Subsidiary Conditions in Covariant Theories*

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We have investigated the effect that coordinate conditions and similar conditions will have on the formal properties of covariant theories. Two distinct types of coordinate conditions were included, those involving first derivatives of the field variables (such as the gauge condition of Lorentz and the coordinate conditions of De Donder) and those algebraic in the field variables (of which the Coulomb gauge is an example).

We have found that with either type of coordinate condition we can construct a variational principle, or a Hamiltonian formalism, which leads to physically meaningful field equations if associated with appropriate initial conditions on a space-like hypersurface. Thus the existence of a properly set Cauchy problem is always assured.

It had been found previously that the infinitesimal invariant transformations of covariant theories form a group and that the coordinate (and similar) transformations represent a normal subgroup. The members of the resulting factor group are in one-to-one correspondence with the true observables of the theory, those dynamical variables which alone possess intrinsic significance without reference to a particular frame of description and whose commutator algebra is presumably reflected in the commutators of the corresponding Hilbert operators of the quantized theory. In this paper we have established the appropriate transformation groups (and their subgroups and factor groups) of a theory with either type of coordinate conditions. We have found that in any of these versions the theory will yield the same observables with the same commutator algebra. One may therefore hope that a quantization scheme based on a theory with subsidiary conditions will be free of the arbitrariness involved in the choice of particular conditions.

1. INTRODUCTION

It is well known that electrodynamics may be developed along three, formally different, lines: (a) without any gauge conditions, (b) with a Lorentz-type gauge condition on the four electromagnetic potentials, and (c) as a theory involving only the three components of the vector potential, the scalar potential being eliminated. In the present paper we shall examine the influence that similar restrictions in the choice of coordinate frame have on general-relativistic theories.

This investigation was motivated by the circumstance that in the absence of any restrictions on the frame of reference, the variables in general-relativistic theories are usually not separable. A highly nonlinear system of partial differential equations that must be solved simultaneously represents a formidable challenge to the physicist who wishes to discuss properties of the solutions. If the problem can be simplified with the help of coordinate conditions, this approach ought to be investigated seriously.[1,2] Aside from the usefulness of coordinate conditions for the solution of the field equations of general relativity and similar theories, one may ask whether coordinate conditions may also facilitate the discovery or construction of so-called *observables*. We define observables as functions (or functionals) of field variables that are invariant with respect to coordinate transformations. Physically, they are the only quantities that lend themselves to observations in (conceptual) experiments that are constructed within the conceptual framework of the theory. Formally, observables are the generators of appropriately defined (infinitesimal) canonical transformations in a covariant theory.[3,4]

The observables in a general-relativistic theory form a Lie algebra. In a corresponding quantum theory they are Hilbert operators and possess expectation values. Other field variables do not have these properties, because they correspond to operations on quantum states that invariably lead outside the Hilbert space of physically permissible states. We are therefore concerned with the question whether in a theory that has been adorned with coordinate conditions, the observables can still be identified as such and whether they form the same Lie algebra. The result of our examination, to be detailed in what follows, is that the Lie group of a given theory can be reconstructed after the adoption of coordinate restrictions, and with it the observables. Unfortunately, there is a corollary to this result: The task of carrying out these constructions is not simplified by coordinate conditions, either. This finding, then, indicates that the adoption of coordinate conditions does not change the fundamentals of the problems involved in the quantization of general-relativistic problems. In some practical situations they may help. Our investigation includes both the Lorentz-type and the algebraic coordinate conditions.

2. LORENTZ-TYPE CONDITIONS

Consider a variational principle that is invariant with respect to a given group of transformations in-

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[1] T. De Donder, *La Gravifique Einsteinienne* (Gauthiers-Villars, Paris, 1921).

[2] V. A. Fock, Revs. Modern Phys. **29**, 325 (1957).

[3] P. G. Bergmann and I. Goldberg, Phys. Rev. **98**, 531 (1955).

[4] Bergmann, Goldberg, Janis, and Newman, Phys. Rev. **103**, 807 (1956).

volving a number of arbitrary functions. Let the most general infinitesimal transformation law for the field variables y_A be[5,6]:

$$\bar{\delta}y_A = c_{Ai}{}^{\rho}\xi^i{}_{,\rho} + c_{Ai}\xi^i - y_{A,\rho}\xi^\rho,$$
$$\xi^\rho \equiv \delta x^\rho = a^\rho{}_i\xi^i \quad (i=1,\cdots,n), \tag{2.1}$$

where the ξ^i, the "descriptors" of the invariant transformation group, are n completely unrestricted functions, except that the requisite (finite) number of derivatives is assumed to exist. The coefficients $c_{Ai}{}^{\sigma}$ are presumed to be given functions of the field variables; the c_{Ai}, functions of the field variables and their first derivatives. The choice of these coefficients is somewhat restricted by the requirement that the infinitesimal transformations form a group.

That the variational principle be invariant with respect to the transformations (2.1) means that under any of these infinitesimal transformations the Lagrangian adds a complete divergence,

$$\bar{\delta}L = Q^\rho{}_{,\rho}. \tag{2.2}$$

It follows immediately that the field equations of the theory,

$$0 = L^A \equiv \delta^A L \equiv \partial^A L - (\partial^{A\rho}L)_{,\rho},$$

where

$$\partial^A \equiv \partial/\partial y_A, \quad \partial^{A\rho} \equiv \partial/\partial y_{A,\rho}, \tag{2.3}$$

satisfy a number of differential identities,

$$(c_{Ai}{}^{\rho}L^A)_{,\rho} + a^\mu{}_i y_{A,\mu}L^A - c_{Ai}L^A \equiv 0. \tag{2.4}$$

Up to this point, we have in no way restricted the choice of frame. As a result, the field equations (2.3) do not contain the second time (x^0) derivatives of all the field variables. Suppose we were to attempt an integration of the field equations throughout (three-dimensional) space along the time axis. If we already possess a solution valid from t_0 to t, then in each space point we can choose arbitrary values for the second time derivatives of a number of field variables equal to the number of descriptors in the theory. The identities (2.4) permit us directly to specify which combinations of second time derivatives do not occur in the field equations. In these identities there occur terms with third time derivatives. The coefficients of these third time derivatives must vanish. They are

$$c_{Ai}{}^0\partial^{A0}\partial^{B0}Ly_{B,000} + \cdots \equiv 0. \tag{2.5}$$

On the other hand, the coefficients of the second time derivatives in the field equations are

$$L^A \equiv \partial^{A0}\partial^{B0}L\ddot{y}_B + \cdots. \tag{2.6}$$

The matrix of these coefficients is singular, and its n null vectors have the components $c_{Ai}{}^0$ $(i=1,\cdots,n)$. Hence, we may form n linear combinations of the second time derivatives of the field variables,

$$A^j \equiv A^{Aj}\ddot{y}_A, \tag{2.7}$$

which are independent among themselves with respect to the null vectors $c_{Ai}{}^0$, i.e., their coefficients satisfy the determinant condition

$$\det|A_i{}^j| \neq 0, \quad c_{Ai}{}^0 A^{Aj} \equiv A_i{}^j. \tag{2.8}$$

Then these linear combinations of second time derivatives (2.7) may be chosen at will. More particularly, they may be chosen to be zero, and it is therefore possible to require that throughout space-time, certain expressions containing no higher than first time derivatives of the field variables should vanish. These requirements, which are to be satisfied by the field in addition to the regular field equations (2.3), we shall write in the form

$$0 = C^j(y_A, y_{A,\rho}) \equiv C^{*j}(y_A) + C^{jA\rho}(y_B)y_{A,\rho}. \tag{2.9}$$

