Remarks on the Hypertriton*

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(Received January 30, 1958)

The structure of the S states possible for a system consisting of a Λ particle and two nucleons is discussed, and criteria are laid down for a trial wave function suitable for describing the orbital motion. In order to obtain an upper bound for the Λ-nucleon interaction strength, variational calculations are carried through for the hypertriton with a simple trial function. Following the discussion of the results, it is concluded that it is very unlikely that there should exist bound states for the hyperdeuteron or for ${}_{\Lambda}He^3$.

1. INTRODUCTION

THE simplest system consisting of nucleons and a Λ particle for which a bound state is known to exist is the hypertriton ${}_{\Lambda}H^3$. Nine clearly identified examples of ${}_{\Lambda}H^3$ decay events are listed in the world survey of hypernuclei recently given by Levi Setti, Slater, and Telegdi.[1] The binding energy B_{Λ} for the Λ particle in the hypertriton is still subject to considerable uncertainty. Its value is obtained from the expression

$$B_{\Lambda}=Q_{\Lambda}-T+B_f-B_d, \tag{1}$$

where B_d and B_f denote the deuteron binding energy and the total binding energy of the nuclear fragments resulting from a π^--mesonic decay of ${}_{\Lambda}H^3$, T denotes the total kinetic energy released in the decay, and Q_{Λ} is the energy released in the charged mode (π^-+p) of free Λ decay. At present the straggling in T, due to the causes discussed by Levi Setti *et al.*, contributes ±0.31 Mev to the uncertainty in B_{Λ}. There is also some doubt about the value of Q_{Λ}. The B_{Λ} values given by Levi Setti *et al.*[1] were obtained by using an early Q_{Λ} value obtained in an emulsion study by Friedlander *et al.*[2]: $Q_{\Lambda}=36.9\pm0.21$ Mev. Application of the range-energy relation used to calculate T for the hypernuclear decay events to the Λ-decay data of Friedlander *et al.* leads to a value[3] of 36.75 ± 0.2 Mev for Q_{Λ}. Further emulsion data obtained by Barkas *et al.*[4] have led (with use of the same range-energy relation) to the value 37.45 ± 0.17 Mev. The average of these last two values, weighted by the number of events (9 for the former, 18 for the latter), is

$$Q_{\Lambda}=37.22\pm0.22 \text{ Mev.} \tag{2}$$

With this value (2), the identified hypertriton events correspond to the binding energy

$$B_{\Lambda}({}_{\Lambda}H^3)=0.6\pm0.4 \text{ Mev.} \tag{3}$$

At present the only definite statement which can be made is that B_{Λ} for the hypertriton is positive and almost certainly less than 1 Mev. Fortunately the qualitative conclusions to be reached in the present work remain the same for any value of B_{Λ} within this range.

The low value of the total binding energy $B_d+B_{\Lambda}\approx2.8$ Mev for the hypertriton means that this system has a very open structure, the mean separation between any pair of particles being at least of order $\hbar/[M(B_d+B_{\Lambda})]^{\frac{1}{2}}\sim3\times10^{-13}$ cm, which is large compared with the range of their nuclear interaction. This means that, while the nuclear interaction is effective between two of the particles, the third particle is on the average relatively distant from them, so that it has no important effect on their mutual interaction.[5] The interaction of each pair of particles therefore takes place under circumstances closely related to those of free two-body collision at low relative energy. For this reason the analysis of the hypertriton is of special interest in the study of the nuclear interaction of the hyperon, since it may be expected to lead to a reliable estimate of some of the low-energy properties of this interaction which do not depend on its detailed form. For example, this analysis, taken together with other evidence on the spin-dependence of the Λ-nucleon interaction,[6] leads to the conclusion that there is no hyperdeuteron (i.e., there is no bound system consisting of a Λ particle and one nucleon), and this is consistent with the absence of any

* This work was begun at the Department of Mathematical Physics, University of Birmingham. At the present institutions, it has been supported by the U. S. Atomic Energy Commission program at the University of Chicago and by the joint program of the U. S. Atomic Energy Commission and the Office of Naval Research at Cornell University.

[1] Levi Setti, Slater, and Telegdi, *Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics, 1957* (Interscience Publishers, New York, 1957), Sec. VIII, p. 6.

[2] Friedlander, Keefe, Menon, and Merlin, Phil. Mag. **45**, 533 (1954).

[3] W. Slater, University of Chicago dissertation, 1958 (unpublished).

[4] Barkas, Giles, Heckman, Inman, Mason, and Smith, University of California Radiation Laboratory Report UCRL-3892, 1957 (unpublished).

[5] This is also the reason why any three-body forces which may exist between a Λ particle and two nucleons will play a very small role in the hypertriton. Such three-body forces could arise, for example, from the pion-pair interaction $\Lambda\rightarrow\Lambda+\pi+\pi$ allowed by charge independence [see R. H. Dalitz, Phys. Rev. **99**, 1475 (1955)], a pion being transferred to each of two neighboring nucleons.

