where for the sake of simplicity the arguments upon which $\tilde{D}^{\beta}$ operates are omitted. By mathematical induction this result can be generalized to yield

$$\tilde{D}^{\beta}\langle\Omega, \mathfrak{T}(x_1\cdots x_n\cdots)\Phi_{\alpha}{}^{(+)}\rangle$$
$$=\sum_{\mu}(-1)^{\mu}\langle\Phi_{\beta/\mu}{}^{(-)}, \mathfrak{T}(x_1\cdots x_n)\Phi_{\alpha/\mu}{}^{(+)}\rangle, \quad \text{(B.19)}$$

and similarly

$$D^{\alpha}\langle\Phi_{\beta}{}^{(-)}, \mathfrak{T}(x_1\cdots x_n\cdots)\Omega\rangle$$
$$=\sum_{\mu}(-1)^{\mu}\langle\Phi_{\beta/\mu}{}^{(-)}, \mathfrak{T}(x_1\cdots x_n)\Phi_{\alpha/\mu}{}^{(+)}\rangle. \quad \text{(B.20)}$$

The recursion formulas for T-products are the direct consequences of (B.19) and (B.20).

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Foldy-Wouthuysen Transformation. Exact Solution with Generalization to the Two-Particle Problem

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The Dirac Hamiltonian for a particle in a nonexplicitly time-dependent field is converted to an even Dirac matrix by means of a single canonical transformation. When the interaction term is an odd Dirac matrix, the transformed Hamiltonian is expressed in a very simple form. An exact transformation is also found for two-particle wave equations of Breit's type. The transformed Hamiltonian is then a uU-separating matrix, in Chraplyvy's sense.

In the nonrelativistic limit expansions in powers of $1/m$ or $1/c$ are made. The approximate wave equations are in agreement with previous transformation results.

INTRODUCTION

A SPIN $\frac{1}{2}$ particle in interaction with various types of external fields is described by a spinor ψ satisfying an equation of the Dirac type.[1-3] For different purposes it is of interest to have this equation converted to a two-component equation of the Pauli type. This was earlier achieved by an elimination method that gives an equation for the large components of ψ. In this equation, however, there are terms nonlinear in $\partial/\partial t$ and non-Hermitian interaction terms like the imaginary electric moment term. Furthermore, the exact interpretation remains in terms of the four-component wave function.

A different treatment is due to Foldy and Wouthuysen.[4] By means of a canonical transformation of the wave equation, a representation is found where the Hamiltonian is an even Dirac matrix. Then the Dirac equation splits into two uncoupled equations of the Pauli type, describing particles in positive- and negative-energy states, respectively. When the particle is free, the transformation is exhibited in a simple, closed form. In the presence of interactions, however, a transformation in closed form has not been found, but an infinite sequence of transformations can be made, each of which makes the Hamiltonian even to one higher order in the expansion parameter $1/m$.

[1] W. Pauli, Revs. Modern Phys. **13**, 203 (1941).
[2] L. L. Foldy, Phys. Rev. **87**, 688 (1952).
[3] W. A. Barker and Z. V. Chraplyvy, Phys. Rev. **89**, 446 (1953).
[4] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

Progress on this point has been made by Case[5] who found the transformation in closed form for spin $\frac{1}{2}$ particles and spin 0 particles in time-independent magnetic fields.

The Foldy-Wouthuysen transformation method has been extended to two-particle wave equations by Chraplyvy,[6,7] who found that in the case of equal masses, the postulate of an *even-even* transformed Hamiltonian is too far-reaching. When the less stringent requirement of a *uU-separating* (or an *lL-separating*) Hamiltonian was introduced, a whole class of usable transformations could be found, but none of them is given explicitly.

In the present paper it is found that the exact transformation of the Dirac equation for one particle, can easily be generalized to two-particle wave equations when Chraplyvy's less stringent requirement is used.

SUMMARY OF THE FOLDY-WOUTHUYSEN TRANSFORMATION

The wave function in the Dirac theory is a column matrix with four components ψ_{ν}, where ψ_1 and ψ_2 are called upper components and ψ_3 and ψ_4 lower components. ψ satisfies the wave equation,

$$i\hbar(\partial/\partial t)\psi = H\psi, \quad (1)$$

the Hamiltonian being a Hermitian four-by-four matrix,

$$H=\beta mc^2+c\boldsymbol{\alpha}\cdot\mathbf{p}+\text{interaction terms.}$$

[5] K. M. Case, Phys. Rev. **95**, 1323 (1954).
[6] Z. V. Chraplyvy, Phys. Rev. **91**, 388 (1953).
[7] Z. V. Chraplyvy, Phys. Rev. **92**, 1310 (1953).

