FIG. 1. Pair components of wave function (3.1).

excitation. Further, since v is a two-body operator only the two-body excitation part of χ gives a contribution. This is

$$\langle\Phi_0|H|\chi\rangle=\tfrac{1}{2}\sum_{m_1m_2}{}'\sum_{p_1p_2}\langle m_1m_2|v|p_1p_2\rangle \times\langle p_1p_2|A|m_1m_2\rangle\equiv\langle vA\rangle. \quad (2.5)$$

Combining (2.3) and (2.5) into (2.2) gives

$$E=\langle T\rangle+\langle v+vA\rangle. \quad (2.6)$$

One usually defines a scattering matrix G rather than A according to

$$\langle m_1m_2|A|p_1p_2\rangle=-\frac{1}{e(p_1,p_2;\,m_1m_2)}\langle m_1m_2|G|p_1p_2\rangle, \quad (2.7)$$

where $e(p_1,p_2;\,m_1m_2)$ is the energy difference between the designated states to be subsequently defined. In terms of G we have

$$\langle m_1m_2|v+vA|m_1m_2\rangle=\langle m_1m_2|v-v(1/e)G|m_1m_2\rangle \equiv\langle m_1m_2|G|m_1m_2\rangle, \quad (2.8)$$

Eq. (2.8), is the usual form defining the scattering matrix $\langle m_1m_2|G|m_1m_2\rangle$. It will turn out in the theory that diagonal elements of G are not specified. Hence one is always free to adopt this definition. This is more than formal, for, in the event of singular potentials, v and $v(1/e)G$ are separately divergent whereas G is finite. In these terms we have

$$E=\langle T\rangle+\langle G\rangle=\sum_{m_1}\langle m_1|T|m_1\rangle +\tfrac{1}{2}\sum_{m_1m_2}{}'\langle m_1m_2|G|m_1m_2\rangle. \quad (2.9)$$

No exchange term is included in G as this is accounted for in the definition (2.8).

Equation (2.9) is the Hartree-Fock form of the energy [see Eq. (2.3)] with v replaced by G. It gives the necessary form that the energy will take if an expansion of the type (2.1) is used. In itself, Eq. (2.9) is empty since the coefficient $\langle m_1m_2|A|p_1p_2\rangle$ is, in general, unknown. Further Φ_0 is completely arbitrary so long as (2.1) is convergent. However, in approximate schemes Φ_0 is obviously chosen to be the Fermi ground state in terms of those $\phi_k(r)$ which give the lowest zeroth order energy. For finite systems, Bethe[2] has shown that the form (2.9) is probably still valid. In this case one varies the $\phi_k(r)$, assuming knowledge of $\langle m_1m_2|A|p_1p_2\rangle$. This yields a set of Hartree-Fock type equations for the ϕ_k in terms of a nonlocal central potential. For infinite systems, translational symmetry imposes that the $\phi_k(r)$ are plane waves.

The main point to be stressed here is that the form (2.9) is prescribed for an infinite medium and in no way involves any approximation.

III. SIMPLE PAIR APPROXIMATION

The simplest imaginable improvement over a product of single particle functions is to include a single pair excitation

$$\psi=\Phi_0+\sum_{m_1,m_2}{}'\langle m_1m_2|A|p_1p_2\rangle \times\mathcal{A}(\phi_{p1}(r_1)\phi_{p2}(r_2)\phi_{m3}(r_3)\cdots\phi_{mN}(r_N)), \quad (3.1)$$

where Φ_0 is normalized to unity. The second term is diagrammed in Fig. 1(a). Actually, this trial function will not lead to any substantial improvement over the Hartree-Fock theory when $\langle\psi|H|\psi\rangle$ is calculated. The reason for this is that Eq. (3.1) permits only 2 nucleons to correlate at a time, whereas it is overwhelmingly probable that when the pair (12) is interacting then some other pairs (ij) are also interacting, completely independently of (12). This dictates that one must include terms of the type $\langle m_1m_2|A|p_1p_2\rangle\langle m_3m_4|A|p_3p_4\rangle$ $(m_1\neq m_2\neq m_3\neq m_4)$ in the wave function. More generally, one must include all possible n-fold products of mutually distinct 2-body A operators. Such terms will be called unlinked clusters. An arbitrary term just contains Fig. 1(a) repeated a certain number of times, an example being Fig. 1(b).

We now evaluate $\langle\psi|H|\psi\rangle/\langle\psi|\psi\rangle$ using the trial function (3.1), *where it is understood that all unlinked clusters are included.*

The first contribution is the Hartree-Fock energy of the model wave function.

$$\langle\Phi_0|H|\Phi_0\rangle/\langle\psi|\psi\rangle. \quad (3.2)$$

Next, we consider the contribution from that part of the product $\langle\psi|H|\psi\rangle$ which arises from identical two-body excitations in ψ^* and ψ (termed diagonal contribution), e.g.,

$$\langle m_1m_2|A|p_1p_2\rangle\int\mathcal{A}(\phi_{p1}{}^*\phi_{p2}{}^*\phi_{m3}{}^*\cdots\phi_{mN}{}^*) \times H\mathcal{A}(\phi_{p1}\phi_{p2}\phi_{m3}\cdots\phi_{mN})\langle p_1p_2|A|m_1m_2\rangle. \quad (3.3)$$