They must satisfy a determinant condition analogous to Eq. (2.8), specifically

$$\det|C_i{}^j| \neq 0, \quad C_i{}^j \equiv c_{Ai}{}^0 C^{jA0}. \tag{2.10}$$

We shall call this type of condition *Lorentz-type conditions*, because they are a natural generalization of the Lorentz gauge condition of electrodynamics. Their choice is restricted in principle only by the inequality (2.10), though considerations of convenience may suggest their form in a particular theory.

In close analogy to Fermi's treatment of electromagnetic theory, we may add a term to the Lagrangian L which makes it possible to satisfy the field equations of the covariant theory $L^A=0$, together with the Lorentz-type conditions (2.9), (2.10), by solving a new variational principle with suitable initial conditions. Let $a_{ij}(y,x)$ be a square array of quantities symmetric in the subscripts and with nonvanishing determinant. Then the new Lagrangian

$$L' = L + \tfrac{1}{2}a_{ij}C^iC^j \quad (i,j=1,\cdots,n) \tag{2.11}$$

has the desired properties. The new field equations are

$$L'^A = L^A + a_{ij}C^i\partial^A C^j + \tfrac{1}{2}\partial^A a_{ij}C^iC^j - (a_{ij}C^i\partial^{A\rho}C^j)_{,\rho}$$
$$\equiv L^A + \lambda^A. \tag{2.12}$$

We shall examine the appearance of second time derivatives in these new equations. We have

$$\lambda^A = -a_{ij}C^{iB0}C^{jA0}\ddot{y}_B + \cdots. \tag{2.13}$$

If we form that linear combination of the new field equations which for the L^A is free of second time deriva-

[5] J. L. Anderson and P. G. Bergmann, Phys. Rev. **83**, 1018 (1951).

[6] P. G. Bergmann and R. Schiller, Phys. Rev. **89**, 4 (1953). Earlier papers are quoted there. Equation (2.1) of the present paper is identical with Eqs. (2.2) and (2.3) of this reference.

tives, we get this time

$$c_{Ak}{}^0 L'^A = c_{Ak}{}^0 \lambda^A + \cdots = -a_{ij} C^{iB0} C_k{}^j \ddot{y}_B + \cdots. \quad (2.14)$$

By assumption, the coefficients C^{iB0} are linearly independent of each other. The determinants of a_{ij} and of $C_k{}^i$ are both nonzero; hence the n sets of coefficients of $\ddot{y}_B$ ($k=1, \cdots, n$) are linearly independent of each other, and *a fortiori* nonzero.

The new field equations (2.12) would be compatible with the original equations (2.3) if the conditions (2.9) were satisfied, because the additional terms λ^A contain as factors the C^j themselves, or their first derivatives. We can obtain a set of n homogeneous second-order partial differential equations for these n conditions by forming with the new field equations the same expressions (2.4) which vanish identically in the L^A. We shall not write these equations out, but merely state that the coefficients of the second time derivatives of the C^j form a matrix with nonvanishing determinant. Hence, if we assume that at some time t_0 the C^j and their first time derivatives vanish everywhere, then they will continue to vanish at all other times. In other words, if the conditions C^j, and their first time derivatives, vanish on some space-like hypersurface, then the solutions of Eqs. (2.12) satisfy both (2.3) and (2.9) throughout space-time.

From what has been said, it follows that the matrix of coefficients $\partial^{A0}\partial^{B0}L'$ no longer has the $c_{Ai}{}^0$ as null vectors, but the accidental appearance of new null vectors is not ruled out *a priori*. If necessary, such new singularities can be removed by an altered choice of the coefficients a_{ij}.

We shall give one example of this generalization of Fermi's treatment. If we replace the usual Lagrangian of the theory of relativity[7] by the expression

$$L' = (-g)^{\frac{1}{2}} g^{\mu\nu} \left(\left\{ \begin{matrix} \sigma \\ \rho\ \sigma \end{matrix} \right\} \left\{ \begin{matrix} \rho \\ \mu\ \nu \end{matrix} \right\} - \left\{ \begin{matrix} \sigma \\ \mu\ \rho \end{matrix} \right\} \left\{ \begin{matrix} \rho \\ \nu\ \sigma \end{matrix} \right\} + \tfrac{1}{2} g^{\rho\sigma} g_{\alpha\beta} \left\{ \begin{matrix} \alpha \\ \mu\ \nu \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ \rho\ \sigma \end{matrix} \right\} \right), \quad (2.15)$$

then it is possible to set as the coordinate conditions

$$\begin{aligned} 0 = C^\rho &\equiv -g^{\alpha\beta} \left\{ \begin{matrix} \rho \\ \alpha\ \beta \end{matrix} \right\} = -g^{\alpha\beta} g^{\rho\sigma} (g_{\alpha\sigma,\beta} - \tfrac{1}{2} g_{\alpha\beta,\sigma}) \\ &= g^{\rho\sigma}{}_{,\sigma} - \tfrac{1}{2} g^{\rho\sigma} g_{\alpha\beta} g^{\alpha\beta}{}_{,\sigma}, \end{aligned} \quad (2.16)$$

and to obtain, besides, the field equations in quasi-separated form.

To return to the Lagrangian (2.11), we may accept L' as being equivalent to L only if the conditions (2.9) and their first time derivatives are set zero on some space-like hypersurface. If we go over to a Hamiltonian treatment, then the resulting Hamiltonian H' and the associated canonical field equations will be equivalent to H and *its* associated field equations only if the same coordinate conditions are satisfied.