The nonvanishing elements of the matrix β are $\beta_{11}=\beta_{22}=1$ and $\beta_{33}=\beta_{44}=-1$. It anticommutes with the α-matrices.

A matrix ω is called even if $\omega_{ul}=\omega_{lu}=0$, and odd if $\omega_{uu}=\omega_{ll}=0$, where $u=1$, 2 and $l=3$, 4. The product of two even or two odd matrices is an even matrix, and the product of an even and an odd matrix is an odd matrix. If the Hamiltonian were a sum of even matrices only, the Dirac equation would split into two sets of equations, one for the upper components and one for the lower components. A necessary and sufficient condition for a matrix to be even (odd) is that it commute (anticommute) with β. Thus the α matrices are odd, and $c\boldsymbol{\alpha}\cdot\mathbf{p}$ is an odd operator.

Any Dirac matrix ω can be written as the sum of an even and an odd matrix,

$$\omega=\tfrac{1}{2}(\omega+\beta\omega\beta)+\tfrac{1}{2}(\omega-\beta\omega\beta),$$

where the first term on the right is the even part of ω, and the second term the odd part of ω.

The Hamiltonian is put in the form,

$$H=\beta mc^2+\mathcal{E}+\mathcal{O},$$

where $\mathcal{E}$ is an even operator and $\mathcal{O}$ is an odd operator. In the present paper both of these are assumed to be nonexplicitly time-dependent. Now consider the canonical transformation,

$$\psi'=e^{iS}\psi,$$

the operator S being Hermitian and nonexplicitly time-dependent. Then ψ' satisfies the wave equation,

$$\begin{aligned}
&i\hbar(\partial/\partial t)\psi'=H'\psi',\\
&H'=e^{iS}He^{-iS}\\
&\quad=H+i[S,H]-\tfrac{1}{2}[S,[S,H]]+\cdots.
\end{aligned}$$

Putting $S=-i\beta\mathcal{O}/2mc^2$, we obtain with $1/m$ as expansion parameter,

$$H'=\beta mc^2+\mathcal{E}+\frac{\beta}{2mc^2}(\mathcal{O}^2+[\mathcal{O},\mathcal{E}])-\frac{1}{m^2c^4}(\tfrac{1}{8}[\mathcal{O},[\mathcal{O},\mathcal{E}]]+\tfrac{1}{3}\mathcal{O}^3)+\cdots.$$

The lowest order odd term is of first order, i.e., of *one* order higher than in the original Hamiltonian. By a sequence of further canonical transformations, the generator of the transformation at each step being chosen to be

$S=-(i/2mc^2)\beta\times$ (odd terms in the Hamiltonian of lowest order in $1/m$),

we can make the Hamiltonian even to any desired order in $1/m$.

When the particle is free, the Hamiltonian is converted to an even operator by a single transformation,

$$H'=e^{iS}(\beta mc^2+c\boldsymbol{\alpha}\cdot\mathbf{p})e^{-iS}=\beta(m^2c^4+c^2p^2)^{\frac{1}{2}},\qquad(2)$$

where S is odd and Hermitian,

$$S=-(i/2p)\beta\boldsymbol{\alpha}\cdot\mathbf{p}\tan^{-1}(p/mc),\quad p=(\mathbf{p}^2)^{\frac{1}{2}}.\qquad(3)$$

We note the property,

$$e^{-iS}\beta=(\cos S-i\sin S)\beta=\beta(\cos S+i\sin S)=\beta e^{iS}.\qquad(4)$$

TRANSFORMATION IN CLOSED FORM

If the operator U is unitary and not explicitly time-dependent, the transformation

$$\psi'=U\psi,\quad H'=UHU^*$$

leaves Eq. (1) in the Hamiltonian form,

$$i\hbar(\partial/\partial t)\psi'=H'\psi'.$$

In order that H' be even, it is necessary and sufficient that

$$[\beta,UHU^*]=0.$$

Multiplying from the left by U^* and from the right by U, the condition reads,

$$[U^*\beta U,H]=0,\qquad(5)$$

where $U^*\beta U$ (like β) is an Hermitian operator with eigenvalues $+1$ and -1.

Now one may ask: Do we know any Hermitian operator which commutes with H and whose only eigenvalues are $+1$ and -1? If we do, it may be possible to identify it with $U^*\beta U$. An operator of this type is:

$$\lambda=H(H^2)^{-\frac{1}{2}}.\qquad(6)$$

If we assume that zero is not an energy eigenvalue, λ is defined in the energy representation by,

$$\lambda\psi_E=H(E^2)^{-\frac{1}{2}}\psi_E=E|E|^{-1}\psi_E,$$

which gives $+\psi_E$ when $E>0$ and $-\psi_E$ when $E<0$.