The integral in Eq. (3.3) has for its basic contribution $\langle\Phi_0|H|\Phi_0\rangle$ which is corrected for by adding on the particle energies and subtracting the hole energies. That is, (3.3) is

$$\langle\Phi_0|H|\Phi_0\rangle|\langle m_1m_2|A|p_1p_2\rangle|^2 +\langle m_1m_2|A|p_1p_2\rangle[\epsilon_{\text{H.F.}}(p_1)+\epsilon_{\text{H.F.}}(p_2) -\epsilon_{\text{H.F.}}(m_1)-\epsilon_{\text{H.F.}}{}^*(m_2)]\langle p_1p_2|A|m_1m_2\rangle, \quad (3.4)$$

where

$$\epsilon_{\text{H.F.}}(p_1) = \langle p_1|T|p_1\rangle + \sum_{m_3} \langle p_1 m_3|v|p_1 m_3\rangle, \quad (3.5)$$

$$\epsilon_{\text{H.F.}}(m_1) = \langle m_1|T|m_1\rangle + \sum_{m_3} \langle m_1 m_3|v|m_1 m_3\rangle. \quad (3.6)$$

Strictly speaking in (3.5) and (3.6) $m_3 \neq m_1, m_2$ should be specified but this is an infinitesimal correction. Also the element $\langle p_1 p_2|v|p_1 p_2\rangle$ has been neglected.

The sum of all diagonal contributions that multiplies $\langle\Phi_0|H|\Phi_0\rangle$ is merely $\langle\Phi_0|H|\Phi_0\rangle[\langle\psi|\psi\rangle]$. Next we consider the effect of the unlinked clusters insofar as they effect the second term in (3.4). Let us take a sample term with ν excited pairs, i.e., that component of the wave function containing the coefficient

$$\langle m_1 m_2|A|p_1 p_2\rangle\langle m_3 m_4|A|p_3 p_4\rangle\cdots \times\langle m_{2\nu-1} m_{2\nu}|A|p_{2\nu-1} p_{2\nu}\rangle. \quad (3.7)$$

The diagonal contribution of this term is

$$\prod_{i=1}^{\nu} |\langle m_{2i-1} m_{2i}|A|p_{2i-1} p_{2i}\rangle|^2] \times[\langle\Phi_0|H|\Phi_0\rangle + \sum_{i=1}^{\nu} (\epsilon_{\text{H.F.}}(p_{2i-1}) + \epsilon_{\text{H.F.}}(p_{2i}) - \epsilon_{\text{H.F.}}(m_{2i-1}) - \epsilon_{\text{H.F.}}(m_{2i})) + (\text{corrections})], \quad (3.8)$$

where $\epsilon(p)$ and $\epsilon(m)$ are defined by (3.5) and (3.6), respectively. The term (corrections) in (3.8) comes about from the fact that in the component (3.7), not all the particles but the two in question (e.g., $2i-1$, $2i$), occupy their hole states. Thus the difference of the diagonal contribution from $\langle\Phi_0|H|\Phi_0\rangle$ may not be expressed in terms of the energies (3.5) and (3.6) alone. The correction term to $\epsilon_{\text{H.F.}}(p_1)$ is thus

$$\sum_{j=2}^{2\nu} [\langle p_1 p_j|v|p_1 p_j\rangle - \langle p_1 m_j|v|p_1 m_j\rangle]. \quad (3.9)$$

This term is, in general, not negligible since $\nu = O(N)$. However, it is clear that if the fraction, ξ, of the mean number of excited pairs is small then a small error of $O(\xi)$ is made in neglecting (3.9). As this approximation is intrinsic to the theory of Brueckner and Bethe we have analyzed this point more closely in Appendix B. Here it is shown that the relatively low density of the nucleus is responsible for keeping ξ small. The ultimate reason for the smallness of ξ or alternatively for low nuclear density is the weakness of the potential of interaction. The success of the theory, in our opinion, depends critically on this fact.

It should be pointed out, however, that the neglected terms of type (3.9) always come in as a difference of quantities. It may thus happen that even if ξ is not particularly small these corrections will tend to fluctuate in sign and cancel out. If this is so, one might hope that the theory is better than one might expect. It should be worthwhile to check this point in terms of Brueckner's numerical results.

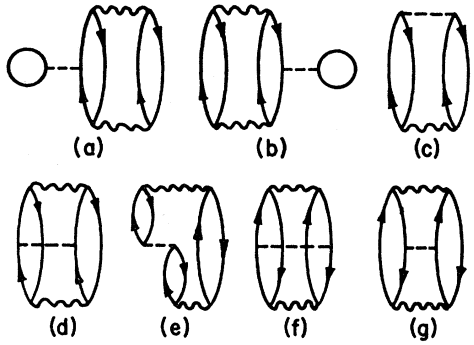

FIG. 2. Diagrams contributing to $\langle\psi|H|\psi\rangle$ from pair components with $N^{m_1 m_2}/N = 1$.

Going back to (3.8), in view of the preceding discussion we shall neglect "(corrections)." In this approximation it is seen that following a given pair term comes a whole sequence of terms all containing the same factors, e.g., $|\langle m_1 m_2|A|p_1 p_2\rangle|^2[\epsilon_{\text{H.F.}}(p_1) + \epsilon_{\text{H.F.}}(p_2) - \epsilon_{\text{H.F.}}(m_1) - \epsilon_{\text{H.F.}}(m_2)]$. The other factor is the normalization factor that would be obtained for $N-2$ particles with the hole states m_1, m_2 omitted. This factor we symbolize as $N^{m_1 m_2}$, and the whole normalization $\langle\psi|\psi\rangle$ is called N.† In summary the total diagonal contribution to $\langle\psi|H|\psi\rangle/\langle\psi|\psi\rangle$ is, to first order in ξ

$$E_0 + \sum\langle m_1 m_2|A|p_1 p_2\rangle e_{\text{H.F.}}(p_1 p_2; m_1 m_2) \times\langle p_1 p_2|A|m_1 m_2\rangle N^{m_1 m_2}/N, \quad (3.10)$$

where $e_{\text{H.F.}}$ = difference in Hartree-Fock energies. The diagrams contributing to (3.10) are given in Figs. 2(a) and (b) for particle and hole states, respectively.