As a first step in our transition to the canonical formalism, we shall introduce the momentum densities associated with L'. In a self-explanatory notation, we shall set

$$\begin{aligned} \pi'^A = \partial^{A0} L' &= \partial^{A0} L + a_{ij} C^i \partial^{A0} C^j \\ &= \pi^A + a_{ij} C^i C^{jA0}, \end{aligned} \quad (2.17)$$

assuming, for the time being, that the conditions (2.9) need not be satisfied. If we now construct the Hamiltonian density according to the usual procedures, we find

$$\begin{aligned} H' = \pi'^A \dot{y}_A - L' &= (\pi^A + a_{ij} C^i C^{jA0}) \dot{y}_A - L - \tfrac{1}{2} a_{ij} C^i C^j \\ &= H + a_{ij} C^i (\dot{y}_A C^{jA0} - \tfrac{1}{2} C^j) \\ &= H + \tfrac{1}{2} a_{ij} C^i (\dot{y}_A C^{jA0} - y_{A,s} C^{jAs} - C^{*j}). \end{aligned} \quad (2.18)$$

In previous papers, it had been pointed out that in any theory of the invariance type here considered, the expressions for the canonical momentum densities can not be solved uniquely with respect to the "velocities," the $\dot{y}_A$, because the matrix of the partial derivatives $(\partial\pi^A/\partial\dot{y}_B)$ is singular; in fact, the sets of coefficients $c_{Ai}{}^0$ form the null vectors of that matrix.[5,6] In the present modified theory, the corresponding derivatives are

$$\frac{\partial \pi'^A}{\partial \dot{y}_B} = \frac{\partial \pi^A}{\partial \dot{y}_B} + a_{ij} C^{iA0} C^{jB0}. \quad (2.19)$$

Because of the inequality (2.10), we are assured that the former null vectors form nonvanishing products with the second term on the right-hand side. Hence it is possible, in principle, to express the "velocity" components in the Hamiltonian (2.18) uniquely in terms of the canonical variables and to arrive at an expression for the Hamiltonian density that is free of arbitrary functions. Accordingly, the grounds for the appearance of "constraints" in the Hamiltonian theory—the fact that the canonical momentum densities π^A are not algebraically independent of each other—are no longer present.

Constraints do, however, reappear in that the conditions (2.9) must be introduced explicitly into the canonical formalism. Our new "primary" constraints are precisely these conditions, and the "secondary" constraints are their Poisson brackets with the Hamiltonian (2.18). Since in our present formulation the canonical formalism is entirely equivalent to the Lagrangian, the canonical field equations automatically provide for the vanishing of the second time derivatives of the conditions (2.9) once these conditions and their first time derivatives have been set equal to zero on one hypersurface. The proof that there will be no tertiary constraints in this theory is, therefore, almost trivial.

[7] P. G. Bergmann, *Introduction to the Theory of Relativity* (Prentice-Hall, Inc., New York, 1942), p. 196.

That the new "constraints," i.e., the Lorentz-type conditions and their Poisson brackets with the Hamiltonian, are equivalent to the primary and secondary constraints of the theory without conditions may be demonstrated immediately with the help of the defining Eqs. (2.17). We multiply these equations by $c_{Ai}{}^0$:

$$c_{Ai}{}^0\pi'^A = c_{Ai}{}^0\pi^A + a_{kl}C_i{}^kC^l. \tag{2.20}$$

The π^A satisfy the primary constraints

$$c_{Ai}{}^0\pi^A - K_i(y_A, y_{A,s}) = 0. \tag{2.21}$$

Hence, the new momentum densities π'^A obey the equations

$$c_{Ai}{}^0\pi'^A - K_i = a_{kl}C_i{}^kC^l. \tag{2.22}$$

By assumption, the determinants of the square arrays a_{ij} and $C_i{}^j$ do not vanish [see Eq. (2.10)]. It follows that to require that the conditions (2.9) be satisfied is equivalent to requiring that the π'^A satisfy the usual primary constraints. The equivalence of the corresponding secondary constraints in both forms of the theory is a straightforward consequence, because in both theories these are defined as Poisson brackets of the primary constraints with the Hamiltonian.

Before concluding this section, we shall make a remark about the group of transformations with respect to which the theory with Lorentz-type conditions is invariant. Suppose we wish to characterize the group of infinitesimal transformations which does not modify the form of the conditions (2.9). Under this transformation group the quantities C^i, considered as fixed functions of their arguments y_A, $y_{A,\rho}$, remain equal to zero. We have

$$\bar{\delta}C^i = \partial^A C^{*i}\bar{\delta}y_A + \partial^B C^{iA\rho}\bar{\delta}y_B y_{A,\rho} + C^{iAs}\bar{\delta}y_{A,s} + C^{iA0}\bar{\delta}\dot{y}_A = 0. \tag{2.23}$$

These equations are a set of linear homogeneous second-order differential equations for the descriptors ξ^ρ, equal in number to the descriptors. The coefficients of the second-order time derivatives of the descriptors, which appear only in the last term of Eq. (2.23), are nonsingular, again according to Eq. (2.10):

$$\begin{aligned} 0 &= \cdots + C^{iA0}c_{Aj}{}^0\xi^j{}_{,00} \\ &= \cdots + C_j{}^i\xi^j{}_{,00}. \end{aligned} \tag{2.24}$$

Accordingly, these differential equations may be solved with respect to the second-order time derivatives of the descriptors. An individual solution is completely determined if the descriptors and their first time derivatives are chosen on one space-like hypersurface; these initial conditions are free of any further restrictions.

3. ALGEBRAIC CONDITIONS

Instead of adding to the Lagrangian a quadratic combination of first-order conditions, it is also possible to introduce, in a similar manner, algebraic conditions. Suppose we require that n algebraic functions of the field variables, $D^i(y_A)$, vanish. These functions must be chosen so that they satisfy the determinantal inequality

$$\det|D_i{}^j| \neq 0, \quad D_i{}^j \equiv c_{Ai}{}^0\partial^A D^j. \tag{3.1}$$

They may be introduced into the Lagrangian by means of a quadratic additional term,

$$L' = L + \tfrac{1}{2}a_{ij}D^iD^j. \tag{3.2}$$

It can be shown easily that this Lagrangian L' yields equations that are equivalent to the original field equations if on one hypersurface the D^i vanish. The modified field equations,

$$L'^A = L^A + a_{ij}D^i\partial^A D^j + \tfrac{1}{2}D^iD^j\partial^A a_{ij}, \tag{3.3}$$

will then assure that the first and all following time derivatives of the D^i will be zero as well. In detail, these expressions are

$$\begin{aligned} &(c_{Ai}{}^\rho L'^A)_{,\rho} + a^\mu{}_i y_{A,\mu}L'^A - c_{Ai}L'^A \\ &\quad\equiv \tfrac{1}{2}\{[c_{Ai}{}^\rho\partial^A(a_{kl}D^kD^l)]_{,\rho} + (a^\rho{}_i a_{kl}D^kD^l)_{,\rho} \\ &\qquad - c_{Ai}\partial^A(a_{kl}D^kD^l)\} \\ &\quad= (c_{Ai}{}^\rho a_{kl}\partial^A D^kD^l)_{,\rho} + \tfrac{1}{2}(c_{Ai}{}^\rho\partial^A a_{kl}D^kD^l)_{,\rho} \\ &\qquad + \tfrac{1}{2}(a^\rho{}_i a_{kl}D^kD^l)_{,\rho} - \tfrac{1}{2}c_{Ai}\partial^A a_{kl}D^kD^l - c_{Ai}a_{kl}\partial^A D^kD^l \\ &\quad= a_{kl}D_i{}^k\dot{D}^l + D^l(a_{kl}D_i{}^k)_{,0} \\ &\qquad + (\delta_s{}^\rho c_{Ai}{}^s a_{kl}\partial^A D^kD^l + \tfrac{1}{2}c_{Ai}{}^\rho\partial^A a_{kl}D^kD^l \\ &\qquad + \tfrac{1}{2}a^\rho{}_i a_{kl}D^kD^l)_{,\rho} - \tfrac{1}{2}c_{Ai}\partial^A a_{kl}D^kD^l - c_{Ai}a_{kl}\partial^A D^kD^l \\ &\quad= 0. \end{aligned} \tag{3.4}$$

Only the first term of this combination of field equations and their first derivatives contains the time derivative of an algebraic condition D^i multiplied by a nonvanishing coefficient.