Obviously, we have the operator identities

$$\lambda^2=1\quad\text{and}\quad\lambda=\lambda^*.$$

$(H^2)^{-\frac{1}{2}}$ can be represented by a binomial expansion,

$$(H^2)^{-\frac{1}{2}}=E_0^{-1}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{H^2-E_0^2}{E_0^2}\right)^n,\qquad(7)$$

with E_0 a positive number. Operating on a wave function ψ, this expansion will produce convergent results provided that all energies E in the spectrum of ψ satisfy

$$-1<(E^2-E_0^2)/E_0^2\lesssim 1,\quad\text{i.e.,}\quad 0<|E|\lesssim E_0\sqrt{2}.$$

In the nonrelativistic limit one should put $E_0=mc^2$. Then $(H^2-E_0^2)/E_0^2$ will be "small" and of first order in $1/m$.

We also mention a representation of $(H^2)^{-\frac{1}{2}}$ in terms of the Poisson integral:

$$(H^2)^{-\frac{1}{2}}=\lim_{N\to\infty}\int_{-N}^{N}d\eta\,\exp(-\pi\eta^2H^2).$$

The exponential operator is defined by means of its Taylor expansion. This form will give convergent results for all finite $E\neq 0$.

Now we look for a unitary operator U with the property

$$U^*\beta U=\lambda. \quad (8)$$

We also postulate

$$U^*\beta=\beta U, \quad (9)$$

which is true for the free-particle Foldy-Wouthuysen transformation, Eq. (4). Then Eq. (8) reads

$$\beta U^2=\lambda, \quad \text{or} \quad U^2=\beta\lambda. \quad (10)$$

Here the operator $\beta\lambda$ is unitary,

$$(\beta\lambda)(\beta\lambda)^*=\beta\lambda\lambda\beta=\beta\beta=1,$$

and the eigenvalues are consequently on the unit circle in the complex plane. As a result an explicit, convergent solution is intricate to attain. The following solution has, however, been found:

$$U=\tfrac{1}{2}(1+\beta\lambda)[1+\tfrac{1}{4}(\beta\lambda+\lambda\beta-2)]^{-\frac{1}{2}}, \quad (11a)$$

i.e.,

$$U^*=\tfrac{1}{2}(1+\lambda\beta)[1+\tfrac{1}{4}(\beta\lambda+\lambda\beta-2)]^{-\frac{1}{2}}. \quad (11b)$$

The $-\frac{1}{2}$ root operator is defined by the binomial expansion in powers of $\frac{1}{4}(\beta\lambda+\lambda\beta-2)$.

To examine the convergence, let us consider an eigenfunction u_δ for $\beta\lambda$,

$$\beta\lambda u_\delta=e^{i\delta}u_\delta.$$

Multiplying from the left by $e^{-i\delta}\lambda\beta$, we get

$$\lambda\beta u_\delta=e^{-i\delta}u_\delta, \quad \tfrac{1}{4}(\beta\lambda+\lambda\beta-2)u_\delta=\tfrac{1}{2}(\cos\delta-1)u_\delta.$$

Thus, the expansion will produce convergent results when

$$-1<\tfrac{1}{2}(\cos\delta-1)\lesseqgtr 1, \quad \text{i.e.,} \quad \cos\delta>-1.$$

If we take it for granted that -1 is not an eigenvalue of $\beta\lambda$ (which is always true when the particle is free or when the interaction term is odd), then the expansion for U is well defined.

In order to prove that the U given above is a solution, we note that $\beta\lambda$ and $\lambda\beta$ commute. Then

$$\begin{aligned} U^2&=\tfrac{1}{4}(1+\beta\lambda)^2[1+\tfrac{1}{4}(\beta\lambda+\lambda\beta-2)]^{-1}\\ &=\beta\lambda[1+\tfrac{1}{4}(\beta\lambda+\lambda\beta-2)][1+\tfrac{1}{4}(\beta\lambda+\lambda\beta-2)]^{-1}=\beta\lambda.\end{aligned}$$

From (11) we see that U^* can be written

$$U^*=\lambda\beta U,$$

Hence,

$$U^*U=\lambda\beta U^2=\lambda\beta\beta\lambda=1.$$

Furthermore, U satisfies the postulated relation (9). This is easily seen from the expressions for U and U^* when one notes that the $-\frac{1}{2}$ root operator is even.

Hence, we can conclude that the transformed Hamiltonian $H'=UHU^*$ is an even operator.