The next terms are those arising from cross-products of the two-body A operator with Φ_0. When the unlinked clusters are taken into account these terms give

$$[\langle m_1 m_2|A|p_1 p_2\rangle\langle p_1 p_2|v|m_1 m_2\rangle + \text{c.c.}]N^{m_1 m_2}/N, \quad (3.11)$$

which is diagramed in Fig. 2(c). Finally the terms that arise from non-identical A operators represent double scattering processes twice by A operators and

† *Note added in proof.*—One should also delete the particle states $p_1 p_2$. However, we shall omit this effect in that for singular potentials it turns out not to be important. The exclusion effects on hole state energies, on the other hand, will be most important as they give the Brueckner self-consistent energy denominator as shown in Sec. IV.

once by a v operator. These are[6]

$$\langle m_1m_2|A|p_1p_2\rangle\langle p_1p_2|v|p_1'p_2'\rangle\langle p_1'p_2'|A|m_1m_2\rangle N^{m_1m_2}/N, \quad \text{particle-particle Fig. 2(d)}, \tag{3.12a}$$

$$\langle m_1m_2|A|p_1p_2\rangle\langle p_1m_3|v|p_1'm_1\rangle\langle p_1'p_2|A|m_2m_3\rangle N^{m_1m_2m_3}/N, \quad \text{particle-hole Fig. 2(e)}, \tag{3.12b}$$

$$\langle m_1m_2|A|p_1p_2\rangle\langle p_1m_3|v|p_1'm_2\rangle\langle p_1'p_2|A|m_1m_3\rangle N^{m_1m_2m_3}/N, \quad \text{particle-hole exchange Fig. 2(f)}, \tag{3.12c}$$

$$\langle m_1m_2|A|p_1p_2\rangle\langle m_3m_4|v|m_1m_2\rangle\langle p_1p_2|A|m_3m_4\rangle N^{m_1m_2m_3m_4}/N, \quad \text{hole-hole Fig. 2(g)}. \tag{3.12d}$$

All diagrams in Fig. 2 are with the normalization ratios set equal to 1. The corrections due to the unlinked clusters as they appear in normalization ratios is discussed in Sec. 4. In (3.12) the factor $N^{m_1m_2m_3}/N$ is the ratio of the normalization factor, with the three states m_1, m_2, m_3 missing, to the correct normalization factor, and similarly for $N^{m_1m_2m_3m_4}$.

We shall now study the effect of variation of the coefficients $\langle p_1p_2|A|m_1m_2\rangle$ in the approximation where the normalization ratios are unity. The factor $\langle p_1p_2|A|m_1m_2\rangle$ appears as a linear coefficient in the diagrams 2(a) to 2(g). Hence variation puts all these diagrams equal to zero with A determined by the integral equation

$$e_{\text{H.F.}}A+v+Av=0, \tag{3.13}$$

where the product Av contains all the intermediate scattering included in diagrams 2(d) to 2(g). Equation (3.13) is put into standard form by defining the scattering matrix G''.

$$\langle m_1m_2|A|p_1p_2\rangle=-\frac{1}{e_{\text{H.F.}}(p_1p_2;\,m_1m_2)}\langle m_1m_2|G''|p_1p_2\rangle. \tag{3.14}$$

In terms of G'', (3.13), when written out in detail, becomes

$$\begin{aligned}\langle m_1m_2|G''|p_1p_2\rangle=\langle m_1m_2|v|p_1p_2\rangle&-\sum_{p_1'p_2'}\langle m_1m_2|G''|p_1'p_2'\rangle\frac{1}{e_{\text{H.F.}}(p_1'p_2';\,m_1m_2)}\langle p_1'p_2'|v|p_1p_2\rangle\\&-\sum_{m_3m_4}\langle m_3m_4|G''|p_1p_2\rangle\frac{1}{e_{\text{H.F.}}(p_1,p_2;\,m_3m_4)}\langle m_1m_2|v|m_3m_4\rangle\\&-\sum_{m_3,p_1'}\langle m_1m_3|G''|p_1'p_2\rangle\frac{1}{e_{\text{H.F.}}(p_1',p_2;\,m_1m_3)}\langle m_2p_1'|v|m_3p_1\rangle\\&-\sum_{m_3p_2'}\langle m_2m_3|G''|p_2'p_1\rangle\frac{1}{e_{\text{H.F.}}(p_2'p_1;\,m_2m_3)}\langle m_1p_2'|v|m_3p_2\rangle\\&-\sum_{m_3p_2'}\langle m_1m_3|G''|p_2'p_1\rangle\frac{1}{e_{\text{H.F.}}(p_2'p_1;\,m_1m_3)}\langle m_2p_2'|v|m_3p_2\rangle\\&-\sum_{m_3p_1'}\langle m_2m_3|G''|p_1'p_2\rangle\frac{1}{e_{\text{H.F.}}(p_1'p_2;\,m_2m_3)}\langle m_1p_1'|v|m_3p_1\rangle.\end{aligned} \tag{3.15}$$

Equation (3.15) differs from the Brueckner-Bethe formulation in two respects: 1. Intermediate states include hole-hole and particle-hole scattering [diagrams 2(e), 2(f), and 2(g)]. 2. Energy denominators contain Hartree-Fock energies defined by Eqs. (3.5) and (3.6). We return shortly to a discussion of these points, after first deriving the expression for the energy.