[6] R. H. Dalitz, *Reports on Progress in Physics* (The Physical Society, London, 1957), Vol. 20, p. 163.

hypernuclear event which requires the existence of a hyperdeuteron for its interpretation.

The Λ particle is an isotopic-spin singlet state ($T=0$). Charge independence then requires[7] that the Λ-neutron and Λ-proton interactions be identical for a given state of spin and relative orbital motion. In terms of the interactions which are known to exist, contributions to these forces may arise from (a) the exchange of two or more pions between the Λ particle and the nucleon (the exchange of a single pion is forbidden by the charge symmetry of the interactions), (b) the exchange of one or more K mesons between the Λ and the nucleon, and (c) the exchange of both pions and K mesons. Each of these contributions gives rise to an interaction whose range is $\hbar/2m_\pi c \approx 0.7\times 10^{-13}$ cm or less, which is shorter than the range of the nucleon-nucleon interaction. The exchange of pions alone (or with an even number of K mesons) would lead to an ordinary force between Λ and nucleon, whereas the exchange of an odd number of K mesons (with or without additional pions) transfers "strangeness" between the Λ and nucleon and will therefore contribute an exchange term to the Λ-nucleon interaction. This difference between these two mechanisms for the Λ-nucleon force may be seen clearly by considering the simplest contribution of each type, namely

(a) $\Lambda+\mathfrak{N}\rightarrow\Lambda+\pi+\pi+\mathfrak{N}\rightarrow\Lambda+\mathfrak{N}$,

(b) $\Lambda+\mathfrak{N}\rightarrow\mathfrak{N}+\bar{K}+\mathfrak{N}\rightarrow\mathfrak{N}+\Lambda$.

The contributions from the two mechanisms could be most readily distinguished by comparison of the interactions between Λ and nucleon in S-wave and P-wave relative motion of given total spin, since the exchange term has a sign proportional to the parity of the relative motion, the ordinary force having a sign independent of the parity of this motion. In a system as lightly bound as the hypertriton, however, the interactions between each pair of particles take place almost entirely in relative S states, so that the distinction between these two types of force has little effect.

The S-wave interaction between Λ and nucleon may still have spin dependence, the strength of the interaction depending on whether the spins are coupled parallel or antiparallel. These interactions may also contain tensor terms, as does the nucleon-nucleon interaction. The short range of the Λ-nucleon forces means, however, that any Λ-nucleon tensor forces would contribute much less to the D state of the hypertriton than does the neutron-proton tensor force, since the centrifugal barriers effective in the D state will prevent the very close Λ-nucleon approach necessary in this state before the short-range Λ-nucleon tensor force comes into play. It is therefore reasonable to expect that the D state of the hypertriton will be considerably smaller in magnitude than the (already small) component of D state in the normal triton. In this situation, the effect of a tensor component in the Λ-nucleon force will be indistinguishable from that of an additional central interaction in this force. What is important for the binding of the hypertriton are the low-energy scattering characteristics of the Λ-nucleon interaction, the zero-energy scattering lengths, and the effective ranges; the central potentials to be used are to be considered only as equivalent potentials giving the same low-energy scattering as the actual Λ-nucleon interaction.

It is now generally accepted that the Λ particle has spin $\frac{1}{2}$. This is consistent with all of the evidence obtained to date[8] on the angular correlations in Λ-particle decay following the π^-+p reactions, although much of the data does not really exclude a spin value of $\frac{3}{2}$. This is also true of the Ruderman-Karplus argument[9] based on analysis of the internal conversion coefficient for nonmesonic decay of helium and of heavier hyperfragments. Also Lee and Yang[10] have pointed out that the large up-down asymmetry recently observed in Λ decay following the π^-+p production reaction excludes high spin values for the Λ particle, being consistent only with a Λ-spin value of $\frac{1}{2}$ or (possibly) $\frac{3}{2}$. Since a Λ spin of $\frac{1}{2}$ turns out to be slightly exceptional in the discussion of the ${}_\Lambda H^3$ system, we shall give the formulas for a general Λ-spin S, although it is now most probable that $S=\frac{1}{2}$ is the physically relevant case.

2. POSSIBLE STATES OF A Λ PARTICLE AND TWO NUCLEONS

The Λ-nucleon interaction will be represented by an equivalent central potential as discussed above. This potential is denoted by V_p for the parallel spin configuration where the Λ-spin S and the nucleon spin couple to a total spin $S+\frac{1}{2}$, and by V_a for the antiparallel configuration with total spin $S-\frac{1}{2}$. With this notation this potential may be written generally

$$V=\frac{\boldsymbol{\sigma}\cdot\mathbf{S}+S+1}{2S+1}V_p+\frac{S-\boldsymbol{\sigma}\cdot\mathbf{S}}{2S+1}V_a, \tag{4}$$

where the coefficients are the spin projection operators for states of total spin $S+\frac{1}{2}$ and $S-\frac{1}{2}$; $\mathbf{S}$ denotes the Λ-spin operator, and $\frac{1}{2}\boldsymbol{\sigma}$ the nucleon-spin operator. With this assumption of equivalent central forces, the wave function for the system will be the product of a spin wave function and a coordinate wave function. In this section we discuss the possible spin wave functions and their properties.

