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Bremsstrahlung of Very Low-Energy Quanta in Elementary Particle Collisions

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It is shown that the first two terms in the series expansion of the differential bremsstrahlung cross section (in powers of the energy loss) may be calculated exactly in terms of the corresponding elastic amplitude and the electromagnetic constants of the participating particles.

1. INTRODUCTION

It is well known that the cross section for bremsstrahlung in a finite-range force field has a dk/k dependence on the energy k of the radiated photon as $k\rightarrow 0$. Except for trivial factors associated with the current producing the radiation, the coefficient of $1/k$ is, in the limit of $k=0$, just the square of the elastic amplitude for the scattering that produced the radiation. (Elastic, as used here, refers to the absence of energy loss to the electromagnetic field rather than to the identity of the initial and final constituents of the collision.) It will be shown in this paper that not only the $1/k$ term but also the term of order zero in k in the bremsstrahlung cross section may be exactly calculated as a function of the corresponding elastic amplitude.

This result is a consequence of electric charge conservation. It appears to have a wide range of validity: it was originally proved for systems interacting through a potential by using the Lippmann-Schwinger formalism, but it can equally well be derived using the formalism of quantum field theory. In this paper we shall restrict ourselves to the latter case since algebraic manipulations are thereby reduced to a minimum, particularly in the relativistic region.

We may imagine the cross section to be expanded in powers of the energy loss k for small k (we use units $\hbar=c=1$ throughout):

$$\sigma=\frac{\sigma_0}{k}+\sigma_1+k\sigma_2+\cdots, \tag{1.1}$$

or

$$k\sigma=\sigma_0+k\sigma_1+k^2\sigma_2+\cdots. \tag{1.2}$$

The results just stated predict a unique value for

$$\sigma_0=\lim_{k\rightarrow 0} k\sigma \tag{1.3}$$

and

$$\sigma_1=\lim_{k\rightarrow 0}\frac{d}{dk}(k\sigma), \tag{1.4}$$

provided the corresponding scattering amplitudes are known. The term σ_1 in the cross section arises from an interference between a term of order $1/k$ and one of order 1 in the amplitude. Unlike the situation in the scattering of light, where there is no interference between the Thompson and magnetic amplitudes for unpolarized targets, there is in general an interference term in this case. The coefficient σ_0 contains all electric multipoles whereas σ_1 contains all magnetic multipoles as well as electric ones. The usefulness of Eqs. (1.3) and (1.4) depends on how close one must come to $k=0$ before the limit is approximately reached. Barring accidentally strong energy dependence, one expects, for the case of strong coupling, that the most important parameter be k/E. Thus we must have

$$k/E\ll 1. \tag{1.5}$$

In a normal radiation problem one expects that the $(l+1)$st multipole amplitude will have the ratio kd to the l'th, where d is a length characteristic of the source, either its linear dimension or, for close collisions, the wavelength of the scattered particles. The bremsstrahlung problem is peculiar in that one finds, instead, the ratio v. The explanation is that there exists another length, that is, the distance a particle can move with energy imbalance $\Delta E=k$. This length is $d'=\Delta tv=v/k$, so that $d'k=v$. Furthermore, unless $d'\gg b$, where b is the range of the force producing the scattering, it must be impossible to separate the scattering from the radiation; we must therefore have, in addition to (1.5),

$$d'/b=v/kb\gg 1,\quad\text{or}\quad kb/v\ll 1. \tag{1.6}$$

For a sufficiently singular potential, presumably b may be replaced by $1/p$, leading back to our condition (1.5).

For a nonrelativistic system the electric dipole amplitude is $\sim(ev/k)T$ where T is the scattering amplitude. We here calculate the k/E (or kb/v, as the case may be) correction to it. The electric quadrupole amplitude is v times as small; we calculate it and also its k/E correction. The magnetic dipole is $k/p=(k/E)v$ times as small as the electric dipole; it is calculated here only in lowest order.

We state in this section our result for two special systems. Formulas for other systems may be derived easily using the methods developed in Secs. 2 and 3.