The integral equation (3.15) has put all the diagrams of Fig. 2 equal to zero. This leaves over the first term E_0 in (3.10) and the complex conjugate term in (3.11). The Hartree-Fock potential energy contribution to E_0 and the term Av that arises in Eq. (3.11) are diagrammed in Fig. 3 together with their sum as given by Eq. (2.6). In accordance with the general theorem of Sec. 2, the energy is expressible in terms of these two terms alone, even though an approximate wave function was used. The energy is given by Eq. (2.9) using (2.8) with Hartree-Fock energy denominators. Equation (3.15) must be solved to find the term $\langle m_1m_2|G''|p_1p_2\rangle$ that enters into the definition (2.8).

[6] In addition to the terms listed above there are a few other ways that unlinked clusters enter into hole state interactions [(3.12b) (c) and (d)], in addition to the simple diagrams listed above. For example, a term entering together with (3.12b) is

$$\langle m_1m_2|A|p_1p_2\rangle\langle p_1m_3|v|p_1'm_1\rangle\langle p_1'p_2|A|m_2m_3\rangle \times\langle m_3m_4|A|p_3p_4\rangle\langle p_3p_4|A|m_1m_4\rangle$$

Such terms lead to considerable combinatorial complication. As we are interested in the derivation of the Brueckner approximation we shall not pursue these terms further in this paper. They constitute refinements on 3-body clusters, already shown to be small in the nuclear problem.

We now return to a discussion of the integral equation (3.15). The intermediate states other than the usual particle-particle scattering are given by diagrams 2(e), 2(f), and 2(g). It is obvious that any theory which handles scattering matrices in an unbiased fashion must come out with these diagrams included since they are merely symmetric counterparts of particle-particle scattering. In the theories of Brueckner and Bethe these higher order interactions of holes are said to be three- and four-body clusters (since the number of particles involved is given by the highest index on m in the integral equation). This is, a somewhat useful classification, though artificial. It is artificial because, in the general theory, holes and particles always are involved symmetrically. The classification is useful because, for hard-core-type interactions, the momentum restrictions in hole interactions (a hole has momentum less than the Fermi momentum) renders these terms negligible. For nonsingular interaction, the hole interactions are as important as the particle interactions. Here, however, the Born approximation is probably fairly good in the nuclear problem as has been emphasized by Swiatecki[7] and Bethe.[2] In this case, the whole question is unimportant. At high density, the Born approximation decidedly breaks down as has been pointed out by Hugenholtz[3] and the hole interactions will certainly play an important role (in fact dominant).

The other interesting point about the integral equation is the presence of Hartree-Fock energy denominators. These were prescribed by the form of the integral equations for the A operator in (3.13) which in turn followed from the analysis leading to the expression (3.10). For nonsingular potentials it is reasonable to accept the Hartree-Fock energy denominator as a first approximation since this accounts in a rough way, for the particle interaction with the medium. For singular potentials, this is out of the question and a more sophisticated theory must be developed. The trial function given by (3.1) will not give finite particle state self energy. The reason for this is that when the trial function admits of the correlation of a pair, then a given member of that pair may not be simultaneously correlated with another third particle in the medium. It is thus that the uncorrelated or Hartree-Fock particle energy enters into the integral equation (3.13). This necessitates broadening the class of admitted trial functions as will be discussed in Sec. V. However, the hole states do have finite self energy, expressible in terms of the scattering matrix G, even for the simple pair function (3.1). These corrections lie in the unlinked clusters as they enter into the normalization ratios $N^{m_1m_2}/N$. The discussion of such terms is the subject of the next section.

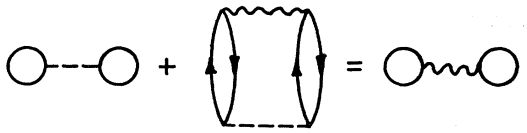

FIG. 3. Diagram of the energy equation (2.8).

IV. HOLE-STATE ENERGIES

To calculate renormalized hole-state energies, it is necessary to evaluate the normalization ratios $N^{m_1m_2}/N$ that appear in (3.10) and the following. $N^{m_1m_2}$ is, by definition, the normalization factor with m_1, m_2 missing which leads to

$$N^{m_1m_2}=1+\sum_{m_3,m_4\neq m_1,m_2}\sum_{p_3p_4}|\langle m_3m_4|A|p_3p_4\rangle|^2 + \sum_{m_3,m_4,m_5,m_6\neq m_1,m_2}\sum_{p_3,p_4,p_5,p_6}|\langle m_3m_4|A|p_3p_4\rangle|^2|\langle m_5m_6|A|p_5p_6\rangle|^2+\cdots. \quad (4.1)$$

Equation (4.1) is conveniently rearranged by adding and subtracting the restricted terms.

$$N^{m_1m_2}=N-\sum_{m_3\neq m_1}|\langle m_1m_3|A|p_1p_3\rangle|^2-\sum_{m_3\neq m_2}|\langle m_2m_3|A|p_2p_3\rangle|^2 - \sum_{m_3\neq m_1}|\langle m_1m_3|A|p_1p_3\rangle|^2\sum_{m_4,m_5\neq m_1,m_3}|\langle m_4m_5|A|p_4p_5\rangle|^2 - \sum_{m_3\neq m_2}|\langle m_2m_3|A|p_2p_3\rangle|^2\sum_{m_4,m_5\neq m_2,m_3}|\langle m_4m_5|A|p_4p_5\rangle|^2-\cdots. \quad (4.2)$$

All p indices are to be summed in the above.