The system of a Λ and two nucleons can exist in

[7] R. H. Dalitz, Phys. Rev. **99**, 1475 (1955).

[8] D. Glaser, *Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics, 1957* (Interscience Publishers, New York, 1957), Sec. 5, p. 24; L. Leipuner and R. K. Adair, Phys. Rev. **109**, 1358 (1958); Alvarez, Bradner, Falk-Vairant, Gow, Rosenfeld, Solmitz, and Tripp, Nuovo cimento **5**, 1026 (1957).

[9] M. Ruderman and R. Karplus, Phys. Rev. **102**, 247 (1956); Schneps, Fry, and Swami, Phys. Rev. **106**, 1062 (1957).

[10] T. D. Lee and C. N. Yang, Phys. Rev. **109**, 1755 (1958).

states of isotopic spin $T=0$ or $T=1$. The $T=0$ states occur only for (Λnp) systems and require parallel spins for the neutron and proton. The Λ-spin S can then combine with the spins of these two nucleons to give states of total spin $S+1$, S, $S-1$. The expectation values of the interactions effective in these states are the following:

$$j=S+1: \quad \bar{V}_{\text{tr}}(np)+2\bar{V}_p(\Lambda\mathfrak{N}), \tag{5a}$$

$$j=S: \quad \bar{V}_{\text{tr}}(np)+\frac{2S}{2S+1}\bar{V}_p(\Lambda\mathfrak{N})+\frac{2(S+1)}{2S+1}\bar{V}_a(\Lambda\mathfrak{N}), \tag{5b}$$

$$j=S-1: \quad \bar{V}_{\text{tr}}(np)+2\bar{V}_a(\Lambda\mathfrak{N}). \tag{5c}$$

In these expressions advantage has been taken of the symmetry of the space wave function with respect to n and p, so that $\bar{V}(\Lambda n)=\bar{V}(\Lambda p)=\bar{V}(\Lambda\mathfrak{N})$. Generally the lowest state for which $T=0$ has spin $S+1$ if parallel Λ-nucleon spins are favored $(V_p>V_a)$, or spin $S-1$ if antiparallel Λ-nucleon spins are favored $(V_a>V_p)$. For $S=\frac{1}{2}$, however, the state $j=S-1$ does not exist; and with $V_a>V_p$, the lowest state will have $j=S=\frac{1}{2}$, the mean Λ-nucleon potential being a combination of both V_a and V_p. The case with $V_a>V_p$ and $S=\frac{1}{2}$ must therefore be regarded as an exceptional case. This situation has been remarked also by Derrick.[11]

In the $T=1$ states, the two nucleons have unit isotopic spin and form a singlet spin system. These states therefore have spin $j=S$ and appear as charge triplets in ${}_\Lambda n^3$, ${}_\Lambda H^3$, and ${}_\Lambda He^3$. The expectation value of the total interaction is here

$$\bar{V}_{\text{sing}}(np)+\frac{2(S+1)}{2S+1}\bar{V}_p(\Lambda\mathfrak{N})+\frac{2S}{2S+1}\bar{V}_a(\Lambda\mathfrak{N}). \tag{6}$$

For the ${}_\Lambda He^3$ system there is an additional Coulomb repulsion between the protons. From (6) it is clear that both the nucleon-nucleon attraction[12] and the Λ-nucleon attractions are weaker in a $T=1$ state than in the lowest $T=0$ configuration. For this reason it is natural to identify the observed ${}_\Lambda H^3$ system with the $T=0$ configuration of lowest energy. This is consistent[7] with the absence of experimental evidence for a ${}_\Lambda He^3$ state corresponding in total energy with the observed hypertriton.

It is also of interest to list the magnetic moments associated with each of these states, since Goldhaber[13] has pointed out the possibility of measuring the magnetic moment of hyperons and hypernuclei by observing the effect of a strong applied field on the axis of any anisotropy in their decay. For the $T=1$ configurations the magnetic moment equals that of the Λ particle, μ_Λ say, since the nucleons are in the singlet state. For $T=0$, the magnetic-moment operator reduces to

$$\mathbf{M}=\tfrac{1}{2}(\mu_p+\mu_n)(\boldsymbol{\sigma}_p+\boldsymbol{\sigma}_n)+(\mu_\Lambda/S)\mathbf{S}. \tag{7}$$

In the state $j=S+1$, the total magnetic moment is simply the sum $(\mu_p+\mu_n+\mu_\Lambda)$, while in the configuration $j=S-1$,

$$\mu(S-1)=\left(1-\frac{1}{S^2}\right)\mu_\Lambda-\left(1-\frac{1}{S}\right)(\mu_p+\mu_n).$$