Let us summarize the properties of U,

$$U^2=\beta\lambda, \quad UU^*=1, \quad U^*\beta=\beta U. \quad (12a,b,c)$$

Furthermore, by pure algebra,

$$U^*\beta U=\lambda,\ U\beta U^*=\beta\lambda\beta,\ \beta U=U\lambda=\lambda U^*=U^*\beta. \quad (13a,b,c)$$

The even character of the transformed Hamiltonian can now be easily verified by direct calculation,

$$\beta H'=\beta UHU^*=U\lambda HU^*=UH\lambda U^*=UHU^*\beta=H'\beta,$$

i.e., $[\beta,H']=0$ and H' is even.

As a consequence of (13b) we have a simple expression for the transform of βmc^2, which is the predominating term in the nonrelativistic limit

$$(\beta mc^2)'=mc^2U\beta U^*=mc^2\beta\lambda\beta. \quad (14)$$

If U is written in exponential form,

$$U=e^{iS}=\cos S+i\sin S,$$

with S Hermitian, we get the following equations for S:

$$\begin{aligned} i\sin S&=\tfrac{1}{2}(U-U^*)\\ &=\tfrac{1}{4}(\beta\lambda-\lambda\beta)[1+\tfrac{1}{4}(\beta\lambda+\lambda\beta-2)]^{-\frac{1}{2}}, \quad (15a)\end{aligned}$$

$$\cos S=\tfrac{1}{2}(U+U^*)=[1+\tfrac{1}{4}(\beta\lambda+\lambda\beta-2)]^{\frac{1}{2}}, \quad (15b)$$

or

$$\sin 2S=2\sin S\cos S=\tfrac{1}{2}i[\lambda,\beta], \quad (16a)$$

$$\cos 2S=2\cos^2 S-1=\tfrac{1}{2}[\lambda,\beta]_+. \quad (16b)$$

Since $\cos S$ is a positive-definite operator, S can always be expressed by means of the first set of equations as a $\sin^{-1}$ function (or a $\tan^{-1}$ function, but then with a reduced domain of convergence). If $\cos 2S$ is positive definite, the second set can be used as well. This is true when the particle is free or when the interaction term is an odd operator, say.

TRANSFORMATION WHEN INTERACTION TERM IS ODD

The Hamiltonian can then be written $H=\beta mc^2+\mathcal{O}$, and the following operators are even:

$$H^2=m^2c^4+\mathcal{O}^2,$$

$$(H^2)^{-\frac{1}{2}}=(m^2c^4+\mathcal{O}^2)^{-\frac{1}{2}},$$

$$\lambda H=(m^2c^4+\mathcal{O}^2)^{\frac{1}{2}}.$$

For the operator $\lambda\beta+\beta\lambda$ we find

$$H(H^2)^{-\frac{1}{2}}\beta+\beta H(H^2)^{-\frac{1}{2}}=(H\beta+\beta H)(H^2)^{-\frac{1}{2}}=2mc^2(H^2)^{-\frac{1}{2}},$$

which shows that the $-\frac{1}{2}$ root operator in U commutes with H. Furthermore,

$$H(1+\lambda\beta)=H+H\lambda\beta=H+\beta H\lambda=(1+\beta\lambda)H,$$

and consequently

$$HU^*=UH.$$

For the transformed Hamiltonian we get the following exact result:

$$H'=UHU^*=U^2H=\beta\lambda H=\beta(m^2c^4+\mathcal{O}^2)^{\frac{1}{2}}. \quad (17)$$

For purposes of illustration let us consider an electron in a magnetostatic field $\mathbf{H}=\text{curl}\mathbf{A}$,

$$H=\beta mc^2+\boldsymbol{\alpha}\cdot(c\mathbf{p}-e\mathbf{A})=\beta mc^2+\mathcal{O},$$

$$H'=\beta[m^2c^4+(c\mathbf{p}-e\mathbf{A})^2-e\hbar c\boldsymbol{\sigma}\cdot\mathbf{H}]^{\frac{1}{2}},\quad \boldsymbol{\sigma}=-\tfrac{1}{2}i\boldsymbol{\alpha}\times\boldsymbol{\alpha}. \tag{18}$$

For a neutron in an electrostatic field $\mathbf{E}$, we use a Dirac equation amplified with a Pauli[1] interaction term,

$$H=\beta Mc^2+c\boldsymbol{\alpha}\cdot\mathbf{p}+i\mu\beta\boldsymbol{\alpha}\cdot\mathbf{E}=\beta Mc^2+\mathcal{O},$$

where μ is the magnetic moment and M the neutron mass.