In (4.2) we have neglected those special terms which arise from m_1, m_2 occurring in the same factor of A. This kind of neglect leads to a finite error in terms which are very high order interactions in (4.2). Again the argument following (3.8) pertains i.e., if a small, albeit finite, fraction of particles are in excited states in the large components of the wave function then the approximation is acceptable.

Equation (4.2) is easily converted into a recurrence relation by noticing that the factors which follow $|\langle m_1m_3|A|p_1p_3\rangle|^2$ are precisely those that enter into the definition of $N^{m_1m_3}$ and similarly for m_2.

Hence, we have the integral equation for $N^{m_1m_2}/N$

[7] W. J. Swiatecki (quoted by H. A. Bethe in reference 2).

given by

$$N^{m_1m_2}/N=1-\sum_{m_3\neq m_1}\sum_{p_1p_3}|\langle m_1m_3|A|p_1p_3\rangle|^2N^{m_1m_3}/N-\sum_{m_3\neq m_2}\sum_{p_2p_3}|\langle m_2m_3|A|p_2p_3\rangle|^2N^{m_2m_3}/N. \quad (4.3)$$

We shall use Eq. (4.3) together with the expression for $\langle\psi|H|\psi\rangle/\langle\psi|\psi\rangle$ found in Sec. III to find variationally the best $\langle m_1m_2|A|p_1p_2\rangle$. We will omit terms involving hole-state interactions [(3.12b)–(3.12d)], as the argument below is not so simple when these terms are included. Inclusion of these terms not only renormalizes hole-state energies but introduces additional complicated terms. The retained terms are those which lead to the Brueckner theory.

For easy reference we write below the expression for $\langle\psi|H|\psi\rangle/\langle\psi|\psi\rangle$ omitting hole-state interactions as well as corrections higher order in ξ as discussed after Eq. (3.8). Thus we take the sum of (3.10), (3.11), and (3.12a)

$$\langle\psi|H|\psi\rangle/\langle\psi|\psi\rangle=E_0+\sum|\langle m_1m_2|A|p_1p_2\rangle|^2e_{\text{H.F.}}(p_1p_2;m_1m_2)N^{m_1m_2}/N$$
$$+\sum[\langle m_1m_2|A|p_1p_2\rangle\langle p_1p_2|v|m_1m_2\rangle+\text{c.c.}]N^{m_1m_2}/N$$
$$+\sum\langle m_1m_2|A|p_1p_2\rangle\langle p_1p_2|v|p_1'p_2'\rangle\langle p_1'p_2'|A|m_1m_2\rangle N^{m_1m_2}/N. \quad (4.4)$$

In (4.4) appear the coefficients $N^{m_1m_2}$ which are functions of the A matrices given by the solutions of Eqs. (4.3). This functional dependence is most easily accounted for by using the method of La Grange multipliers. We consider the $N^{m_im_j}$ to be independent variables in (4.4) and the Eqs. (4.3) are taken to be conditions among the variables. Introducing the multipliers $\lambda_{m_im_j}$, we shall minimize the function $[\langle\psi|H|\psi\rangle/\langle\psi|\psi\rangle-\sum\lambda_{m_im_j}F_{m_im_j}]$ where

$$F_{m_im_j}\equiv 1-N^{m_im_j}-\sum_{m_k,p_k}[\sum_{p_i}|\langle m_im_k|A|p_ip_k\rangle|^2N^{m_im_k}+\sum_{p_j}|m_jm_k|A|p_ip_k\rangle|^2N^{m_jm_k}]. \quad (4.5)$$

Varying $\langle p_1p_2|A|m_1m_2\rangle$ leads to the equation

$$\langle m_1m_2|A|p_1p_2\rangle[e_{\text{H.F.}}(p_1p_2;m_1m_2)+\sum_{m_3}\lambda_{m_1m_3}+\sum_{m_3}\lambda_{m_2m_3}]$$
$$+\langle m_1m_2|v|p_1p_2\rangle+\sum\langle m_1m_2|A|p_1'p_2'\rangle\langle p_1'p_2'|v|p_1p_2\rangle=0, \quad (4.6)$$

and variation of $N^{m_1m_2}$ leads to

$$\sum_{p_1p_2}\{|\langle m_1m_2|A|p_1p_2\rangle|^2e_{\text{H.F.}}(p_1p_2;m_1m_2)+[\langle m_1m_2|A|p_1p_2\rangle\langle p_1p_2|v|m_1m_2\rangle+\text{c.c.}]\}$$
$$+\langle m_1m_2|AvA|m_1m_2\rangle+\lambda_{m_1m_2}+\sum_{m_3;\,p_1p_2}\lambda_{m_1m_3}|\langle m_1m_2|A|p_1p_2\rangle|^2+\sum_{m_3;\,p_1p_2}\lambda_{m_2m_3}|\langle m_1m_2|A|p_1p_2\rangle|^2=0. \quad (4.7)$$

The last three terms in (4.7) come from the coefficients of $N^{m_1m_2}$ in $F_{m_1m_2}$, $F_{m_1m_3}$ and $F_{m_2m_3}$, respectively, these being the only terms that contain $N^{m_1m_2}$. Equation (4.7) is easily rearranged into more recognizable form by putting the terms containing $\lambda_{m_1m_3}$ and $\lambda_{m_2m_3}$ together with $e_{\text{H.F.}}(p_1p_2;m_1m_2)$. Using (4.6) it is then seen that almost all terms in (4.7) cancel, leaving the equality

$$\lambda_{m_1m_2}=-\sum_{p_1p_2}\langle m_1m_2|v|p_1p_2\rangle\langle p_1p_2|A|m_1m_2\rangle. \quad (4.8)$$

Equation (4.8) determines the Lagrange multipliers λ.