In the intermediate configuration $j=S$ (of interest only for $S=\frac{1}{2}$), the magnetic moment is

$$\mu(S)=\left(1-\frac{1}{S(S+1)}\right)\mu_\Lambda+\frac{1}{S+1}(\mu_p+\mu_n).$$

3. TRIAL WAVE FUNCTION AND VARIATIONAL CALCULATION FOR ${}_\Lambda H^3$

Since the low binding energy of the Λ particle in the hypertriton means that the Λ particle is distant from the nucleons for a large fraction of the time, it is tempting to consider a product form,

$$\psi=\phi_{np}(r)g(s), \tag{8}$$

as being appropriate for this system, where r is the n-p separation and s is the distance of the Λ particle from the n-p center of mass. This has the advantage of giving the correct asymptotic form to ψ for large s if ϕ_{np} is chosen to be the free-deuteron wave function; in this case only the small separation energy B_Λ is left to be accounted for by the Λ-nucleon interactions. If this ψ is taken as a trial function for a variational calculation of the Λ-nucleon interaction to produce a given B_Λ, then it is clear that this will lead at least to an upper bound for the actual strength of the Λ-nucleon interaction; a variational calculation of this kind has been made by Derrick.[11] The upper bound obtained in this way will, however, be a poor estimate, becoming increasingly worse the shorter is the range of the Λ-nucleon interaction. The wave function (8) does not have sufficient flexibility to allow strong correlation in position between the Λ and a nucleon. The Λ-nucleon interaction is effective only when these two particles are close together, so that the function (8) does not lead to a sufficiently large estimate of the attraction due to the Λ-nucleon interactions. This defect becomes increasingly serious with decreasing range for these interactions. This may be seen in the following way. Taking the form $Vf(r/a)$ for the Λ-nucleon potential, the average potential seen by the Λ particle in the motion described by the function (8) is obtained by folding this Λ-nucleon potential with the probability distribution $\phi_{np}^2(r)$ for the n-p separation. When a is sufficiently small, it is clear that this effective potential is given by $\phi_{np}^2(2s)$ multiplied by a factor proportional to the volume

[11] G. H. Derrick, Nuovo cimento **5**, 565 (1956).
[12] This has been emphasized by J. T. Jones and J. K. Knipp, Nuovo cimento **2**, 857 (1955).
[13] M. Goldhaber, Phys. Rev. **101**, 1828 (1956).

integral Va^3 of the Λ-nucleon interaction. For a given B_Λ, therefore, the trial function (8) requires a potential strength V proportional to a^{-3} for small a. On the other hand, it is known that the correct potential strength V has the asymptotic form Ca^{-2} for small a,[14] the correct wave function having strong correlations between the particles. The upper bound for V obtained with (8) therefore becomes increasingly worse as a decreases; this was already realized by Derrick.[11]

The wave function (8) may also be used allowing distortion of $\phi_{np}(r)$ from the deuteron form. This can be advantageous despite the corresponding increase in the n-p relative energy, since the more concentrated average potential seen by the Λ particle can give the increased Λ binding necessary for a given total B_d+B_Λ with a smaller value of V. If both ϕ and g are permitted to vary, however, it is essential that a finite range be used for the Λ-nucleon potential, which corresponds to some physically reasonable mechanism. As was first shown by Thomas,[14] a zero-range interaction allows the system to collapse, as small a volume integral Va^3 as desired being obtained for a fixed B_Λ by considering sufficient collapse. On account of the ease with which the n-p system may be distorted, not even a local minimum in any physically reasonable region will be found for the volume integral of the Λ-nucleon potential corresponding to a given B_Λ (see Appendix).

The following features appear to be necessary in a variational calculation in order that the upper limit obtained be reasonably close to the physically correct value:

(i) It is essential that finite-range interactions, which are appropriate to the physical situation, should be used (although it will be shown that there are certain parameters whose determination seems relatively insensitive to the range assumed, in the region of physical interest).

(ii) The trial wave function must have sufficient flexibility to allow strong correlations in position between the Λ particle and each nucleon; this is necessary in order that the contribution of the strong short-range Λ-nucleon interactions to the total potential energy should be estimated adequately. The wave function should also have a long tail for separation of the Λ particle, corresponding to the low separation energy B_Λ, in order that the Λ kinetic energy should not be overestimated.

(iii) The separation energy (B_d+B_Λ) of each nucleon being considerably larger than the Λ separation energy B_Λ, an appropriate trial function must clearly allow a lack of symmetry between the Λ particle and the nucleons; this is also required by the rather different ranges of the Λ-nucleon and nucleon-nucleon interactions.

[14] L. H. Thomas, Phys. Rev. **47**, 903 (1935); N. Svartholm, thesis, Lund, 1945 (unpublished).

