quantities with which we are dealing. In point of fact, we know how to interpret these quantities as true observables. (An apt analog in electromagnetic theory may clarify this point of view. The transverse components of the vector potential may be considered as the gauge-invariant true observables; or they may be considered as the components of the vector potential in a particular gauge, namely the radiation gauge.) An investigation of the quantization of general relativity from the point of view of considering (2.4) as a conventional coordinate condition is now in progress.

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Spectral Representations in Perturbation Theory. I. Vertex Function*

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The vertex operator is examined in lowest order perturbation theory. It is found that, as a function of the invariant momentum transfer $q^2=\mathbf{q}^2-q_0^2$, it is analytic in a cut plane with the branch point on the negative real axis. A spectral representation (dispersion relation) may therefore be inferred. The threshold of the spectrum depends on the masses of all fields involved unless certain inequalities hold between the masses of the incident and outgoing particles on one hand and the particles in intermediate states on the other; in that case the threshold depends only on the intermediate masses.

DISPERSION relations and spectral representations in terms of physically accessible intermediate states have recently supplemented perturbation expansions as a theoretical tool in the study of elementary-particle interactions.[1] It has been possible to derive them as general consequences of a causal, Lorentz-invariant field theory, but only with the imposition of rather curious restrictions on the masses of the interacting particles.[2] We have in mind particularly the nucleon electromagnetic form factor and the nucleon-nucleon scattering amplitude, both as determined by coupling to the pion field, for which dispersion relations hold only if

$$m_\pi > (\sqrt{2}-1)M_N, \tag{1}$$

where m_π and M_N are the pion and nucleon masses, respectively. Unfortunately, the observed masses do not satisfy this inequality; the question, whether the quantities mentioned have the desired spectral representations, therefore remains unanswered.

As a guide in this matter we may appeal to perturbation theory[3]; and it is easily established that at least the low-order contributions calculated from pseudoscalar meson theory indeed have the analyticity properties from which the dispersion relations can be obtained. This result has given rise to speculation that the exact solution to the field-theoretical problem may have properties quite different from those of the perturbation series taken term by term.[4]

We believe that the explanation is far simpler: the perturbation expansion contains implicitly information about the interacting system that has not been used in the derivation of the dispersion relations. While the general discussions are based on selection rules derivable from invariance principles and the stability criterion that the rest mass of all intermediate states to which a single particle is coupled must exceed the rest mass of that particle, the perturbation expansions contain explicit lower limits on the mass of each particle in each intermediate state. For example, in the calculation of the nucleon form factor, each intermediate state coupled to the nucleon contains at least one particle with a mass equal to or greater than the nucleon mass (nucleon conservation). This is the decisive element which results in the validity of the perturbation-theoretical dispersion relations for the form factor and nucleon-nucleon scattering. We may point out here that nucleon conservation does not have such strong

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[1] Some references on the use of dispersion relations are Goldberger, Federbush, and Treiman (to be published); M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958); J. Bernstein and M. L. Goldberger, Revs. Modern Phys. **30**, 465 (1958); Chew, Karplus, Gasiorowicz, and Zachariasen, Phys. Rev. **110**, 265 (1958); and Chew, Low, Goldberger, and Nambu, Phys. Rev. **106**, 1337 (1957).
[2] Bogoliubov, Medvedev, and Polivanov, Uspekhi Mat. Nauk (to be published); Bremermann, Oehme, and Taylor, Phys. Rev. **109**, 2178 (1958); R. Jost and H. Lehmann, Nuovo cimento **5**, 1598 (1957); F. Dyson, Phys. Rev. **110**, 1460 (1958).
[3] Y. Nambu, Nuovo cimento **6**, 1064 (1957).
[4] See, for instance, *Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics, 1957* (Interscience Publishers, Inc., New York, 1957), Session IV.

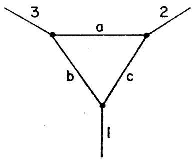

FIG. 1. Feynman diagram for vertex operator.

implications for the heavier baryons[5]; in some cases the perturbation formulas have spectral representations, but with a modified threshold, different from the one obtained in the general arguments.

In the discussion of dispersion relations it is necessary to decompose the amplitudes into irreducible covariants multiplied by scalar functions of the momenta. Since the derivations, from this point, depend on the properties of these scalar functions only, we may restrict our investigation to scalar field theories without loss of generality.