Case A.—Two spin-zero bosons, with initial four-momenta p_1 and p_2, final four-momenta p_1' and p_2'. Particle one carries charge e, particle two is neutral. The S-matrix element is

$$\langle kp_1'p_2'|S|p_1p_2\rangle=-(2\pi)^4i\delta(p_1'+p_2'+k-p_1-p_2)\times(32kE_1'E_2'E_1E_2)^{-\frac{1}{2}}M_\mu\epsilon_\mu,$$

with

$$M_\mu=e\left(\frac{p_{1\mu}'}{p_1'\cdot k}-\frac{p_{1\mu}}{p_1\cdot k}\right)T(\nu,\Delta)+e\left(\frac{p_{1\mu}'p_2'\cdot k}{p_1'\cdot k}-p_{2\mu}'+\frac{p_{1\mu}p_2\cdot k}{p_1\cdot k}-p_{2\mu}\right)\frac{\partial T(\nu,\Delta)}{\partial\nu}+O(k),\tag{1.7}$$

where $\nu=p_1\cdot p_2+p_1'\cdot p_2'$, $\Delta=(p_2-p_2')^2$, T is the invariant scattering amplitude, and $p_1\cdot p_2=p_{1\mu}p_{2\mu}$. The photon polarization is ϵ_μ. For a nonrelativistic system ν is proportional to the average (of the initial and final) kinetic energy and $\sqrt{\Delta}$ is the momentum transfer of the neutral particle.

Case B.—One neutral spin-zero boson, of four-momenta q_0 and q_f, and one spin-one-half fermion of charge e, anomalous moment λ, mass m, and four-momenta p_0 and p_f:

$$M_\mu=\bar{u}(p_f)\left\{(ie\gamma_\mu+i\lambda\sigma_{\mu\nu}k_\nu)\frac{1}{i\gamma\cdot(p_f+k)+m}T+T\frac{1}{i\gamma\cdot(p_0-k)+m}(ie\gamma_\mu+i\lambda\sigma_{\mu\nu}k_\nu)+e\left(\frac{q_f\cdot kp_{f\mu}}{p_f\cdot k}-q_{f\mu}+\frac{q_0\cdot kp_{0\mu}}{p_0\cdot k}-q_{0\mu}\right)\frac{\partial T}{\partial\nu}\right\}u(p_0)+O(k),\tag{1.8}$$

where T is the invariant scattering amplitude, expressed in the form $T=A(\nu,\Delta)+\frac{1}{2}i\gamma\cdot(q_0+q_f)B(\nu,\Delta)$, and where $\nu=p_0\cdot q_0+p_f\cdot q_f$ and $\Delta=(q_0-q_f)^2$. Equations (1.7) and (1.8) are correct to order k.

In general, the methods developed in this paper make it clear that, given the amplitude for any radiationless process, the amplitude for the process accompanied by radiation may be calculated to order k^0 in terms of it.

We give finally the nonrelativistic limit of Eqs. (1.7) and (1.8). In this limit $\Delta\approx(p_2-p_2')^2\approx(p_1-p_1')^2\approx(\mathbf{p}_1-\mathbf{p}_1')^2$ and $\nu\approx-(m_1+m_2)(E+E')$, where $\mathbf{p}_1$ and $\mathbf{p}_1'$ are the initial and final momenta of particle 1 in the c.m. system and E and E' are the initial and final energies of the particles in the c.m. system. One finds easily:

$$\mathbf{M}=\frac{e_1}{m_1}\left[\frac{(\mathbf{p}_1'-\mathbf{p}_1)}{k}T\left(\frac{E+E'}{2},\Delta\right)-\tfrac{1}{2}(\mathbf{p}_1+\mathbf{p}_1')\frac{\partial T}{\partial E}(E,\Delta)\right].\tag{1.7 N.R.}$$

2. DERIVATION FOR TWO SPIN-ZERO PARTICLES

We shall for simplicity consider here radiation by a system of two spin-zero bosons, of which only one is charged. The generalization to two charged particles, charge exchange scattering, etc., is obvious; the generalization to the case where at least one of the particles has spin $\frac{1}{2}$ is less obvious and will be discussed in the next section.