One form of trial function which allows these conditions to be met makes use of the triangular coordinate system (r_1,r_2,r_3) appropriate to a three-particle system. Here r_3 will be taken to be the neutron-proton separation, r_1 and r_2 being the distances of the Λ particle from each nucleon. These coordinates are subject to the usual triangular inequalities, $r_1+r_2\geqslant r_3$, $r_2+r_3\geqslant r_1$, $r_3+r_1\geqslant r_2$. With these coordinates an appropriate trial function,

$$\psi=u(r_3)v(r_1)v(r_2), \tag{9}$$

was proposed by Wigner[15] for three-particle systems, and this has been used by a number of authors[16] for the study of the H^3 and He^3 systems. According to the discussion in the Introduction, these functions u and v should be expected to have substantially the same form as the low-energy scattering (or bound state) wave function for the appropriate pair of particles, at least over the region of their nuclear interaction.

With triangular coordinates, the variational principle for the bound state system takes the form

$$8\pi^2\int r_1dr_1r_2dr_2r_3dr_3\Bigg\{\frac{\hbar^2(M+M_\Lambda)}{2MM_\Lambda}\times\Bigg[\sum_{i=1}^{3}\left(\frac{\partial\psi}{\partial r_i}\right)^2+t(123)+t(231)+t(312)\Bigg]+\frac{\hbar^2(M_\Lambda-M)}{2MM_\Lambda}\Bigg[\left(\frac{\partial\psi}{\partial r_3}\right)^2-t(123)+t(231)+t(312)\Bigg]-Vf(2\kappa r_3)\psi^2-U[g(2\lambda r_1)+g(2\lambda r_2)]\psi^2-(B_d+B_\Lambda)\psi^2\Bigg\}\geqslant 0, \tag{10}$$

with

$$t(ijk)=\left(\frac{r_i^2+r_j^2-r_k^2}{2r_ir_j}\right)\frac{\partial\psi}{\partial r_i}\frac{\partial\psi}{\partial r_j}. \tag{11}$$

This inequality (10) will be used here to provide a variational principle for the strength U of the Λ-nucleon potential effective in the configuration considered. The masses of Λ and nucleon have been denoted by M_Λ and M, respectively.

Calculations have been carried through only for exponential and Yukawa shapes for f and g, the forms of the nucleon-nucleon and Λ-nucleon potentials. For Yukawa shape these functions are written $e^{-2\kappa r}/2\kappa r$, where $2\kappa=2.1196/b$ in terms of the intrinsic range b; for the exponential shape, $e^{-2\kappa r}$ where $2\kappa=3.5412/b$. The parameters for the nucleon-nucleon potentials were taken from the low-energy p-p scattering data[17]

[15] E. P. Wigner, Phys. Rev. **43**, 252 (1933).

[16] See, for example, R. D. Present, Phys. Rev. **50**, 635 (1936); Fröhlich, Huang, and Sneddon, Proc. Roy. Soc. (London) **A191**, 61 (1947).

[17] J. D. Jackson and J. M. Blatt, Revs. Modern Phys. **22**, 77 (1950).

TABLE I. Λ-nucleon interaction in the $T=0$ $_\Lambda H^3$ configuration.

(a) Yukawa potential shapes (B_d+B_Λ) Mev	(i) Intrinsic range 0.8411×10^{-13} cm α (10^{13} cm^{-1})	β (10^{13} cm^{-1})	s	$\bar{U}_2$ [Mev $\times(10^{-13}$ cm$)^3$]	a (10^{-13} cm)	(ii) Intrinsic range 1.4843×10^{-13} cm α (10^{13} cm^{-1})	β (10^{13} cm^{-1})	s	$\bar{U}_2$ [Mev $\times(10^{-13}$ cm$)^3$]	a (10^{-13} cm)
2.226	0.62	0.69	0.76	489	−1.8	0.36	0.65	0.68	766	−2.2
2.626	0.64	0.71	0.77	494	−1.9	0.39	0.65	0.70	786	−2.4
3.226	0.67	0.71	0.78	501	−2.0	0.43	0.66	0.72	813	−2.6
(b) Exponential potential shapes (B_d+B_Λ) Mev	(i) Intrinsic range 0.8411×10^{-13} cm α (10^{13} cm^{-1})	β (10^{13} cm^{-1})	s	$\bar{U}_2$ [Mev $\times(10^{-13}$ cm$)^3$]	a (10^{-13} cm)	(ii) Intrinsic range 1.4843×10^{-13} cm α (10^{13} cm^{-1})	β (10^{13} cm^{-1})	s	$\bar{U}_2$ [Mev $\times(10^{-13}$ cm$)^3$]	a (10^{-13} cm)
2.226	0.52	0.52	0.75	492	−1.9	0.29	0.52	0.64	742	−2.0
2.626	0.55	0.53	0.76	500	−2.0	0.33	0.52	0.66	772	−2.2
3.226	0.58	0.53	0.77	510	−2.1	0.37	0.52	0.70	810	−2.6