The additional terms on the right-hand side of Eq. (3.3) contribute no time derivatives, and these equations are, by themselves, not any simpler in their mathematical structure than the unmodified equations L^A. They can be solved with respect to their highest (second) time derivatives if we add to them the twice-differentiated algebraic conditions D^i.

The implications of algebraic conditions become much clearer if we introduce a transformation of variables that makes n of the field variables equal to the algrbraic conditions D^i. In other words, if these conditions are satisfied, then n of the field variables will vanish at all world points, and the number of *de facto* field variables is reduced by that number. Accordingly, we shall introduce new variables Y^M, consisting of two classes, η^a and D^i, which are algebraically independent functions of the original field variables y_A,

$$\det\left|\frac{\partial(\eta^a,D^i)}{\partial y_A}\right|\neq 0. \tag{3.5}$$

It is very easy to show that the new field equations are linear combinations of the original field equations and that they satisfy similar differential identities. The equations themselves are

$$L_M=\partial_M y_A L^A, \tag{3.6}$$

and their identities may be written in the form

$$\begin{aligned}(c^M{}_i{}^\rho L_M)_{,\rho}+a^\rho{}_i Y^M{}_{,\rho}L_M-c^M{}_i L_M\equiv 0,\\ c^M{}_i{}^\rho=\partial^A Y^M c_{Ai}{}^\rho,\\ c^M{}_i=\partial^A Y^M c_{Ai}.\end{aligned} \tag{3.7}$$

If we separate in these identities the summations over the index M into separate summations over the indices a (belonging to the variables η^a) and j (belonging to the D^j), then Eq. (3.7) takes the form

$$\begin{aligned}(D_i{}^j L_j)_{,0}+(c^j{}_i{}^s L_j)_{,s}+(c^a{}_i{}^\rho L_a)_{,\rho}+\cdots\equiv 0,\\ c^j{}_i{}^0\equiv D_i{}^j.\end{aligned} \tag{3.8}$$

In other words, if we satisfy everywhere the field equations L_a, then the remaining field equations, L_j, satisfy a set of n first-order linear and homogeneous equations. It is then sufficient to satisfy the L_j on one space-like hypersurface to assure that they will be satisfied everywhere else. It is again essential for this result that the determinant of the $D_i{}^j$ does not vanish.

We can draw this conclusion: If in the original action principle we introduce the new field variables Y^M, if we then set the n variables D^j equal to zero, and if we vary only with respect to the remaining $(N-n)$ field variables η^a, then the new field equations will be equivalent to the original field equations if we satisfy the field equations L_j on one space-like hypersurface. All that is left of the variables we have "killed" is a set of n initial conditions.

We shall now go over to the canonical formalism. If we operate with all N field variables Y^M, we obtain from the Lagrangian (3.2) the Hamiltonian

$$H'=\pi_M\dot{Y}^M-L(Y^M,\dot{Y}^M,Y^M{}_{,s})-\tfrac{1}{2}a_{ij}D^iD^j. \tag{3.9}$$

If we are to satisfy the conditions D^j, then not only the D^j but their time derivatives as well must vanish.

From previous investigations[8] it is known that the Hamiltonian (3.9) is not a unique function of the canonical field variables in the absence of conditions, because under the most general circumstances the $\dot{Y}^M$ are not unique functions of the canonical field variables, either. In the presence of the conditions D^j, however, we shall be able to resolve this lack of uniqueness. The algebraic relationships between the canonical field variables that we have previously called the primary constraints are of the form

$$\begin{aligned}g_i\equiv c^M{}_i{}^0\pi_M-K_i\\ \equiv D_i{}^j\pi_j+c^a{}_i{}^0\pi_a-K_i\equiv 0.\end{aligned} \tag{3.10}$$

They can be solved with respect to the n variables π_j. Suppose now we wish to choose the Hamiltonian so that the time derivatives of the conditions D^j vanish. This requirement is equivalent to requiring that the Hamiltonian should be free of the canonical momentum densities π_j. All that needs to be done is to obtain some expression for the Hamiltonian density (3.9) that is free of velocities, and then to eliminate the π_j wherever they may occur by means of the constraints (3.10). This procedure leads to a unique expression, no matter which particular expression for the Hamiltonian density is chosen as the point of departure. The resulting Hamiltonian density then is free of the π_j; hence the D^j are constants of the motion. If they are set zero on one space-like hypersurface they will remain zero permanently.

There is, of course, some difference between the Hamiltonian density obtained from L', Eq. (3.1), and that obtained from the original Lagrangian L, even when in both expressions the π_j have been eliminated. That difference is the last term, $-\frac{1}{2}a_{ij}D^iD^j$. Its effect is zero if the D^i have been set zero; otherwise this additional term will appear in the equations for the time dependence of the π_j, and in the equations for those π_a whose conjugate η^a occur in the a_{ij}. At any rate, if the D^i are satisfied, then there is only one Hamiltonian density, the π_j occur nowhere in the system of the canonical field equations, and the total number of canonical field variables has been reduced by $2n$. As the result of this amputation, the primary constraints have disappeared from the theory.

The transformation law of the D^j is

$$\bar{\delta}D^j=\bar{\delta}y_A\partial^A D^j=D_i{}^j\xi^i+\cdots. \tag{3.11}$$

Hence, if the D^j are satisfied, then there exists a well-defined transformation group that maintains them. The descriptors of the infinitesimal group satisfy n first-order differential equations, which can be solved with respect to their time derivatives; the members of the infinitesimal group may be characterized by the initial values of the descriptors ξ^i at one space-like hypersurface, i.e., by a set of n arbitrary functions of the three spatial coordinates. On the other hand, we also have a set of n secondary constraints to satisfy, again on one hypersurface.

4. TRUE OBSERVABLES

We call true observables of a classical (i.e., non-quantum) theory those quantities that are in principle

[8] Bergmann, Penfield, Schiller, and Zatzkis, Phys. Rev. **80**, 81 (1950).

measurable within the theory. For instance, in electrostatics the (scalar) electric potential at a given point in space and time with respect to infinity is observable, whereas in electrodynamics it is not, because of the freedom of gauge transformations. In order to make the concept of measurability more precise, let us say that the value of a true observable at a time t can be predicted (at least in principle) from a sufficient set of data at an earlier time t_0.[9] Otherwise a quantity whose behavior is completely unpredictable within the framework of the theory is nevertheless measurable; but then one would search for a more complete theory, in which the quantity in question would also be predictable.