$$H'=\beta[M^2c^4+c^2\mathbf{p}^2-\mu\hbar c\beta\,\text{div}\,\mathbf{E} -\mu c\beta\boldsymbol{\sigma}\cdot(\mathbf{E}\times\mathbf{p}-\mathbf{p}\times\mathbf{E})+\mu^2\mathbf{E}^2]^{\frac{1}{2}}. \tag{19}$$

If we write U in exponential form, $U=e^{iS}$, then S shall satisfy

$$\sin 2S=\tfrac{1}{2}i[\lambda,\beta]=-i\beta\mathcal{O}(H^2)^{-\frac{1}{2}},$$
$$\cos 2S=\tfrac{1}{2}[\lambda,\beta]_+=mc^2(H^2)^{-\frac{1}{2}}.$$

Here $\cos 2S$ is positive-definite so we have the solution,

$$S=-\tfrac{1}{2}\sin^{-1}[i\beta\mathcal{O}(H^2)^{-\frac{1}{2}}], \tag{20}$$

or, with a reduced domain of convergence,

$$S=-\tfrac{1}{2}\tan^{-1}(i\beta\mathcal{O}/mc^2). \tag{21}$$

If we put $\mathcal{O}=\boldsymbol{\alpha}\cdot(c\mathbf{p}-e\mathbf{A})$, then the last expression gives the transformation found by Case.[5] For $\mathbf{A}=0$ we get

$$S=-\tfrac{1}{2}\tan^{-1}(i\beta\boldsymbol{\alpha}\cdot\mathbf{p}/mc),$$

which is equal to the operator given by Foldy and Wouthuysen[4]:

$$S=-(i/2p)\beta\boldsymbol{\alpha}\cdot\mathbf{p}\tan^{-1}(p/mc),\quad p=(\mathbf{p}^2)^{\frac{1}{2}}. \tag{3}$$

This can be seen by means of the power expansion for $\tan^{-1}$ and the relation $(\boldsymbol{\alpha}\cdot\mathbf{p})^2=p^2$.

TRANSFORMATION IN THE NONRELATIVISTIC LIMIT

In the Hamiltonian,

$$H=\beta mc^2+\mathcal{E}+\mathcal{O},$$

βmc^2 is now the predominating term. We take $1/m$ as expansion parameter and assume $\mathcal{E}$ and $\mathcal{O}$ to be of zeroth order in $1/m$. Choosing $E_0=mc^2$ in the expansion (7) for $(H^2)^{-\frac{1}{2}}$, we have to third order:

$$\beta\lambda=1+\frac{\beta}{mc^2}\mathcal{O}+\frac{1}{2m^2c^4}([\mathcal{O},\mathcal{E}]-\mathcal{O}^2) +\frac{\beta}{4m^3c^6}([\mathcal{O},[\mathcal{O},\mathcal{E}]]-[\mathcal{E},[\mathcal{O},\mathcal{E}]]-2\mathcal{O}^3). \tag{22}$$

If we put

$$\beta\lambda=1+q,\quad\text{i.e.,}\quad \lambda\beta=1+q^*,$$

then q and q^* are "small" and of first order. In terms of q and q^* the transformation matrix reads

$$U=(1+\tfrac{1}{2}q)[1+\tfrac{1}{4}(q+q^*)]^{-\frac{1}{2}}.$$

Since $q+q^*=-qq^*$, the term $\frac{1}{4}(q+q^*)$ is actually of order $(1/m)^2$ so we have, to second order,

$$U=(1+\tfrac{1}{2}q)[1-\tfrac{1}{8}(q+q^*)+\cdots]$$
$$=1+\tfrac{1}{2}q-\tfrac{1}{8}(q+q^*)+\cdots$$
$$=1+\frac{\beta}{2mc^2}\mathcal{O}+\frac{1}{4m^2c^4}([\mathcal{O},\mathcal{E}]-\tfrac{1}{2}\mathcal{O}^2). \tag{23}$$

By means of the expansions written down here, H' can be found to second order:

$$H'=UHU^*=mc^2\beta\lambda\beta+U(\mathcal{E}+\mathcal{O})U^*$$
$$=\beta mc^2+\mathcal{E}+\frac{\beta}{2mc^2}\mathcal{O}^2-\frac{1}{8m^2c^4}[\mathcal{O},[\mathcal{O},\mathcal{E}]], \tag{24}$$

which is in agreement with the Foldy-Wouthuysen expansion for the transform of a nonexplicitly time-dependent Hamiltonian.