and from the deuteron binding energy[18] $B_d=2.226$ Mev. For the Yukawa shape the intrinsic range was taken to be 2.4995×10^{-13} cm, the volume integral of the potential then being 1404 Mev$\times(10^{-13}$ cm$)^3$ for the triplet state and 952 Mev$\times(10^{-13}$ cm$)^3$ for the singlet state. For the exponential shape an intrinsic range of 2.4938×10^{-13} cm was taken with volume integrals of 1533 Mev$\times(10^{-13}$ cm$)^3$ for the triplet state and 962 Mev$\times(10^{-13}$ cm$)^3$ for the singlet state. Two cases were considered with each shape for the Λ-nucleon potential: an intrinsic range $b=0.8411\times10^{-13}$ cm, corresponding to a Yukawa potential with range parameter $2\lambda=(\hbar/m_Kc)^{-1}$ such as K exchange would produce; and an intrinsic range $b=1.4843\times10^{-13}$ cm corresponding to a Yukawa potential with range parameter $2\lambda=(\hbar/2m_\pi c)^{-1}$, appropriate to the exchange of two pions.

In these preliminary calculations a very simple form has been used for ψ:

$$\psi=Ne^{-\alpha(r_1+r_2)}e^{-\beta r_3}. \tag{12}$$

This form of trial function has been used[16] in variational calculations for the normal triton, where it has provided much better results than other trial functions of comparable simplicity which have been explored. The integrals required to form the expression (10) may all be expressed in terms of the basic integral

$$I(a,b,c)=8\pi^2\int dr_1dr_2dr_3\exp[-(ar_1+br_2+cr_3)]$$
$$=16\pi^2/[(a+b)(b+c)(c+a)]. \tag{13}$$

For example, the normalization factor N is expressed as

$$N^2J(2\alpha,2\alpha,2\beta)=1,$$

where $J(a,b,c)=-\partial^3I(a,b,c)/\partial a\partial b\partial c$. The kinetic energy terms may be written down explicitly:

$$\frac{\hbar^2(M+M_\Lambda)}{2MM_\Lambda}\frac{(\alpha+\beta)(16\alpha^3+9\alpha^2\beta+4\alpha\beta^2+\beta^3)}{8\alpha^2+5\alpha\beta+\beta^2}+\frac{\hbar^2(M_\Lambda-M)}{2MM_\Lambda}\frac{\beta(\alpha+\beta)(5\alpha^2+4\alpha\beta+\beta^2)}{8\alpha^2+5\alpha\beta+\beta^2}. \tag{14}$$

For exponential potentials, the expectation value may be written down in terms of the function J; for example, the two Λ-nucleon potential terms are each

$$U_EJ(2\alpha+2\lambda,2\alpha,2\beta)/J(2\alpha,2\alpha,2\beta). \tag{15}$$

Expectation values for Yukawa potentials may be expressed in terms of $K=\partial^2I/\partial a\partial b$; for example, the two Λ-nucleon potential terms are then each

$$U_YK(2\beta,2\alpha,2\alpha+2\lambda)/2\lambda J(2\alpha,2\alpha,2\beta). \tag{16}$$

The expression thus obtained for U from (10) has been minimized with respect to the parameters α and β. The final results for the $T=0$ configuration are given in Table I.

In Table I the upper bound for the Λ-nucleon potential as a function of B_Λ has been specified in three ways: (a) by the volume integral $\bar{U}_2$ for both Λ-nucleon interactions in the system, (b) by the zero-energy scattering length for the Λ-nucleon potential, and (c) by its well-depth parameter s. This well-depth parameter s is the ratio of the potential strength to the strength necessary to give a bound Λ-nucleon system at zero energy. Although $\bar{U}_2$ has considerable range dependence, the well-depth parameter and also the scattering length a vary relatively little with variation of range and shape for a given B_Λ. Apart from the exceptional case $S=\frac{1}{2}$ and $V_a>V_p$, the value obtained for a may therefore be regarded as a reliable upper bound (in magnitude) for the scattering length a appropriate to zero-energy Λ-nucleon scattering in the most strongly attractive spin configuration. These results imply that it is rather safe to conclude that the hyperdeuteron does not form a bound state, with the possible exception of the case $S=\frac{1}{2}$, $V_a>V_p$. In the latter case the amplitude a no longer has direct physical significance, the mean Λ-nucleon interaction being the combination $(3V_a+V_p)/4$. The well-depth parameter[19] s_a for the potential V_a can be deduced only with knowledge of the relative strengths of V_a and V_p. There is evidence[6] from the analysis of the binding energies for light hypernuclei that there is

[18] J. M. Blatt and J. D. Jackson, Phys. Rev. 76, 18 (1949).