We define the renormalized hole state energies by

$$e(m_1)=t(m_1)+\sum_{m_3}\langle m_1m_3|v-v(1/e')G'|m_1m_3\rangle, \quad (4.9)$$

where

$$e'=e_{\text{H.F.}}(p_1)+e_{\text{H.F.}}(p_2)-e(m_1)-e(m_2),$$
$$G'=e'A. \quad (4.10)$$

The operation (4.9) is diagrammed in Fig. 4. The integral equation (4.6) becomes

$$G'=v-G'(1/e')v. \quad (4.11)$$

It is thus seen that the simple pair wave function

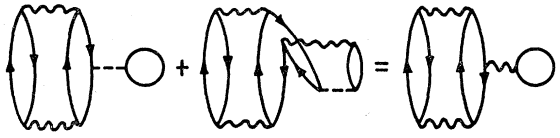

FIG. 4. Diagram of the definition of the self energy of a hole state [Eq. (4.7a)].

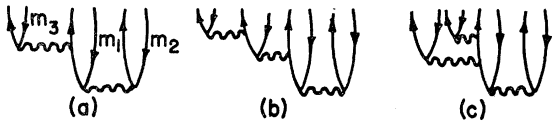

FIG. 5. The first few components added to the wave function to modify the particle-state self energy.

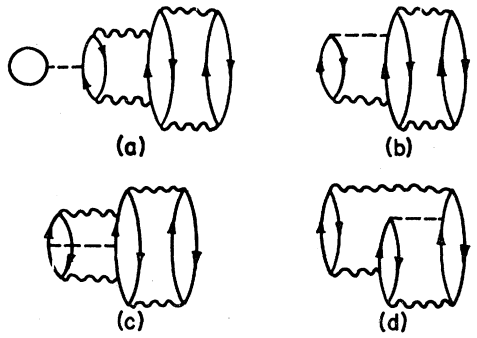

FIG. 6. Energy diagrams arising from the presence of diagram 7(a) in the wave function.

leads to an integral equation for the two-body scattering matrix by a variational procedure valid if the fraction of excited pairs is small, and hole-state interactions are neglected. The energy denominators involving hole states are renormalized according to the Brueckner self-consistent prescription (4.9). The particle-state energies are still Hartree-Fock.

V. PARTICLE SELF ENERGY

In order to convert the Hartree-Fock particle energies in Eq. (4.9) to "G matrix" energies, it is necessary to enlarge upon the original trial function (3.1). One must admit components of the wave function which permit interaction of the excited states with the medium. Thus a typical added component diagrammed in Fig. 5(a) is

$$\langle m_1 m_2 | A | p_1 p_2 \rangle \langle p_1 m_3 | A | p_1' p_3 \rangle \times \mathcal{A}(\phi_{p_1'} \phi_{p_2} \phi_{p_3} \phi_{m_4} \cdots \phi_{m_N}). \quad (5.1)$$

Together with all the unlinked clusters of two- and three-particle excitations [i.e., unlinked combinations of Figs. 1(a) and 5(a)].

The addition of component (5.1) to the wave function introduces various new terms into $\langle \psi | 1 + | \psi \rangle$. There are *six* new bubble diagrams, a sample of which is shown in Fig. 6(a). There are two diagrams which close off the AA product of (5.1) with a vA product. One of these is shown in Fig. 6(b). There are six interaction terms which are the analogs of Figs. 2(d) to 2(g). A sample is the particle-particle interaction shown in Fig. 6(c). Finally there are diagrams of the type 6(d) in which particle "3," having interacted with particle "1" through an A matrix, is scattered back to its hole state by particle "2." We now introduce the fundamental approximation stated below.

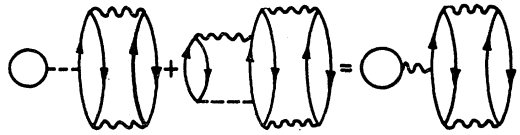

FIG. 7. Diagram of the definition of the self energy of a particle state.

Approximation.—Those diagrams in which more than two particle-hole loops are directly connected [as in Fig. 6(d)] will be neglected. Thus if a particle ν interacts with a particle ν' involving a change of state, the inverse of this process again involving the particles ν and ν' must occur in the diagram either through a dotted line or a wavy line. With this approximation it is simple to find an integral equation for $\langle p_1 m_3 | A | p_1' p_3 \rangle$ by variation. We first define

$$\langle p_1 m_3 | G' | p_1' p_3 \rangle_{m_1;m_2 p_2} \equiv [\epsilon_{\text{H.F.}}(p_1') + \epsilon_{\text{H.F.}}(p_2) + \epsilon_{\text{H.F.}}(p_3) - \epsilon(m_1) - \epsilon(m_2) - \epsilon(m_3)] \langle p_1 m_3 | A | p_1' p_3 \rangle. \quad (5.2)$$

The subscripts m_1, m_2, p_2 in Eq. (5.2) indicate the holes and excited particles that appear in a given diagram other than those specifically involved in the particular scattering process. This is a necessary notation as is witnessed by the presence of the *six* possible bubble diagrams one of which is Fig. 6(a). The integral equation is then (we again omit hole-state interaction)

$$\langle p_1 m_3 | G' | p_1' p_3 \rangle_{m_1;m_2 p_2} = \langle p_1 m_3 | v - G'(1/e') v | p_1' p_3 \rangle. \quad (5.3)$$

The e' in Eq. (5.3) is the threefold energy difference that appears in the definition (5.2). Equation (5.3) contains the so-called off-energy-shell propagator due to the presence of the difference $e(p_2) - e(m_2)$ in the scattering equation for particles "1" and "3."