TWO-PARTICLE PROBLEM: TERMINOLOGI AND NOTATION

We consider the wave equation for two Dirac particles,

$$i\hbar(\partial/\partial t)\psi=\mathcal{H}\psi,$$

with the Hamiltonian written in the form

$$\mathcal{H}=\beta^{\text{I}}m_{\text{I}}c^2+\beta^{\text{II}}m_{\text{II}}c^2+(\mathcal{E}\mathcal{E})+(\mathcal{E}\mathcal{O})+(\mathcal{O}\mathcal{E})+(\mathcal{O}\mathcal{O}). \tag{25}$$

Quantities referring to each of the two particles are labeled by Roman numbers I and II, respectively. The wave function has sixteen components ψ_{kK} with $k, K=1, 2, 3, 4$. The lower case subscript refer to particle I and the capital index to particle II. The components are classified as upper-upper ψ_{uU}, upper-lower ψ_{uL}, lower-upper ψ_{lU}, and lower-lower ψ_{lL} with $u, U=1, 2$; $l, L=3, 4$. The Hamiltonian is composed of terms which are direct products of Dirac matrices, labeled by I and II, respectively. When a matrix is single, the direct product with the unit matrix is understood. We distinguish even-even terms, even-odd terms, odd-even terms, and odd-odd terms. $(\mathcal{E}\mathcal{E})$ stands for the sum of all even-even terms, $(\mathcal{E}\mathcal{O})$ stands for the sum of all even-odd terms, and so forth.

When a matrix $\omega^{\text{I}}\omega^{\text{II}}$ acts on ψ, the first matrix affects the index k, and the second matrix affects the index K.

$$(\omega^{\text{I}}\omega^{\text{II}}\psi)_{nN}=\omega_{nk}{}^{\text{I}}\omega_{NK}{}^{\text{II}}\psi_{kK}.$$

Thus, any Dirac matrix labeled by I commute with any of those labeled by II.

According to Chraplyvy,[7] a matrix Ω^{us} is called uU-separating when the components $(\Omega^{us}\psi)_{uU}$ are expressed by ψ_{uU} only; whereas, $(\Omega^{us}\psi)_{uL}$, $(\Omega^{us}\psi)_{lU}$ and $(\Omega^{us}\psi)_{lL}$ do not depend on ψ_{uU}. If the Hamiltonian were a uU-separating matrix, there would be a separate four-component equation for the upper-upper components of ψ, and all other components could be put equal to zero.

TWO-PARTICLE PROBLEM: TRANSFORMATION IN CLOSED FORM

We shall limit our considerations to the problem of finding a transformation in closed form which makes the Hamiltonian uU-separating.

According to Chraplyvy,[7] the most general expression of a uU-separating matrix is

$$\Omega^{us}=\Omega-\tfrac{1}{4}(1+\beta^{\text{I}})(1+\beta^{\text{II}})\Omega-\tfrac{1}{4}\Omega(1+\beta^{\text{I}})(1+\beta^{\text{II}})+\tfrac{1}{8}(1+\beta^{\text{I}})(1+\beta^{\text{II}})\Omega(1+\beta^{\text{I}})(1+\beta^{\text{II}}), \quad (26)$$

with Ω an arbitrary matrix.

As a consequence we have the commutation relation

$$[(1+\beta^{\text{I}})(1+\beta^{\text{II}}),\Omega^{us}]=0. \quad (27)$$

This is also a sufficient condition for a matrix to be uU-separating because if Ω is a matrix which satisfies this commutation relation, then Ω can be expressed in terms of itself as the right-hand side of the general expression.

Now we put

$$(1+\beta^{\text{I}})(1+\beta^{\text{II}})=2B+2,$$

i.e.,

$$B=\tfrac{1}{2}(\beta^{\text{I}}+\beta^{\text{II}}+\beta^{\text{I}}\beta^{\text{II}}-1). \quad (28)$$

Then, a necessary and sufficient condition for a matrix to be uU-separating is

$$[B,\Omega^{us}]=0. \quad (29)$$

The role played by B in the two-particle theory is analogous to that played by β in the single-particle theory. By means of (28) one can easily verify that $B^2=1$.

We make a canonical transformation of the wave equation by means of a unitary matrix U, such that the transformed Hamiltonian,

$$\mathcal{H}'=U\mathcal{H}U^*,$$

becomes uU-separating, which means that $U\mathcal{H}U^*$ commutes with B, or, equivalently,

$$[U^*BU,\mathcal{H}]=0. \quad (30)$$

As in the single-particle theory we look for an operator which can be put equal to U^*BU. The only eigenvalues must be $+1$ and -1, and the operator must commute with $\mathcal{H}$. We put

$$U^*BU=P(P^2)^{-\frac{1}{2}}\equiv\Lambda, \quad (31)$$

where $P=P(\mathcal{H})$ is a real function of $\mathcal{H}$. The eigenvalues of P are assumed to be different from zero. Operating on an energy eigenfunction ψ_E, Λ gives $+\psi_E$ when $P(E)$ is positive and $-\psi_E$ when $P(E)$ is negative.