[19] It is of interest to note here that, in the present case, s_a (for Yukawa shape) cannot exceed 1.15 for $2\lambda=(\hbar/m_Kc)^{-1}$ nor 1.27 for $2\lambda=(\hbar/2m_\pi c)^{-1}$. This follows from the stability of $_\Lambda H^3$ against disintegration to $_\Lambda H^2+n$.

some spin dependence in the Λ-nucleon interaction, but with $V_a > V_p$ it appears very unlikely[20] that V_p should be less than $V_a/3$. In this case s_a will not exceed 0.9 in any of the possible situations, and the existence of a bound hyperdeuteron is excluded.

It is of interest to compare briefly the results of this paper with calculations in which other simple wave functions for ${}_{\Lambda}H^3$ have been used. Comparison may be made with Derrick's result[11] $s=0.94$ for $B_{\Lambda}=0$ and a Yukawa potential of range parameter $(\hbar/m_K c)^{-1}$, a value which is about 24% higher than given in Table I for this case. Jones and Keller[21] used an Irving wave function $\exp[-\alpha(r_1^2+r_2^2+r_3^2)^{\frac{1}{2}}]$; for $B_{\Lambda}=0.5$ Mev and a Yukawa potential of range parameter $(\hbar/m_K c)^{-1}$, they find a well-depth parameter 16% above the corresponding value in Table I.

The simple wave function (12) has a number of obvious defects. For small B_{Λ} the volume integral $\bar{U}_2(B_{\Lambda})$ should have the expansion $\bar{U}_2(0)+C\sqrt{B_{\Lambda}}+O(B_{\Lambda})$, whereas the tabulated $\bar{U}_2$ is almost linear in B_{Λ} itself. This defect may be traced to the incorrectness of the asymptotic form of (12) for large separation of the Λ from the nucleons, especially for very small B_{Λ}; the value of α obtained represents a compromise between the small value of α appropriate to small B_{Λ} and the large value of α required to give adequate probability for the Λ and nucleon to be found within the range of their nuclear interaction. For this reason the upper bound obtained for $\bar{U}_2$ from (12) may be expected to be at its poorest for $B_{\Lambda}=0$. Moreover, the neutron-proton wave function in (12) is simply $e^{-\beta r_3}$ when the Λ particle is far distant, and a simple exponential function gives a very poor estimate of the relative neutron-proton energy in the deuteron. In fact, with a Yukawa potential and the nuclear parameters given above, this function gives at best 1.626 Mev for the deuteron binding energy. The energy of relative motion between the Λ particle and the n-p system, being the difference between $(2.226+B_{\Lambda})$ Mev and the internal n-p energy, is therefore required to be much too large with this choice of trial function; this also results in a serious defect for the asymptotic form, especially for the low B_{Λ} of physical interest. These are criticisms which do not apply to the use of this wave function for H^3, where the neutron binding energy is much larger (about 6.3 Mev) and comparable with the proton binding energy (about 8.5 Mev). It appears that a substantial improvement in these upper bounds for s, $\bar{U}_2$, etc., should result from the use of a more flexible trial function for ${}_{\Lambda}H^3$. For example, the wave function (17) given below[22] would have sufficient flexibility to avoid the defects just discussed and would also allow the integrations of expression (10) to be carried through in terms of the same functions J, K, and other derivatives of I:

$$\psi=N(e^{-ar_1}+xe^{-br_1})(e^{-ar_2}+xe^{-br_2})(e^{-\alpha r_3}+ye^{-\beta r_3}). \quad (17)$$

The possibility of excited $T=0$ states of ${}_{\Lambda}H^3$ should also be considered. This will be discussed briefly here for the case $S=\frac{1}{2}$ only. If $V_a > V_p$ and spin $\frac{1}{2}$ holds for the ground state of ${}_{\Lambda}H^3$, then the spin $\frac{3}{2}$ excited state will be bound only if the potential V_p has well-depth parameter exceeding $s(0)$, the potential $(3V_a+V_p)/4$ effective in the ground state corresponding to $s(B_{\Lambda})$. Even in the most favorable case possible ($B_{\Lambda}=1.0$ Mev), this requires V_p to have at least 86% of the strength of V_a. (This figure refers to $\hbar/2m_{\pi}c$ range; for $\hbar/m_K c$ range V_p is required to be at least 95% of V_a.) This appears very unlikely on the basis of the present evidence from other hypernuclear binding energies.[20] Similarly with $V_p > V_a$, the spin $\frac{1}{2}$ excited state could be bound only if the potential $(3V_a+V_p)/4$ had well-depth parameter exceeding $s(0)$, the potential V_p corresponding to $s(B_{\Lambda})$. This requires V_a to have strength at least 85% of V_p for $\hbar/2m_{\pi}c$ range (95% again for $\hbar/m_K c$ range), whereas the binding-energy data require that V_a be repulsive for this case. Even if such excited states did exist, they would be expected to decay to the ground state of ${}_{\Lambda}H^3$ with emission of an $M1$ photon (energy Δ Mev) in a time of order $\Delta^{-3}10^{-14}$ sec, a time generally short relative to the hypernuclear decay time.