Once the matrix $\langle p_1 m_3 | G | p_1' p_3 \rangle_{m_1;m_2 p_2}$ is known it may be substituted into the complex conjugate diagram of Fig. 6(b). This is then combined with the dotted-line bubble diagram of Fig. 2(a). The result is a wavy-line bubble diagram as indicated in Fig. 7. Thus the particle-state energy is

$$[e(p_1)]_{m_1;m_2 p_2} \equiv \langle p_1 | T | p_1 \rangle + \sum_{m_3} \langle p_1 m_3 | G' | p_1 m_3 \rangle_{m_1;m_2 p_2}, \quad (5.4)$$

$$\langle p_1 m_3 | G' | p_1 m_3 \rangle_{m_1;m_2 p_2} \equiv \langle p_1 m_3 | v | p_1 m_3 \rangle - \sum \langle p_1 m_3 | v | p_1' p_3 \rangle \times \frac{1}{\epsilon_{\text{H.F.}}(p_1') + \epsilon_{\text{H.F.}}(p_2) + \epsilon_{\text{H.F.}}(p_3) - \epsilon(m_1) - \epsilon(m_2) - \epsilon(m_3)} \langle p_1' p_3 | G | p_1 m_3 \rangle. \quad (5.5)$$

The hole energies are still given by Eq. (4.9).

It is now necessary to add to the trial function terms which will convert the dotted-line bubble diagrams 6(a) to wavy-line bubble diagrams. These terms are diagrammed in Figs. 5(b) and 5(c), respectively, defining the symbols $\langle m_4 p_1' | G' | p_4 p_1'' \rangle_{m_1;m_2 p_2;m_3 p_3}$ and $\langle m_4 p_3 | G' | p_4 p_3' \rangle_{m_1 p_1';m_2 p_2;m_3}$. With the use of our fundamental approximation, these terms introduce diagrams

which convert the bubble diagrams of Fig. 6(a) to wavy-line bubble diagrams. Hence the integral equation (5.3) must be rewritten now containing "G matrix" particle state energies as well as "G matrix" hole-state energies. The hole-state energies are given by an equation of the form (4.9) with G' replaced G. $\langle m_1m_2|G|m_1m_2\rangle$ is defined in terms of $\langle m_1m_2|G|p_1p_2\rangle$ as in Eq. (2.8) with the energy denominator expressed completely in terms of "G matrix" energies.

The general pattern is then to add successive particle-hole pair components to all particle lines that appear in ψ. For a component possessing ν excited pairs, there are all possibilities of diagrams containing ν scatterings of one particle to one scattering each of $\nu+1$ particles. Each such scattering defines a G matrix with a subscript defining the previous history of the diagram before the particular scattering in question. This subscript labels the number of holes and particles present in the wave function at the time of the scattering in question. Variation of each of these G matrices leads to an infinite hierarchy of integral equations.

As this hierarchy of equations has already been presented in the literature we shall not duplicate it here. Reference may be made to Brueckner and Gammel[8] where it is shown how the entire hierarchy may be reduced to a single equation. We would like to stress here, that this hierarchy of equations is not derivable variationally even to first order in the parameter ξ introduced in Sec. III. The theory here depends completely on the proof of the negligibility of the cross-linked clusters such as in Fig. 6(d). All that has been shown here is that the Brueckner-Bethe hierarchy of equations minimizes a part of $\langle\psi|H|\psi\rangle/\langle\psi|\psi\rangle$ (the non-cross-linked diagrams and intermediate particle-state interaction only) if ξ is small.

VI. CONCLUDING REMARKS

We briefly summarize the results of this paper. The simple trial wave function introduced in Sec. III succeeds in formulating the energy in terms of a diagonal G matrix. In the approximation that only a small but finite fraction of the particles are excited in the important part of the wave function and intermediate hole-state interactions are omitted, it is possible to find by variation an integral equation for the nondiagonal G matrix. However, this integral equation contains Hartree-Fock energies for the particle states. The hole-state energies are given by diagonal G matrices as proved in Sec. 4. In order to convert the Hartree-Fock particle-state energies to "G matrix" particle energies, the trial wave function must be completely modified. These modifications destroy the simplicity of the simple pair theory by introducing cross-linked clusters such as Fig. 6(d). If the approximation is made of neglecting all such cross-linked terms, but retaining all "self-energy" terms, the simplicity of the pair theory is recovered. All particle state energies are then converted to "G matrix" energies. These particle state energies, in addition to being functions of the state in question, are also functions of all holes and particles that exist in a particular diagram at the time the particular pair containing the particle in question is created. This circumstance necessitates defining an infinite hierarchy of integral equation. The result is that the part of $\langle\psi|H|\psi\rangle$ containing non-cross-linked diagrams and particle state interactions only is minimized to first order in ξ. Only in this sense are the Brueckner-Bethe integral equations derivable from a wave function.

To the extent that the omitted terms are small, variational thinking is valid and Brueckner's procedure of minimizing the energy with respect to the density is permissible.

[8] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).

ACKNOWLEDGMENTS

I should like to thank Mr. D. J. Thouless for a number of important remarks. I am particularly indebted to Professor E. E. Salpeter for many helpful suggestions, conversations and a critical reading of the manuscript. To Professor H. A. Bethe, I owe the suggestion of the interpretation of the theory in terms of diagrams.