Accordingly, we introduce the scalar three-point function (vertex operator) calculated from the Feynman diagram[6] (Fig. 1). It describes three scalar fields ϕ_1, ϕ_2, and ϕ_3 coupled to three intermediary fields ϕ_a, ϕ_b, and ϕ_c, as follows:

$$\mathcal{L}_{\text{int}}=g_1\phi_1\phi_b\phi_c+g_2\phi_2\phi_a\phi_c+g_3\phi_3\phi_a\phi_b. \tag{2}$$

We have in mind the conversion of a particle of field ϕ_2 (mass M_2) to one of field ϕ_3 (mass M_3) by a virtual quantum of the field ϕ_1, with invariant momentum transfer $q^2=\mathbf{q}^2-q_0^2$. The masses of the intermediate fields ϕ_a, ϕ_b, and ϕ_c are m_a, m_b, and m_c, respectively. For the nucleon form factor, for instance, $m_a=M_2=M_3=M_N$, $m_b=m_c=m_\pi$. Stability of the field is insured by the requirements

$$M_2<m_a+m_c,\quad M_3<m_a+m_b. \tag{3}$$

Except for a constant factor, the vertex operator as a function of complex q^2 is

$$F(q^2)=\int_0^1 d\alpha\int_0^1 d\beta\int_0^1 d\gamma \times\frac{\delta(1-\alpha-\beta-\gamma)}{[m_a^2\alpha+m_b^2\beta+m_c^2\gamma+q^2\beta\gamma-M_2^2\alpha\gamma-M_3^2\alpha\beta]}. \tag{4}$$

It is clear that $F(q^2)$ will be regular for complex q^2 because the denominator can then not vanish in the region of integration. One can infer the same about positive real q^2 from the stability condition (3), and, further, that $F(q^2)$ is real there.[7] As q^2 approaches a negative real q_0^2, however, the denominator may vanish or may not vanish. In the latter case, $F(q_0^2)$ is real, while in the former it is complex, and complex conjugate values are obtained on approaching the real axis from the upper and lower half-planes. The range in q_0^2 over which $F(q_0^2)$ is complex is determined by the conditions

$$m_a^2\alpha_0+m_b^2\beta_0+m_c^2\gamma_0+q_0^2\beta_0\gamma_0-M_2^2\alpha_0\gamma_0-M_3^2\alpha_0\beta_0=0 \tag{5}$$

for some α_0, β_0, and γ_0 such that

$$0\leq\alpha_0;\quad 0\leq\beta_0;\quad 0\leq\gamma_0;\quad \alpha_0+\beta_0+\gamma_0=1.$$

The function $F(q^2)$ is therefore analytic in the cut q^2 plane. The cut extends along the real axis from $q^2=-\infty$ to the branch point $q^2=-\mu^2<0$, beyond which it is no longer possible to satisfy Eq. (5) with the stated conditions on α_0, β_0, and γ_0. The branch point is therefore the largest value of q_0^2 allowed by Eq. (5). It may be a true maximum or it may occur at the end of the range of one of the parameters. In either case, $F(q^2)$ has the spectral representation

$$F(q^2)=-\frac{1}{2\pi i}\int_{\mu^2}^{\infty}\lim_{\epsilon\to0+}\left[\frac{F(-m^2+i\epsilon)-F(-m^2-i\epsilon)}{m^2+q^2}\right]d(m^2)$$
$$=\frac{1}{\pi}\lim_{\epsilon\to0+}\int_{\mu^2}^{\infty}\frac{\operatorname{Im}[F(-m^2-i\epsilon)]}{m^2+q^2}d(m^2). \tag{6}$$

The expectation is that the branch point occurs at

$$q^2=-\mu^2=-(m_b+m_c)^2 \tag{7}$$

because that is the threshold for the production by the quantum ϕ_1, of a real intermediate state, of particles ϕ_b and ϕ_c to which it is coupled. It is easy to verify that this is indeed the largest value of q^2 as obtained from Eq. (5):

$$q_0^2=-\frac{1}{\beta_0\gamma_0}\times[m_a^2\alpha_0+m_b^2\beta_0+m_c^2\gamma_0-M_2^2\alpha_0\gamma_0-M_3^2\alpha_0\beta_0], \tag{8}$$

when $\alpha_0=0$, at the end point of its range, and

$$\beta_0=m_c/(m_b+m_c),\quad \gamma_0=m_b/(m_b+m_c). \tag{9}$$

The location of this branch point is of course independent of M_2 and M_3.

We must still investigate the possibility that q^2 in Eq. (8) has a maximum when all three parameters are in the region of integration. It is straightforward to verify that such a maximum does indeed occur when

$$\frac{m_bM_2^2+m_cM_3^2}{m_b+m_c}>m_a^2+m_bm_c. \tag{10}$$

The quite complicated general formula for its value is given in the appendix. Here we shall only state the

[5] Dr. Nambu has described similar conclusions in a recent letter to us (to be published).