A quantity can fail to be a true observable only within the framework of a theory possessing a group of invariant transformations which depend on arbitrary functions of the time. We shall now demonstrate that those quantities which are not themselves invariant under this group of transformations are not true observables: Assume that at a time t_0 we have a sufficient set of data to enable us, by use of the field equations, to predict the future behavior of some quantity A. Let us perform an invariant transformation leaving the initial data unchanged, a procedure which is always possible because of the arbitrary time depencence of the functions on which the transformation depends. The field equations will also remain unchanged under this transformation, for that is what is meant by an invariant transformation. If A is not invariant under this transformation, there will be some time t at which the values of A before and after the transformation are not the same. But, since neither the field equations nor the initial data at the time t_0 change, the value predicted for A at the time t cannot change. Hence, if A is not invariant it cannot be a true observable.

Let us now consider a quantity A which is a true observable. We have seen that if we are given a sufficient set of data at one time, t_0, we can predict the development of A in the course of time. In other words, we may solve the field equations and their derivatives for all those time derivatives of A which are not determined by the initial data. Accordingly, we can construct a constant of the motion whose value is equal to the value of A at the time t_0; if we restrict ourselves to consideration of only those quantities which are constants of the motion, we lose no information about the true observables.

Let us now specify that the field equations are to be derived from an action principle, so that it is possible to speak of the generator of a transformation.[6] Then the constants of the motion generate the group of all the invariant transformations[10] of the theory, which include not only the transformations depending on arbitrary functions, which we have been discussing up to now, and which are, in fact, generated by zero generators (at least, zero *modulo* the field equations), but also all the other invariant transformations generated by nonzero true observables. It may be shown[4] that the transformations generated by zero generators form a normal subgroup of all the invariant transformations, so that in order to obtain a relization of the true observables we go to the generators of the factor group formed by using this subgroup as a normal divisor.

5. TRUE OBSERVABLES IN THE PRESENCE OF COORDINATE CONDITIONS

Before we show how to identify the true observables when the theory is modified by coordinate conditions, let us give a mathematical formulation to the ideas presented in the previous section. In what follows, we shall gain simplicity of notation by restricting ourselves to a particle theory with a finite number of degrees of freedom, rather than a field theory, and we shall consider Lagrangians which are at most quadratic in the velocities.

We shall denote by $\mathcal{L}$ the group of all the invariant transformations of the Lagrangian, L. The group $\mathcal{L}$ may be defined as consisting of those transformations under which the change in the *value* of the Lagrangian (for a given physical state) and the change in the *form* of the Lagrangian (as a function of the arguments q_k, $\dot{q}_k$) are given by

$$\bar{\delta}L=\dot{Q}, \quad \delta' L=0, \tag{5.1}$$

respectively, where Q is an arbitrary function of the q_k and $\dot{q}_k$. In order to relate the requirements (5.1) to a condition on the transformation quantities $\bar{\delta}q_k$ we use the general relation between $\bar{\delta}F$ and $\delta' F$ for an arbitrary function $F(q_k,\dot{q}_k)$:

$$\begin{gathered}\delta' F=\bar{\delta}F-\partial^k F\bar{\delta}q_k-\partial^{k\cdot}F\bar{\delta}\dot{q}_k,\\ \partial^k\equiv\partial/\partial q_k, \quad \partial^{k\cdot}\equiv\partial/\partial\dot{q}_k.\end{gathered} \tag{5.2}$$

The requirements (5.1) imply then that the generator C of such a transformation is related to the transformation quantities $\bar{\delta}q_k$ by the equation[6]

$$\dot{C}+L^k\bar{\delta}q_k=0, \tag{5.3}$$

where L^k stands for the left-hand sides of the equations of motion:

$$L^k\equiv\partial^k L-\frac{d}{dt}(\partial^{k\cdot}L). \tag{5.4}$$

The subgroup of $\mathcal{L}$ consisting of those transformations which are generated by zero generators we shall

[9] In quantum theory the same statement holds for the expectation value of a true observable; moreover, the uncertainty of this expectation value can be reduced below any fixed finite bound by a suitable selection of the data available at the time t_0.

[10] Here, and in what follows, we mean infinitesimal transformations, thus ruling out discontinuous transformations such as parity, time reversal, etc. However, in quantum theory it is meaningful to speak of an infinitesimal parity transformation, for example, as the transformation which takes $\psi(x,t)$ into $\psi(x,t)+\epsilon\psi(-x,t)$. Such infinitesimal transformations have no classical analogs.

call $\mathcal{C}$. If we form the factor group $\mathcal{L}/\mathcal{C}$, the zero generators merge into the identity element; the generators of the remaining elements are the true observables of the theory; and the relationship between the elements of $\mathcal{L}/\mathcal{C}$ and the true observables is reversibly unique.

The existence of nontrivial transformations in the subgroup $\mathcal{C}$ (that is, the existence of a group of invariant transformations depending on arbitrary functions of the time) implies that the equations of motion cannot be solved for all of the accelerations (see Sec. 2) unless one is willing to introduce coordinate conditions. As in Secs. 2 and 3, we introduce a modified Lagrangian L' which is quadratic in the coordinate conditions:

$$L'=L+\tfrac{1}{2}a^{(rs)}D_{(r)}D_{(s)}. \tag{5.5}$$

We shall use $D_{(r)}$ to stand for either Lorentz-type or algebraic conditions; in the former case the $D_{(r)}$ are to be linear in the velocities $\dot{q}_k$, in the latter case they are to be independent of the $\dot{q}_k$. In either case the $a^{(rs)}$ shall depend only on the undifferentiated coordinates q_k (and possibly the time coordinate t).

We determine the group of transformations that is to be considered within the framework of the modified theory as follows. First, if we are to maintain our coordinate conditions, the transformations must leave them invariant:

$$\bar{\delta}D_{(r)}=0, \quad \delta' D_{(r)}=0. \tag{5.6}$$

Such transformations have been discussed in the closing paragraphs of Secs. 2 and 3.