Let p_1, p_2 be the initial and p_1' and p_2' the final four-momenta of the two particles. The photon has energy-momentum k, so that

$$p_1+p_2=p_1'+p_2'+k.\tag{2.1}$$

The bremsstrahlung matrix element may be written

$$\epsilon_\mu M_\mu(32kE_1'E_2'E_1E_2)^{-\frac{1}{2}},\tag{2.2}$$

where $M_\mu=M_\mu^{(1)}+M_\mu^{(2)}$ and $M_\mu^{(1)}$ consists of the sum of all Feynman diagrams in which the photon is emitted before or after the interaction; $M_\mu^{(2)}$ consists of all other diagrams. As $k\rightarrow 0$, $M^{(1)}\sim 1/k$, $M^{(2)}\sim$constant, independent of how $k_\mu\rightarrow 0$.

The current that emits a final photon (let particle 1 carry the charge) is

$$J_\mu^{(f)}=ep_{1\mu}'/p_1'\cdot k,\tag{2.3}$$

whereas the current that emits an initial photon is

$$J_\mu^{(i)}=-ep_{1\mu}/p_1\cdot k.\tag{2.4}$$

Equations (2.3) and (2.4) are exact. To prove this, we use the relation

$$\Delta_{F_c}(p_2)(p_2-p_1)_\mu T_\mu(p_2,p_1)\Delta_{F_c}(p_1)=e[\Delta_{F_c}(p_1)-\Delta_{F_c}(p_2)],\tag{2.5}$$

where Δ_{F_c} is the exact renormalized boson propagation function and T_μ the exact renormalized electromagnetic vertex operator. It follows from (2.5) that

$$(p_2-p_1)_\mu J_\mu(p_2,p_1)=-e,\tag{2.6}$$

where $J_\mu(p_2,p_1)$ is the final current and where

$$p_2^2+m^2=0,\quad\text{i.e.,}\quad J_\mu=T_\mu(p_2,p_1)\Delta_{F_c}(p_1).\tag{2.7}$$

Now

$$J_\mu=(p_2+p_1)_\mu f(p_2^2+m^2,\,p_1^2+m^2,\,(p_2-p_1)^2)+(p_2-p_1)_\mu g(p_2^2+m^2,\,p_1^2+m^2,\,(p_2-p_1)^2).\tag{2.8}$$

In our application, $p_2=p_1'$, $p_1=p_1'+k$, so that the coefficient g multiplies k_μ and may be ignored. Equation (2.6) tells us that

$$-2k\cdot p_1'f(0,\,2k\cdot p_1',\,0)=-e,$$

or $f(0,x,0)=e/2x$ so that (2.3) holds. The proof for (2.4) is identical.

The coefficients of $J_\mu^{(f)}$ and $J_\mu^{(i)}$ are the appropriate invariant scattering amplitudes, which we shall call T: thus

$$M_\mu^{(1)}=e\left[\frac{p_{1\mu}'}{p_1'\cdot k}\langle p_1'+k,\,p_2'|T|p_1,p_{2k}-\frac{p_{1\mu}}{p_1\cdot k}\langle p_1',p_2'|T|p_1-k,\,p_2\rangle\right].\tag{2.9}$$

The T matrices in Eq. (2.9) conserve momentum and energy but not mass. Let us call the initial and final masses of which the T matrix is a function M_1 and M_1', respectively; the actual masses of the two particles are m_1 and m_2. The T matrices with which we have to deal depend on the two mass values assumed by m_1 as well as on two other variables, essentially energy and angle. It is convenient to choose these last to be $\nu=p_1\cdot p_2+p_1'\cdot p_2'$ and $\Delta=(p_2'-p_2)^2$. At $M_1=M_1'=m_1$ the T matrix is a known linear function of the scattering amplitude. We write

$$T=T(M_1'^2,M_1^2,\nu,\Delta). \tag{2.10}$$

Then

$$M_\mu{}^{(1)}=\frac{ep_{1\mu}'}{p_1'\cdot k}T(m_1^2-2k\cdot p_1',\, m_1^2,\, p_1\cdot p_2+p_1'\cdot p_2'+k\cdot p_2',\, (p_2-p_2')^2)$$
$$-\frac{ep_{1\mu}}{p_1\cdot k}T(m_1^2,\, m_1^2+2p_1\cdot k,\, p_1\cdot p_2+p_1'\cdot p_2'-k\cdot p_2,\, (p_2-p_2')^2). \tag{2.11}$$