[6] For a description of perturbation methods, see J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison Wesley Press, Cambridge, 1955), Chap. 8.

[7] Actually, we are here choosing one branch of the function $F(q^2)$ which we may call the physical branch.

result for the most interesting case

$$M_2=M_3=M;\quad m_b=m_c=m_0. \tag{11}$$

Then the inequality (10) becomes

$$M^2>m_a^2+m_0^2, \tag{12}$$

and the branch point is located at[8]

$$q^2=-\mu^2=-\frac{1}{m_a^2}[(m_0+m_a)^2-M^2]\times[M^2-(m_0-m_a)^2]>-4m_0^2. \tag{13}$$

We may observe that Eqs. (13) and (7) give the same value,

$$\mu^2=4m_0^2, \tag{14}$$

when

$$M^2=m_a^2+m_0^2. \tag{15}$$

We find, therefore, that the lower limit in the representation (6) depends on the value of M_2 and M_3,

$$\mu^2=\begin{cases}4m_0^2; & M^2\leqq m_a^2+m_0^2\\ \dfrac{1}{m_a^2}[(m_0+m_a)^2-M^2][M^2-(m_0-m_a)^2]; & M^2\geqq m_a^2+m_0^2.\end{cases} \tag{16}$$

Does the branchpoint Eq. (13) correspond to a "physical" threshold as did Eq. (7)? Since these phenomena occur in the so-called nonphysical region, in which momentum and energy can be conserved only for reactions between particles with complex momenta, it is not possible to present a completely convincing intuitive argument. Consider the Feynman diagram Fig. 2, where the energy-momentum vectors of the particles are given in the coordinate system in which the virtual quantum of the field ϕ_1 is at rest. If we now require that the intermediate particles, as well as the final particles, be on the mass shell, we obtain the equations

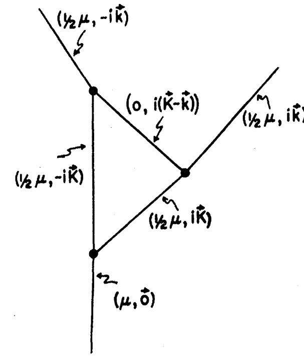

FIG. 2. Feynman diagram for vertex operator; nonphysical region.

$$\begin{gathered}0<\kappa^2=m^2-\tfrac{1}{4}\mu^2,\\ 0<k^2=M^2-\tfrac{1}{4}\mu^2,\\ (\boldsymbol{\kappa}-\mathbf{k})^2=m_a^2,\end{gathered} \tag{17}$$

for the complex vectors $\mathbf{k}$ and $\boldsymbol{\kappa}$, which can be satisfied only if μ^2 is greater than the value given in Eq. (13). In this sense does the imposition of the free particle energy-momentum relation, i.e., the vanishing of energy denominators, give rise to a threshold.

We may note that the deduction leading from Eq. (17) to Eq. (13) can be carried out even if the inequality (12) on M is not satisfied, but it is then misleading. For in that case the branch point does not occur in the physical branch of the multivalued function $F(q^2)$, but only in the other branches. This may be shown by explicit construction of $F(q^2)$ or by reference to Eq. (8) from which $\alpha_0<0$ is obtained.

The general derivations of dispersion relations[1,2] are based on a study of Eq. (6) as a function of M_2. The relation is first derived for values of this mass that do not lead to a "nonphysical" region. An analytic continuation with respect to M_2 is then employed to give Eq. (6) with the desired mass. It is clear that the behavior of μ, Eq. (16), permits this continuation only in a limited range, a range which leads to conditions such as Eq. (1).

We shall now show that Eq. (16) is equivalent to Eq. (1) for the nucleon form factor when one only imposes the condition that the lowest mass state coupled to the photon is the two-pion state, and the lowest mass state coupled to the nucleon is the meson-nucleon state:

$$m_a+m_0\geq M_N+m_\pi;\quad m_0\geq m_\pi. \tag{18}$$

From Eq. (16) we see that the region to which the representation (6) may be extended as a function of M is limited by

$$M_N^2<m_a^2+m_0^2. \tag{19}$$

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[8] Nambu[5] has suggested that these considerations be applied to weakly bound two-particle systems (e.g., the deuteron) whose binding energy ($\hbar=c=1$),

$$\epsilon=m_a+m_0-M,$$

is small compared to the masses, i.e., the nonrelativistic case.