APPENDIX A

The diagram notation of Goldstone as used here is the following. The sequence of events in a matrix element reads from left to right in the notation of this paper. Correspondingly a diagram reads from down to up. Particle (hole) states are represented by lines directed up (down). An A matrix of the type $\langle m_1m_2|A|p_1p_2\rangle$ creates two particle-hole pairs. These are created on the same level of the diagram and are connected by an "A" line which is a wavy line. This explains the notation of Fig. 1(a). Some A matrices create a pair while merely scattering a second particle from one particle state to another, i.e., $\langle p_1m_3|A|p_1'p_3\rangle$. This is diagrammed in Fig. 7. The same notation is used for v matrices, but the wavy line is replaced by a dotted line.

Some interactions scatter particles from one state to the next. These are represented by dotted (for v matrices) or wavy (for A matrices) lines terminating on the hole and particle lines that are scattered such as Fig. 4(d).

Other interactions are diagonal in character and leave the particle in the same state it was in before. In this paper both direct and exchange elements are grouped as one and are represented by one bubble diagram attached to the particular particle or hole line in question. An example is Fig. 2(a).

APPENDIX B

In this appendix, we will find an approximate expression for the fraction of excited pairs in the pair wave function (3.1) together with all unlinked clusters. Let

us consider all terms in the normalization integral $\langle\psi|\psi\rangle$ that have P pairs excited. These number $N!/(N-2P)!P!2^P$; $P\leq N/2$. Their average coefficient is $\langle\sum_{p_1p_2}|\langle m_1m_2|A|p_1p_2|^2\rangle_{\text{Av}}{}^P$; where the average is over hole states. Calling this latter coefficient ξ/N, we have for the contribution of all terms with P pairs excited the value

$$\frac{N!}{(N-2P)!P!N^P}\left(\frac{\xi}{2}\right)^P\cong\frac{N^P}{P!}\left(\frac{\xi}{2}\right)^P \quad \text{for} \quad \frac{P}{N}\ll 1. \tag{B-1}$$

Let us assume (B-1) to be valid for all P. Then (B-1) defines a Poisson distribution and we have for the mean fraction of excited pairs the value

$$\langle P\rangle_{\text{Av}}/\tfrac{1}{2}N=\xi. \tag{B-2}$$

If $\xi\ll 1$, then (B-1) is a good approximation. Further the relative dispersion about the $\langle P\rangle_{\text{Av}}$ term is small like $O(1)/N$, i.e., $\langle P^2\rangle_{\text{Av}}/\langle P\rangle_{\text{Av}}{}^2=1/\langle P\rangle_{\text{Av}}$. In this respect $\langle\psi|\psi\rangle$ is quite like the cluster expansion of the partition function in statistical mechanics where the relative dispersion about the mean cluster populations is again $O(1)/N$.

An expression for ξ is found by observing that for an isolated pair the wave function of the particles in the pair is

$$\psi_{k_1k_2}=\mathfrak{A}[\phi_{k_1}\phi_{k_2}+\textstyle\sum_q\langle k_1k_2|A|k_1+q;k_2-q\rangle \times\phi_{k_1+q}\phi_{k_2-q}, \tag{B-3}$$

so that

$$\xi=\frac{1}{v}\int|\langle|\psi_{\text{pair}}|^2\rangle_{\text{Av}}-1| \tag{B-4}$$

$$=\frac{\text{volume over which a pair is correlated}}{\text{volume per nucleon}}. \tag{B-5}$$

Bethe and Goldstone[4] have shown that for a hard-core interaction of the correlation range is of the order of the core radius. Further, they expect that an attractive well outside the core will not change this substantially. This would give an estimate for ξ from (B-5) of $\xi\approx 1/10$.

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Longitudinal Polarization of Bremsstrahlung and Pair Production*

KIRK W. McVOY
Brookhaven National Laboratory, Upton, New York
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The Born approximation bremsstrahlung (and pair production) cross sections, valid at all energies and angles, are given for all possible states of longitudinal polarization of the particles involved.

When the photon-incoming electron angle (θ_0) and the photon-outgoing electron angle (θ) are both zero, a cancellation of Feynman diagrams causes *all* cross sections to vanish in Born approximation. Further, if both θ_0 and θ are small compared to m^2/E_0^2, the "spin-flip" cross sections are small (of order θ^2) relative to the "non-spin-flip" ones. When account is taken of the above cancellation, angular momentum conservation is sufficient to determine this small-angle behavior, but it explains neither the sign nor the magnitude of the bremsstrahlung circular polarization.

SINCE the recognition by Goldhaber *et al.*[1] that the circular polarization of bremsstrahlung can serve as a useful means of measuring longitudinal electron polarization, considerable interest has developed in the bremsstrahlung cross sections for specific polarization states of the incoming and outgoing particles. Although several cross-section calculations are now available,[2] little emphasis has been given to the differential cross sections for the most general combinations of longitudinal polarization, nor has an attempt been made to understand the physical origin of the polarization effects. We shall present these cross sections in detail, and call attention to their seemingly anomalous behavior at small angles. The explanation of this anomaly provides some physical insight into the details of the process, and brings to light the rather unexpected role played by orbital angular momentum in the polarization phenomena.

In both bremsstrahlung and pair production, the total number of incoming and outgoing particles (apart from the static nucleus) is three, and since each particle has two states of longitudinal polarization, there are eight possible cross sections. However, by Lenard's theorem,[3] two cross sections which differ from each other only in having all three spin directions reversed are equal in Born approximation, so to this approximation there are only four distinct cross sections. If only one of the outgoing particles is to be

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[1] Goldhaber, Grodzins, and Sunyar, Phys. Rev. **106**, 826 (1957).

[2] A definitive calculation of those integrated cross sections which are of most immediate experimental interest has recently been supplied by C. Fronsdal and H. Überall [Phys. Rev. **111**, 580 (1958)], which contains references to previous calculations.

[3] A. Lenard, Phys. Rev. **107**, 1712 (1957).