Since the over-all current is conserved, we have

$$k_\mu M_\mu=0, \tag{2.12}$$

so that

$$k_\mu M_\mu{}^{(2)}=-k_\mu M_\mu{}^{(1)} \tag{2.13}$$

$$=e[T_2(2p_1\cdot k)+T_1(2p_1'\cdot k)-T_3(p_2+p_2')\cdot k]+O(k^2), \tag{2.14}$$

where T_n means differentiation with respect to the nth of the four arguments given by Eq. (2.10). The derivatives in (2.14) may be evaluated at $k=0$ to give us the necessary accuracy. From (2.14) and the absence of singularities of $M_\mu{}^{(2)}$ as $k\rightarrow 0$, we have

$$M_\mu{}^{(2)}=e[2p_{1\mu}T_2+2p_{1\mu}'T_1-(p_{2\mu}+p_{2\mu}')T_3]. \tag{2.15}$$

If we now expand Eq. (2.11) in power of k, and add Eq. (2.11) and Eq. (2.15), we find, keeping only the first two terms:

$$M_\mu=e\left(\frac{p_{1\mu}'}{p_1'\cdot k}-\frac{p_{1\mu}}{p_1\cdot k}\right)T(m_1^2,m_1^2,\nu,\Delta)$$
$$+e\left[\frac{p_{1\mu}}{p_1\cdot k}p_2\cdot k-p_{2\mu}+\frac{p_{1\mu}'}{p_1'\cdot k}p_2'\cdot k-p_{2\mu}'\right]\times\frac{\partial T}{\partial\nu}(m_1^2,m_1^2,\nu,\Delta)+O(k), \tag{2.16}$$

where $\nu=p_1\cdot p_2+p_1'\cdot p_2'$ and $\Delta=(p_2-p_2')^2$. It will be seen that the derivatives with respect to M^2 have disappeared from the final formula; that is, the matrix-element has been expressed, to order k^0, as a function of the (mass-shell) scattering amplitude and its derivatives with respect to energy and angle. Before going on to more complicated processes we may emphasize that although the details of the correction terms depend on the system under consideration, the cancellation of the derivatives of the T matrix with respect to the masses is a very general result. Thus, although we shall only give the result for two special cases, the procedure is sufficiently straightforward so that it can be applied to bremsstrahlung in any kind of process. As our second example, we consider a boson-fermion collision.

3. DERIVATION FOR ONE SPIN-ZERO AND ONE SPIN-ONE-HALF PARTICLE

Here, again for simplicity, we consider scattering of a charged spin $-\frac{1}{2}$ fermion of four-momentum p, mass m, and anomalous magnetic moment λ by a neutral boson of four-momentum q, mass μ, and spin zero. The case of a charged boson may be treated by the same methods as were introduced in Sec. 2. Although the case of fermion-fermion scattering is considerably more complicated, it will be clear that the same methods can be applied there as here.

Our procedure parallels that of Sec. 2. As before, we write the radiation matrix element as

$$\epsilon_\mu M_\mu(32kE_fE_0\omega_f\omega_0)^{-\frac{1}{2}}, \tag{3.1}$$

where $M_\mu=M_\mu{}^{(1)}+M_\mu{}^{(2)}$ with

$$M_\mu{}^{(1)}=J_\mu{}^{(f)}\langle p_f+k,\, q_f|T|p_0,\, q_0\rangle-\langle p_f,\, q_f|T|p_0-k,\, q_0\rangle J_\mu{}^{(i)}, \tag{3.2}$$

and $M_\mu{}^{(2)}$ again to be determined by Eq. (2.12). Here we can no longer calculate the J_μ's exactly; however, we can still calculate that part of the J_μ's which has a singularity as $k\rightarrow 0$; it is, for $J_\mu{}^{(f)}$,

$$J_\mu{}^{(f)}=\bar{u}(p_f)(ie\gamma_\mu+i\lambda\sigma_{\mu\nu}k_\nu)\frac{1}{i\gamma\cdot(p_f+k)+m}, \tag{3.3}$$

and for $J_\mu{}^{(i)}$,

$$J_\mu{}^{(i)}=\frac{1}{i\gamma\cdot(p_i-k)+m}(ie\gamma_\mu+i\lambda\sigma_{\mu\nu}k_\nu)u(p_i). \tag{3.4}$$