In the absence of coordinate conditions, one ordinarily requires that the total change in the value of a Lagrangian be a time derivative $\dot{Q}$, so that the variation of the action integral may lead to the same physical results. But in the presence of coordinate conditions, to require that $\bar{\delta}L'$ be a total time derivative would be too severe, as the Lagrangian L' will correspond to physical reality only when the coordinate conditions are satisfied; that is, L' is the Lagrangian only *modulo* the equations

$$D_{(r)}=0. \tag{5.7}$$

The appropriate requirement is then that part of L' which corresponds to the actual physical situation transform by a total time derivative. Considering the first Eq. (5.6), we have for the total change in the value of L' the expression

$$\bar{\delta}L'=\dot{Q}+\tfrac{1}{2}\bar{\delta}a^{(rs)}D_{(r)}D_{(s)}. \tag{5.8}$$

We have seen in Sec. 4 that we lose no information about the true observables if we restrict ourselves to invariant transformations of the Lagrangian, since the invariant transformations are generated by constants of the motion, and a constant of the motion can be constructed corresponding to every true observable. If our coordinate conditions are of the Lorentz type, the Lagrangian L' is chosen in such a manner that the modified equations of motion can be solved explicitly for all of the accelerations; hence it is possible to construct a constant of the motion corresponding to any arbitrary function of the q_k and $\dot{q}_k$. If the coordinate conditions are algebraic, we use them to reduce the number of coordinates in such a manner that the remaining accelerations are determined by the modified equations of motion; as for the coordinates that have been eliminated, we shall see in the discussion at the end of this section that they carry no information about the true observables. Hence there is no loss in generality if we restrict ourselves to invariant transformations of the Lagrangian L':

$$\delta' L'=0. \tag{5.9}$$

One might wonder, in view of the fact that $\bar{\delta}L'$ is not a total time derivative, whether it is still true that invariant transformations are generated by constants of the motion. We shall see in Eqs. (5.12) that this relationship still holds.

We shall use the notation $\mathcal{L}'$ for the group of transformations defined by Eqs. (5.6), (5.8), and (5.9).

We shall now demonstrate that the group $\mathcal{L}'$ is homomorphic to $\mathcal{L}/\mathcal{C}$, the factor group which we have used to define the true observables. This homomorphism implies[11] that there is a factor group of $\mathcal{L}'$, say $\mathcal{L}'/\mathcal{E}$, which is isomorphic to $\mathcal{L}/\mathcal{C}$, where $\mathcal{E}$ denotes the transformations in $\mathcal{L}'$ which map into the identity element of $\mathcal{L}/\mathcal{C}$.

We shall begin by showing that $\mathcal{L}'$ is a subgroup of $\mathcal{L}$. We see from the definition (5.5) of the Lagrangian L', together with the requirements (5.6), (5.8), and (5.9), which define the group $\mathcal{L}'$, that under a transformation in $\mathcal{L}'$ the original Lagrangian L transforms according to the equations

$$\bar{\delta}L=\dot{Q}, \quad \delta' L=-\tfrac{1}{2}\delta' a^{(rs)}D_{(r)}D_{(s)}. \tag{5.10}$$

If we fix the transformation properties of $a^{(rs)}$ by requiring that $\delta' a^{(rs)}=0$, then we see that Eqs. (5.10) reduce to Eqs. (5.1), the defining equations for $\mathcal{L}$.

We define a mapping of $\mathcal{L}'$ onto $\mathcal{L}/\mathcal{C}$ as follows: We note that each element of the factor group $\mathcal{L}/\mathcal{C}$ is a set of elements of $\mathcal{L}$ which differ from one another by coordinate transformations, and each element of $\mathcal{L}$ appears in one and only one such set. Since $\mathcal{L}'$ is a subgroup of $\mathcal{L}$, we may define the mapping by associating with each element of $\mathcal{L}'$ that element of $\mathcal{L}/\mathcal{C}$ in which it appears. The law of group multiplication is obviously preserved under this mapping; so all that remains to be shown is that each element of $\mathcal{L}/\mathcal{C}$ has at least one counter image in $\mathcal{L}'$. To this end we choose an arbitrary element of $\mathcal{L}/\mathcal{C}$, and from the set of elements of $\mathcal{L}$ which comprise it we choose an arbitrary transformation. If this transformation leaves the coordinate conditions invariant, then it is a member of $\mathcal{L}'$, and we have found a counter image. If it does not, i.e., if as a result

[11] See, for example, L. Pontrjagin, *Topological Groups* (Princeton University Press, Princeton, 1946), p. 11.

of the chosen transformation the $D_{(r)}$ change their values, then we shall multiply our first transformation by a *coordinate transformation* taking us into such a frame that the $D_{(r)}$ resume their original set of values. The product of the two transformations will belong to $\mathcal{L}'$; inasmuch as the original transformation differs from our product only by a coordinate transformation, the two belong to the same element of $\mathcal{L}/\mathcal{C}$, and the proof is complete.[12]

It remains to discuss the problem of finding the generators of the transformations in $\mathcal{L}'$. From Eqs. (5.8) and (5.9), which specify the way the Lagrangian L' is to transform under a transformation in $\mathcal{L}'$, we have

$$\partial^k L' \bar{\delta} q_k + \partial^{k\cdot} L' \bar{\delta} \dot{q}_k = \dot{Q} + \tfrac{1}{2}\partial^k a^{(rs)} \bar{\delta} q_k D_{(r)} D_{(s)}. \quad (5.11)$$

Differentiation by parts of the second term on the left-hand side of Eq. (5.11) yields

$$\dot{C}' + M^k \bar{\delta} q_k = 0, \quad (5.12a)$$

where

$$C' = -Q + \partial^{k\cdot} L' \bar{\delta} q_k, \quad (5.12b)$$

and

$$M^k = \partial^k L' - \frac{d}{dt}(\partial^{k\cdot} L') - \tfrac{1}{2}\partial^k a^{(rs)} D_{(r)} D_{(s)}. \quad (5.12c)$$

We may consider the equations $M^k = 0$ as an alternative form of the equations of motion. When the coordinate conditions are satisfied, we have $M^k = L^k = L'^k$, Eqs. (2.12), (3.3).

In the following discussion, it will be convenient to treat the two types of coordinate conditions separately.

(*A*) *Lorentz-type conditions.*—In this case, one may solve the modified equations of motion for all of the accelerations. Hence, given an arbitrary function of the q_k and $\dot{q}_k$, say $F(q_k,\dot{q}_k)$, we can construct a $C'(q_k,\dot{q}_k,t)$, satisfying Eq. (5.12a), which reduces to $F(q_k,\dot{q}_k)$ at a specified time t_0. However, the transformations appearing in Eqs. (5.12) are not necessarily members of $\mathcal{L}'$ unless we also impose the requirements (5.6), which express the invariance of the coordinate conditions under a transformation in $\mathcal{L}'$. If Eqs. (5.6) are satisfied as well, then we should expect to find a connection between the generator C' appearing in Eqs. (5.12) and the generator C appearing in Eq. (5.3). In fact, we can obtain this connection by noting that Eqs. (5.1) and (5.3) imply that

$$C = -Q + \partial^{k\cdot} L \bar{\delta} q_k. \quad (5.13)$$

Comparison of Eq. (5.13) with Eq. (5.12b) shows that

$$C = C' - a^{(rs)} D_{(r)} \partial^{k\cdot} D_{(s)} \bar{\delta} q_k. \quad (5.14)$$

Because not all the transformations satisfying Eqs. (5.12) satisfy the requirements (5.6), we must find those linear combinations which do. We may obtain equations for the generators of these combinations as follows.