By Fourier transformation one may define a coordinate-space particle density $\rho(r_0)$ as function of r_0, the particle distance from the center of mass. The exponent governing its asymptotic behavior for large r_0 is

$$\rho(r_0)\sim e^{-\mu r_0}.$$

It may be verified from Eq. (13) that this result is the same as obtained from the wave functions ψ describing the bound state,

$$\rho(r_0)=|\psi(r)|^2\sim e^{-2\kappa r},$$

where $r=r_0(m_a+m_0)/m_a$ is the interparticle spacing and

$$\kappa^2=2\frac{m_0m_a}{m_0+m_a}\epsilon.$$

To obtain the particle density associated with an inelastic amplitude that causes the composite particle to make a transition from the state with wave function $\psi_1(r)$ and binding energy ϵ_1 to that with $[\psi_2(r),\epsilon_2]$, we may use the formulas in the Appendix to show that

$$\rho(r_0)=|\psi_2^*(r)\psi_1(r)|\sim e^{-(\kappa_1+\kappa_2)r},$$

where

$$\kappa_i^2=2\frac{m_0m_a}{m_0+m_a}\epsilon_i.$$

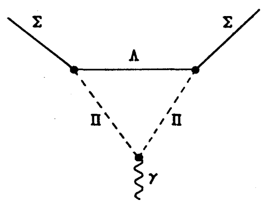

FIG. 3. Feynman diagram for Σ-particle form factor contribution.

The most restrictive inequality is obtained when m_a and m_0 are equal,

$$m_a=m_0=\tfrac{1}{2}(M_N+m_\pi), \tag{20}$$

which leads directly to Eq. (1) when inserted into Eq. (19). It is true that assumption (20) seems rather artificial; but one must realize that in the general theory the intermediary fields will have a mass spectrum, and that Eq. (18) contains the only conditions on this spectrum.

Since the usual perturbation-theory calculations recognize nucleon conservation, which implies

$$m_a \geq M_N \quad \text{or} \quad m_0 \geq M_N, \tag{21}$$

Eq. (19) is satisfied trivially and there are no conditions imposed on the meson mass.

There are some theories, however, in which the complications of Eq. (16) manifest themselves. Consider, for instance, the form factor of the Σ particle resulting from the coupling term $\bar{\psi}_\Sigma\psi_\Lambda\phi_\pi$ (Fig. 3). The mass parameters have the values

$$M=M_\Sigma, \quad m_a\geq M_\Lambda, \quad m_0\geq m_\pi, \tag{22}$$

which satisfy the inequality

$$M_\Sigma^2 > M_\Lambda^2+m_\pi^2. \tag{23}$$

The threshold of the spectrum therefore is

$$\mu^2=4m_\pi^2\left[1-\left(\frac{M_\Sigma^2-M_\Lambda^2-m_\pi^2}{2M_\Lambda m_\pi}\right)^2\right]<4m_\pi^2. \tag{24}$$

Similar results are obtained when one considers the form factors of Λ, Σ, and Ξ brought about by the strong coupling terms $\bar{\psi}_\Lambda\psi_N\phi_K$, $\bar{\psi}_\Sigma\psi_N\phi_K$, and $\bar{\psi}_\Xi\psi_\Lambda\phi_K$. In other words, nucleon conservation is a selection rule that should permit the extension of the general derivation of spectral representations, but it will not permit all cases to be treated.

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Conversations with J. G. Taylor, R. Jost, and F. Dyson have aided us greatly in our investigation of these questions. We are also indebted to them for prepublication copies of their results.

APPENDIX

With arbitrary mass values, the threshold μ^2 of the spectral representation is

$\mu^2=(m_b+m_c)^2$ when

$$\frac{m_bM_2^2+m_cM_3^2}{m_b+m_c}<m_a^2+m_bm_c, \tag{A-1}$$

and is the larger root of the quadratic equation

$$A(\mu^2)^2+B\mu^2+C=0, \tag{A-2}$$

where

$$A=m_a^2,$$
$$B=-m_a^2(m_b+m_c)^2+\left[\frac{m_bM_2^2+m_cM_3^2}{m_b+m_c}-(m_a^2+m_bm_c)\right]^2-\frac{C}{(m_b+m_c)^2}, \tag{A-3}$$
$$C=(M_2^2m_b^2-M_3^2m_c^2)(M_2^2-M_3^2+m_b^2-m_c^2)-m_a^2(M_2^2-M_3^2)(m_b^2-m_c^2),$$

when

$$\frac{m_bM_2^2+m_cM_3^2}{m_b+m_c}>m_a^2+m_bm_c. \tag{A-4}$$