Now consider the operator

$$\frac{1}{i\gamma\cdot(p_f+k)+m}=\frac{1}{2p_f\cdot k}[-i\gamma\cdot P_f+M_f+m-M_f], \tag{3.5}$$

where

$$P_f=p_f+k,\quad M_f^2=-P_f^2=m^2-2p_f\cdot k. \tag{3.6}$$

Since $m-M_f\approx(p_f\cdot k)/m$, the $m-M_f$ in Eq. (3.5) cancels the $1/k$ singularity, so that its effect may be included in $M_\mu{}^{(2)}$. Thus for the purpose of calculating $M_\mu{}^{(1)}$, we may let

$$J_\mu{}^{(f)}=\bar{u}(p_f)\frac{(ie\gamma_\mu+i\lambda\sigma_{\mu\nu}k_\nu)}{2p_f\cdot k}(-i\gamma\cdot P_f+M_f), \tag{3.7}$$

with a similar equation for $J_\mu{}^{(i)}$. The point now is that

$i\gamma\cdot P_f$ acting to the right of $J_\mu{}^{(f)}$ gives $-M_f$, since

$$(-i\gamma\cdot P_f+M_f)i\gamma\cdot P_f=P_f^2+M_f(i\gamma\cdot P_f)$$
$$=-M_f^2+M_f i\gamma\cdot P_f=(-i\gamma\cdot P_f+M_f)(-M_f).$$

Therefore the T matrices in Eq. (3.2) have the same number of independent functions as the real T matrix, that is, two:

$$T=A+\tfrac{1}{2}i\gamma\cdot(q_f+q_0)B, \tag{3.8}$$

where, exactly as in Eq. (2.10),

$$A=A(M_f,M_0,\nu,\Delta),\quad B=B(M_f,M_0,\nu,\Delta),$$
$$\nu=p_f\cdot q_f+p_0\cdot q_0,\quad \Delta=(q_f-q_0)^2. \tag{3.9}$$

We now write out $M_\mu{}^{(1)}$:

$$M_\mu{}^{(1)}=\bar{u}(p_f)\bigg\{(ie\gamma_\mu+i\lambda\sigma_{\mu\nu}k_\nu)$$
$$\times\left(\frac{-i\gamma\cdot P_f+M_f}{2p_f\cdot k}\right)\bigg[\big[A(M_f^2,m^2,\nu+q_f\cdot k,\Delta)$$
$$+\frac{i\gamma\cdot(q_0+q_f)}{2}B(M_f^2,m^2,\nu+q_f\cdot k,\Delta)\bigg]$$
$$-\bigg[A(m^2,M_0^2,\nu-q_0\cdot k,\Delta)+\frac{i\gamma\cdot(q_0+q_f)}{2}$$
$$\times B(m^2,M_0^2,\nu-q_0\cdot k,\Delta)\bigg]\left(\frac{-i\gamma\cdot P_0+M_0}{2p_0\cdot k}\right)$$
$$\times(ie\gamma_\mu+i\lambda\sigma_{\mu\nu}k_\nu)\bigg\}u(p_0). \tag{3.10}$$

In Eq. (3.10), we have $P_0=p_0-k$, $M_0^2=m^2+2p_0\cdot k$, $\nu=p_0\cdot q_0+p_f\cdot q_f$, and $\Delta=(q_0-q_f)^2$. We proceed exactly as in Sec. 2 by calculating $k_\mu M_\mu{}^{(2)}=-k_\mu M_\mu{}^{(1)}$. The only new algebraic points included are the following:

$$\bar{u}(p_f)i\gamma\cdot k\left(\frac{-i\gamma\cdot P_f+M_f}{2p_f\cdot k}\right)$$
$$=\bar{u}(p_f)\frac{i\gamma\cdot k}{2p_f\cdot k}\left[-i\gamma\cdot(p_f+k)+m-\frac{p_f\cdot k}{m}\right]$$
$$=\bar{u}(p_f)\left(1-\frac{i\gamma\cdot k}{2m}\right), \tag{3.11}$$

$$\left(\frac{-i\gamma\cdot P_0+M_0}{2p_0\cdot k}\right)i\gamma\cdot k u(p_0)=\left(1+\frac{i\gamma\cdot k}{2m}\right)u(p_0). \tag{3.12}$$