Suppose we have found a complete set of modified constants of the motion, satisfying Eqs. (5.12) (e.g., the constants of the motion corresponding to all the q_k and $\dot{q}_k$). Let us label the members of this set by a subscript, as C'_A. Any function of the C'_A, say $F(C'_A)$, is also a modified constant of the motion, since its time derivative is given by

$$\dot{F} = \frac{\partial F}{\partial C'_A}\dot{C}'_A = -\frac{\partial F}{\partial C'_A} M^k \bar{\delta}_A q_k, \quad (5.15)$$

where $\delta_A q_k$ is the transformation generated by C'_A. Hence

$$\bar{\delta}_F q_k = \frac{\partial F}{\partial C'_A}\bar{\delta}_A q_k, \quad (5.16)$$

where $\bar{\delta}_F q_k$ is the transformation generated by F. If we require that F generate a transformation in $\mathcal{L}'$, then $\bar{\delta}_F q_k$ must satisfy Eqs. (5.6), which may be rewritten in the form

$$\partial^k D_{(r)} \bar{\delta} q_k + \partial^{k\cdot} D_{(r)} \bar{\delta} \dot{q}_k = 0. \quad (5.17)$$

Finally, by substituting Eq. (5.16) into Eq. (5.17) and rearranging terms, we obtain the partial differential equation for F:

$$(\partial^k D_{(r)} \bar{\delta}_A q_k + \partial^{k\cdot} D_{(r)} \bar{\delta}_A \dot{q}_k)\frac{\partial F}{\partial C'_A} = \partial^{k\cdot} D_{(r)} M^l \bar{\delta}_A q_k \bar{\delta}_B q_l \frac{\partial^2 F}{\partial C'_A \partial C'_B}. \quad (5.18)$$

(*B*) *Algebraic conditions.*—As in Sec. 3, we may introduce a new set of coordinates such that the $D_{(r)}$ are among the new coordinates. We may then set the coordinates which are equal to the $D_{(r)}$ equal to zero, and consider only the remaining coordinates and the equations of motion obtained by varying with respect to them, provided that we impose certain initial conditions. These initial conditions are obtained from the equations of motion corresponding to the "zero" coordinates by simply setting these latter coordinates identically equal to zero, and requiring that what is left be satisfied at a particular time. We shall denote these conditions by $\bar{D}_{(r)}$.

Since the transformations in $\mathcal{L}'$ leave the coordinate conditions invariant, it is clear that these transformations preserve the separation of the coordinates into the "zero" coordinates and the remaining ones; that is, under a transformation in $\mathcal{L}'$, the "zero" coordinates are unchanged and the remaining coordinates transform among themselves. Furthermore, since these transformations also leave the form of the equations of motion unchanged, the $\bar{D}_{(r)}$ will be left invariant. Hence, the group $\mathcal{L}'$ may be characterized as those invariant transformations of the "reduced" Lagrangian

[12] It could, of course, happen that more than one transformation in $\mathcal{L}'$ maps into each element of $\mathcal{L}/\mathcal{C}$. For example, in electromagnetic theory with the Fermi form of the Lagrangian there are gauge transformations within the Lorentz gauge, and two transformations in $\mathcal{L}'$ differing only by such a gauge change would map into the same element of $\mathcal{L}/\mathcal{C}$.

(i.e., the Lagrangian obtained by setting the prescribed number of coordinates equal to zero) which leave the initial conditions $\bar{D}_{(r)}$ invariant.

We may now proceed along lines analogous to the discussion given for Lorentz-type conditions. As in that case, we may construct constants of the motion corresponding to any arbitrary function of the q_k and $\dot{q}_k$, where now k runs over the reduced number of variables, since the "reduced" equations of motion can be solved for all of the remaining accelerations. We may also confirm that this process of reduction does not lose any information about the true observables: If for some C' Eq. (5.12a) were to involve an M^i belonging to one of the "zero" coordinates, then C' would generate a transformation in which $\bar{\delta}q_i$ had a nonzero value, in violation of the conditions (5.6); hence the transformation would not belong to $\mathfrak{L}'$, and C' would not be a true observable.

Again, if we have found a complete (reduced) set of constants of the motion, we may proceed to deduce the requirements to be imposed in order to preserve the conditions $\bar{D}_{(r)}$. We shall again be led to equations of the form (5.18), where now we have $\bar{D}_{(r)}$ instead of $D_{(r)}$, and the sums run only over the reduced variables.

6. AN EXAMPLE

One of the simplest Lagrangians exhibiting invariance under a group of transformations depending on an arbitrary function is

$$L=\tfrac{1}{2}[(\varphi\mathbf{k}+d\mathbf{A}/dt)^2-(\mathbf{k}\times\mathbf{A})^2], \tag{6.1}$$

which is essentially the Lagrangian for the electromagnetic field in $\mathbf{k}$ space. It is clearly invariant under the gauge transformation

$$\bar{\delta}\mathbf{A}=G\mathbf{k}, \quad \bar{\delta}\varphi=-dG/dt, \tag{6.2}$$

where G is an arbitrary function of the time. The equations of motion are

$$\begin{aligned} L^\varphi &\equiv k^2\varphi+\mathbf{k}\cdot(d\mathbf{A}/dt)=0, \\ L^\mathbf{A} &\equiv (\mathbf{k}\cdot\mathbf{A})\mathbf{k}-k^2\mathbf{A}-(d\varphi/dt)\mathbf{k}-d^2\mathbf{A}/dt^2=0. \end{aligned} \tag{6.3}$$

It is well known that the true observables for this problem (that is, the gauge-invariant quantities) are the transverse parts of $\mathbf{A}$ and $d\mathbf{A}/dt$. The combination $k\varphi+dA_1/dt$, where A_1 is the longitudinal part of $\mathbf{A}$, is also gauge invariant, but it is clearly zero *modulo* the equations of motion. By way of illustration, let us obtain these true observables by the methods of Sec. 5.

To illustrate the Lorentz-type coordinate condition, we use the Lorentz gauge itself, which is given by

$$D\equiv d\varphi/dt-\mathbf{k}\cdot\mathbf{A}=0. \tag{6.4}$$

We then write the modified (Fermi) Lagrangian as

$$L'=L-\tfrac{1}{2}D^2. \tag{6.5}$$

We see that the modified equations of motion are

$$\begin{aligned} M^\varphi &\equiv k^2\varphi+d^2\varphi/dt^2=0, \\ M^\mathbf{A} &\equiv -k^2\mathbf{A}-d^2\mathbf{A}/dt^2=0. \end{aligned} \tag{6.6}$$

As the first step in finding the true observables, we construct a complete set of modified constants of the motion. The constant of the motion corresponding to φ is given by

$$C'_\varphi=\varphi-(d\varphi/dt)t-\tfrac{1}{2}k^2\varphi t^2+\frac{1}{3!}k^2(d\varphi/dt)t^3+\cdots. \tag{6.7}$$

We see that

$$dC'_\varphi/dt=-M^\varphi\bar{\delta}_\varphi\varphi, \tag{6.8}$$

where

$$\bar{\delta}_\varphi\varphi=t-\frac{1}{3!}k^2t^3+\cdots. \tag{6.9}$$

We obtain similar results for the modified constants of the motion corresponding to $d\varphi/dt$, the three components of $\mathbf{A}$, and the three components of $d\mathbf{A}/dt$, giving a total of eight such constants with their corresponding transformations.