Furthermore, we postulate the property,

$$U^*B=BU, \quad (32)$$

which then gives the equation,

$$BU^2=\Lambda \quad \text{or} \quad U^2=B\Lambda. \quad (33)$$

As in the one-particle theory, we have the solution

$$U=\tfrac{1}{2}(1+B\Lambda)[1+\tfrac{1}{4}(B\Lambda+\Lambda B-2)]^{-\frac{1}{2}}, \quad (34)$$

which is unitary with the property (32), and which is convergent when -1 is not an eigenvalue of $B\Lambda$. The properties which were summarized in (12) and (13) can be directly translated to the two-particle transformation matrix by putting B for β and Λ for λ.

We want P to be a function of $\mathcal{H}$ such that

$$P(\mathcal{H})\rightarrow B \quad \text{when} \quad \mathcal{H}\rightarrow\beta^{\text{I}}m_{\text{I}}c^2+\beta^{\text{II}}m_{\text{II}}c^2.$$

Then in the nonrelativistic limit the predominating term of BP and P^2 will be 1.

The simplest function having this property is a polynomial of third degree,

$$P=\frac{M^2}{4m_{\text{I}}m_{\text{II}}}\left[-1-\left(\frac{m_{\text{I}}-m_{\text{II}}}{M}\right)^2\frac{\mathcal{H}}{Mc^2}+\left(\frac{\mathcal{H}}{Mc^2}\right)^2+\left(\frac{\mathcal{H}}{Mc^2}\right)^3\right], \quad M=m_{\text{I}}+m_{\text{II}}. \quad (35)$$

For equal masses there is a solution of second degree as well:

$$P=-1+\frac{\mathcal{H}}{Mc^2}+\left(\frac{\mathcal{H}}{Mc^2}\right)^2, \quad M=2m. \quad (36)$$

TWO-PARTICLE PROBLEM: NONRELATIVISTIC LIMIT

In the case of equal masses the transformed Hamiltonian has been calculated to second order in $1/c$ under the assumption that $(\mathcal{E}\mathcal{E})$ and $(\mathcal{O}\mathcal{O})$ is of order c^0 and $(\mathcal{E}\mathcal{O})$ and $(\mathcal{O}\mathcal{E})$ of order c^1. The calculation was done for each of the two polynomials above. Use was made of the fact that any matrix M can be written as the sum,

$$M=\tfrac{1}{2}(M+BMB)+\tfrac{1}{2}(M-BMB),$$

where the first part commutes and the second part anticommutes with B. P was written in the form,

$$P=B+\epsilon+\omega,$$

where ϵ commutes and ω anticommutes with B. For either of the polynomials ϵ was of second order and ω of first order in $1/c$. In this notation, to fourth order,

$$B\Lambda=1+B\omega-\tfrac{1}{2}\omega^2-\tfrac{1}{2}B\omega^3+\tfrac{1}{2}[\omega,\epsilon]+\tfrac{3}{8}\omega^4+\tfrac{1}{4}B[\omega,[\omega,\epsilon]], \quad (37)$$

$$\tfrac{1}{4}(B\Lambda+\Lambda B-2)=-\tfrac{1}{4}\omega^2+\tfrac{3}{16}\omega^4+\tfrac{1}{8}B[\omega,[\omega,\epsilon]], \tag{38}$$

$$U=1+\tfrac{1}{2}B\omega-\tfrac{1}{8}\omega^2-\tfrac{3}{16}B\omega^3+\tfrac{1}{4}[\omega,\epsilon] +(11/128)\omega^4+\tfrac{1}{16}B[\omega,[\omega,\epsilon]]. \tag{39}$$