The calculation of $k\cdot M^{(1)}$ is now straightforward and yields

$$k\cdot M^{(1)}=e\bar{u}(p_f)\bigg\{\left(1-\frac{i\gamma\cdot k}{2m}\right)T(M_f^2,m^2,\nu+q_f\cdot k,\Delta)$$
$$-T(m^2,M_0^2,\nu-q_0\cdot k,\Delta)\left(1+\frac{i\gamma\cdot k}{2m}\right)\bigg\}u(p_0). \tag{3.13}$$

Therefore

$$M_\mu{}^{(2)}=e\bar{u}(p_f)\bigg\{2p_{f\mu}T_1+2p_{0\mu}T_2$$
$$+(i\gamma_\mu T+Ti\gamma_\mu)\frac{1}{2m}-(q_f+q_0)_\mu T_3\bigg\}u(p_0), \tag{3.14}$$

where, as in Sec. 2, subscripts refer to differentiation, and all quantities in (3.14) may be evaluated at $k=0$.

We turn now to $M_\mu{}^{(1)}$. We may replace $(-i\gamma\cdot P_f+M_f)/2p_f\cdot k$ by $(-i\gamma\cdot P_f+m)/(2p_f\cdot k)-1/2m$. The second term, $-1/2m$, will cancel the next to the last term in (3.14) and may be ignored. Also, we expand A and B in powers of k; associated with the first-order term we may set $(-i\gamma\cdot P_f+m)=(-i\gamma\cdot p_f+m)$ since the difference is of first order and will result in a second-order product. Finally we note that

$$\bar{u}(p_f)i\gamma_\mu(-i\gamma\cdot p_f+m)=\bar{u}(p_f)\,2p_{f\mu},$$

and, of course, that

$$\frac{-i\gamma\cdot(p_f+k)+m}{2p_f\cdot k}=\frac{1}{i\gamma\cdot(p_f+k)+m}.$$

With these remarks, (3.10) becomes

$$M_\mu{}^{(1)}=\bar{u}(p_f)\bigg\{(ie\gamma_\mu+i\lambda\sigma_{\mu\nu}k_\nu)\frac{1}{i\gamma\cdot(p_f+k)+m}T$$
$$+T\frac{1}{i\gamma\cdot(p_0-k)+m}(ie\gamma_\mu+i\lambda\sigma_{\mu\nu}k_\nu)$$
$$+e\left(\frac{q_f\cdot kp_{f\mu}}{p_f\cdot k}+\frac{q_0\cdot kp_{0\mu}}{p_0\cdot k}\right)\frac{\partial T}{\partial\nu}$$
$$-e(2p_{f\mu}T_1+2p_{0\mu}T_2)\bigg\}u(p_0). \tag{3.15}$$

We now add (3.14) and (3.15) to get M_μ:

$$M_\mu=M_\mu{}^{(1)}+M_\mu{}^{(2)},$$

or

$$M_\mu=\bar{u}(p_f)\bigg\{(ie\gamma_\mu+i\lambda\sigma_{\mu\nu}k_\nu)\frac{1}{i\gamma\cdot(p_f+k)+m}T$$
$$+T\frac{1}{i\gamma\cdot(p_0-k)+m}(ie\gamma_\mu+i\lambda\sigma_{\mu\nu}k_\nu)$$
$$+e\left(\frac{q_f\cdot k}{p_f\cdot k}p_{f\mu}-q_{f\mu}+\frac{q_0\cdot k}{p_0\cdot k}p_{0\mu}-q_{0\mu}\right)\frac{\partial T}{\partial\nu}\bigg\}u(p_0)$$
$$+O(k). \tag{3.16}$$

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