We must now find those generators which generate transformations leaving the gauge condition (6.4) invariant; that is, the transformations must satisfy the equation

$$\bar{\delta}(d\varphi/dt)-\mathbf{k}\cdot\bar{\delta}\mathbf{A}=0. \tag{6.10}$$

It is easily found that the following six combinations of the generators such as (6.7) generate transformations satisfying Eq. (6.10):

$$\begin{array}{llll} \text{(a)} & kC'_\varphi+C'_{dA_1/dt}, & \text{(d)} & C'_{dA_2/dt}, \\ \text{(b)} & C'_{d\varphi/dt}-kC'_{A_1}, & \text{(e)} & C'_{A_3}, \\ \text{(c)} & C'_{A_2}, & \text{(f)} & C'_{dA_3/dt}. \end{array} \tag{6.11}$$

We shall show that the first two generate transformations which map into the identity element of the factor group $\mathfrak{L}/\mathfrak{C}$, whereas the last four serve to identify the true observables of the theory.

In order to show that the generators (6.11a) and (6.11b) generate transformations which map into the identity element of $\mathfrak{L}/\mathfrak{C}$, it is sufficient to show that under the correspondence indicated in Eq. (5.14) we obtain a quantity that is zero *modulo* the original equations of motion. In terms of the present example, Eq. (5.14) becomes

$$C=C'+(d\varphi/dt-\mathbf{k}\cdot\mathbf{A})\bar{\delta}\varphi. \tag{6.12}$$

If we put in for C' in Eq. (6.12) the generator (6.11a), we obtain

$$C=L^\varphi(k^{-1}-\tfrac{1}{2}kt^2+\cdots). \tag{6.13}$$

We obtain a similar expression corresponding to the generator (6.11b).

We find that for the remaining four generators the correspondence (6.12) reduces to $C=C'$. The four generators we thus obtain are simply the constants of the motion corresponding to the transverse parts of **A** and $d\mathbf{A}/dt$, the true observables of the theory.

As an illustration of algebraic coordinate conditions, we may use the Coulomb gauge, which is given by

$$D \equiv \varphi = 0. \tag{6.14}$$

The "reduced" Lagrangian is then

$$L' = \tfrac{1}{2}[(d\mathbf{A}/dt)^2 - (\mathbf{k}\times\mathbf{A})^2], \tag{6.15}$$

and the corresponding equations of motion are

$$M^{\mathbf{A}} \equiv (\mathbf{k}\cdot\mathbf{A})\mathbf{k} - k^2\mathbf{A} - d^2\mathbf{A}/dt^2 = 0. \tag{6.16}$$

The initial condition which must be satisfied is seen to be

$$\bar{D} \equiv \mathbf{k}\cdot(d\mathbf{A}/dt) = 0. \tag{6.17}$$

In analogy to our treatment of the Lorentz-type condition, we construct the modified constants of the motion corresponding to the three components of **A** and the three components of $d\mathbf{A}/dt$. If we then require that the condition $\bar{D}$ be left invariant, we see that the transformations must satisfy the equation

$$\mathbf{k}\cdot\bar{\delta}(d\mathbf{A}/dt) = 0. \tag{6.18}$$

We find that the requirement (6.18) rules out only the one generator C'_{A_1}. Of the remaining five generators, it is seen that $C'_{dA_1/dt}$ is zero *modulo* the condition (6.17), which implies that the corresponding transformation maps into the identity element of $\mathfrak{L}/\mathfrak{C}$. Once again we are left with the transverse parts of **A** and $d\mathbf{A}/dt$ as the true observables.

7. CONCLUSION

The use of coordinate conditions leads to a suitable definition of true observables, but not to an actual prescription for obtaining them in a given theory. What we have done, in essence, is to replace one problem by another. Without coordinate conditions it is difficult to find constants of the motion; with coordinate conditions the construction of (modified) constants of the motion is almost trivial, but among them we must find those constants of the motion whose corresponding transformations leave the coordinate conditions invariant. In the example of the last section, we obtained these generators with little difficulty. However, in a theory of greater formal complexity, such as general relativity, it would be necessary to solve Eqs. (5.18) by approximate means in order to obtain expressions for the true observables.

One possible approach might be to use an approximation method analogous to that used by Newman and Bergmann.[13] They have shown how an approach analogous to that of Einstein, Infeld, and Hoffmann[14] (EIH) can be used to obtain the true observables in the canonical formalism. The original EIH method was designed to obtain the equations of motion for point singularities from the field equations of general relativity. At each order of the expansion conditions on the singularities arise out of the solution for the previous order. Similarly, Newmann and Bergmann expand the Lagrangian in such a manner that the problem may be completely solved for the zeroth-order case. Then the true observables at each order are found from those of the previous order by a specified procedure. If the theory contains M true observables, for example, then at the pth order one would obtain Mp true observables, corresponding to the p terms in the expansion of each true observable to that order. If this scheme were to be applied to the formalism of the present paper, then it is a direct result of certain theorems proven by Newman and Bergmann that of the $M(p+1)$ observables at the $(p+1)$ order, Mp of them are precisely those of the pth order. We are thus left with the problem of finding M new observables at each order. Since at each order the new variables enter only linearly, it is an easier task to find the observables in this stepwise fashion than to attempt to find the M exact observables all at once.[15]

In closing, we should like to mention one other approach to the problem of true observables, which is due to Komar[16] and Géhéniau and Debever.[17] This method is based specifically on the properties of metric spaces. The essence of their approach is to label points in the space by the values of four independent scalars instead of the four conventional coordinates. If we then consider any other scalar as a function of the first four, it would be an invariant under transformations of the conventional coordinates. The systematic reconstruction of general relativity in terms of these observables is being actively pursued at this time.

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[13] E. Newman and P. G. Bergmann, Revs. Modern Phys. **29**, 443 (1957).

[14] See, for example, L. Infeld, Revs. Modern Phys. **29**, 398 (1957), where earlier references are given.

[15] We should like to express thanks to Dr. Newman for pointing out the preceding line of argument.

[16] A. Komar, Phys. Rev. **99**, 662(A) (1955); Bull. Am. Phys. Soc. Ser. II, **3**, 68 (1958); Phys. Rev. (to be published).

[17] J. Géhéniau and R. Debever, *Jubilee of Relativity Theory*, edited by A. Mercier and M. Kervaire (Birkhäuser Verlag, Basel, 1956), p. 101.