The following expression for the transformed Hamiltonian, correct to second order, was obtained:

$$\mathcal{H}'=\mathcal{H}_a'+\mathcal{H}_b'+\mathcal{H}_c', \tag{40}$$

where

$$\mathcal{H}_a'=\beta^{\rm I}mc^2+\beta^{\rm II}mc^2+(\mathcal{E}\mathcal{E})+\frac{\beta^{\rm I}(1+\beta^{\rm II})}{4mc^2}(\mathcal{O}\mathcal{E})^2 +\frac{\beta^{\rm II}(1+\beta^{\rm I})}{4mc^2}(\mathcal{E}\mathcal{O})^2+\frac{1+\beta^{\rm II}}{16m^2c^4}[[(\mathcal{O}\mathcal{E}),(\mathcal{E}\mathcal{E})],(\mathcal{O}\mathcal{E})] +\frac{1+\beta^{\rm I}}{16m^2c^4}[[(\mathcal{E}\mathcal{O}),(\mathcal{E}\mathcal{E})],(\mathcal{E}\mathcal{O})]-\frac{\beta^{\rm I}(1+\beta^{\rm II})}{16m^3c^6}(\mathcal{O}\mathcal{E})^4 -\frac{\beta^{\rm II}(1+\beta^{\rm I})}{16m^3c^6}(\mathcal{E}\mathcal{O})^4+\frac{\beta^{\rm I}+\beta^{\rm II}}{8mc^2}(\mathcal{O}\mathcal{O})^2, \tag{40a}$$

$$\mathcal{H}_b'=+\frac{\beta^{\rm I}+4\beta^{\rm II}+3\beta^{\rm I}\beta^{\rm II}}{64m^2c^4}[[(\mathcal{O}\mathcal{E}),(\mathcal{O}\mathcal{O})]_+,(\mathcal{E}\mathcal{O})]_+ +\frac{4\beta^{\rm I}+\beta^{\rm II}+3\beta^{\rm I}\beta^{\rm II}}{64m^2c^4}[[(\mathcal{E}\mathcal{O}),(\mathcal{O}\mathcal{O})]_+,(\mathcal{O}\mathcal{E})]_+ +\frac{1-\beta^{\rm II}}{64m^2c^4}[[(\mathcal{O}\mathcal{E}),(\mathcal{O}\mathcal{O})],(\mathcal{E}\mathcal{O})] +\frac{1-\beta^{\rm I}}{64m^2c^4}[[(\mathcal{E}\mathcal{O}),(\mathcal{O}\mathcal{O})],(\mathcal{O}\mathcal{E})] +\frac{\beta^{\rm I}+\beta^{\rm II}+2\beta^{\rm I}\beta^{\rm II}}{64m^3c^6}[(\mathcal{O}\mathcal{E}),(\mathcal{E}\mathcal{O})]^2 -\frac{3+2\beta^{\rm I}+2\beta^{\rm II}+\beta^{\rm I}\beta^{\rm II}}{64m^3c^6}[(\mathcal{O}\mathcal{E})^2,(\mathcal{E}\mathcal{O})^2]_+ +\frac{3+4\beta^{\rm I}+\beta^{\rm I}\beta^{\rm II}}{64m^3c^6}(\mathcal{E}\mathcal{O})(\mathcal{O}\mathcal{E})^2(\mathcal{E}\mathcal{O}) +\frac{3+4\beta^{\rm II}+\beta^{\rm I}\beta^{\rm II}}{64m^3c^6}(\mathcal{O}\mathcal{E})(\mathcal{E}\mathcal{O})^2(\mathcal{O}\mathcal{E}), \tag{40b}$$

$$\mathcal{H}_c'=+\tfrac{1}{2}(1-\beta^{\rm I})[(\mathcal{E}\mathcal{O})+\textstyle\sum \mathfrak{N}_{eo}]+\tfrac{1}{2}(1-\beta^{\rm II})[(\mathcal{O}\mathcal{E}) +\textstyle\sum \mathfrak{N}_{oe}]+\tfrac{1}{2}(1-\beta^{\rm I}\beta^{\rm II})[(\mathcal{O}\mathcal{O})+\textstyle\sum \mathfrak{N}_{oo}]. \tag{40c}$$

A large number of terms of even-odd, odd-even and odd-odd type are only indicated by means of the symbols $\sum \mathfrak{N}_{eo}$, $\sum \mathfrak{N}_{oe}$, and $\sum \mathfrak{N}_{oo}$, respectively. The terms (40c) will not be present in the reduced wave equation for the upper-upper components at all, because of the left multipliers which give zero.

For either of the two polynomials which define P, the transformed Hamiltonian can be written as the expression above, but $\sum \mathfrak{N}_{oo}$ is not the same in the two cases. The difference is of second order.

Putting $m_{\rm I}=m_{\rm II}=m$ in the Hamiltonian found by Chraplyvy[7] by means of the "least change" transformation, one will find complete agreement between (40a) and corresponding terms in Chraplyvy's Hamiltonian. Terms corresponding to (40b) will also be found, but with different left multipliers. Identical results are, however, obtained when $\beta^{\rm I}$ and $\beta^{\rm II}$ are set equal to 1.

Thus, for equal masses and to second order, the reduced wave equations will agree.