4. DISCUSSION OF THE $T=1$ HYPERNUCLEAR CONFIGURATIONS

There have not been any hypernuclear decay events observed to date whose interpretation requires the existence of a bound $T=1$ hypernuclear configuration of mass 3. This situation is particularly clear for ${}_{\Lambda}He^3$ whose decay should involve a relatively high branching ratio for the mode

$${}_{\Lambda}He^3 \to \pi^- + p + p + p,$$

an event which would readily allow a unique identification for this hypernucleus. Stability of this ${}_{\Lambda}He^3$ complex against nuclear disintegration would require only that its total binding energy B be positive; with very small B, however, the formation of this hypernucleus might well be relatively rare. This same remark applies to the neutral counterpart ${}_{\Lambda}n^3$, whose detection would be more difficult since it would generally decay in flight with decay modes $\pi^-+p+n+n$ and, less frequently, π^-+H^3. A corresponding $T=1$ state would then exist for ${}_{\Lambda}H^3$ also; however, this state would generally have a rather short lifetime for radiative $M1$ decay to the ground state ${}_{\Lambda}H^3$ or for dissociation to $H^2+\Lambda$ if $B < B_d$. (This dissociation is forbidden by the selection rules based on charge symmetry, but it would still compete successfully with radiative decay as a result of nucleon mass differences and virtual electromagnetic effects in the Λ-nucleon interaction, for which charge symmetry does not hold.)

[20] R. H. Dalitz and B. W. Downs, Phys. Rev. (to be published).
[21] J. T. Jones and J. M. Keller, Nuovo cimento 4, 1329 (1956).
[22] It is of interest to note that the function $(e^{-\alpha r_3}+ye^{-\beta r_3})$ leads to a deuteron binding energy of 2.221 Mev for the nuclear parameters given above, with $\alpha=0.38$, $\beta=1.12$, and $y=2.27$.

TABLE II. Λ-nucleon interaction for zero binding in the $T=1$ systems. $\bar{U}_2'$ is given in units Mev$\times(10^{-13}\text{ cm})^3$; α and β are in units of 10^{13} cm^{-1}.

Potential shape	Intrinsic range	α	β	$\bar{U}_2'$	$\bar{U}_2'$ (Coulomb)
Yukawa	0.8411×10^{-13} cm	0.84	0.51	584	596
	1.4843×10^{-13} cm	0.49	0.41	973	1008
Exponential	0.8411×10^{-13} cm	0.71	0.39	594	608
	1.4843×10^{-13} cm	0.41	0.34	973	1014

An upper bound for the strength which must be exceeded by the mean Λ-nucleon potential effective in the $T=1$ configuration in order to produce a positive binding energy B may be obtained by use of the variational principle (10). In this case, the nucleon-nucleon potential appropriate to the singlet state must be inserted and the total energy of the system replaced by zero. Calculations have been made using the simple trial function (12), and the results have been listed in Table II for the various potential shapes and ranges considered. The volume integral $\bar{U}_2'$ for both Λ-nucleon interactions is given both for ${}_\Lambda He^3$, including the Coulomb repulsion between the protons, and for the case where there is no Coulomb repulsion. (The parameters α and β refer to the latter case.) Quite considerable reduction in these estimates may be expected with the use of a trial function more flexible than (12), because the criticisms made in the last section concerning its application to the hypertriton apply even more strongly here where the binding energy is zero for each particle of the system. In each case, the estimate obtained for $\bar{U}_2'$ greatly exceeds the volume integral $\bar{U}_2$ for the Λ-nucleon interaction effective in ${}_\Lambda H^3$. This means that, even if the Λ-nucleon interaction were spin-independent, the low binding energy for ${}_\Lambda H^3$ excludes the possibility of a bound $T=1$ system for ${}_\Lambda n^3$, ${}_\Lambda H^3$, ${}_\Lambda He^3$, in which the nucleon-nucleon attraction is considerably less than in $T=0$ ${}_\Lambda H^3$. In fact, with spin dependence in the Λ-nucleon potential, the potential strength available for $\bar{U}_2'$ is necessarily less than that for $\bar{U}_2$, since only the spin-average potential is effective in the $T=1$ system, whereas full advantage can be taken of the spin dependence of the potential only in the $T=0$ ground state.

5. ACKNOWLEDGMENTS

It is a pleasure for the authors to thank Professor H. A. Bethe and Professor E. E. Salpeter for some helpful discussions. We are particularly grateful to Joanne Downs for the computations reported in Tables I and II.

APPENDIX. USE OF A PRODUCT WAVE FUNCTION WITH δ-FUNCTION POTENTIALS

The use of a product wave function $\psi_{\text{nucleus}}g(s)$ for the representation of a large hypernucleus finds considerable justification as a result of the successes of shell-model wave functions for the ground states of nuclei. It is then plausible to go on and obtain an estimate of the volume integral $\bar{U}$ of the Λ-nucleon potential by representing this potential as a δ function, because the range of the interaction is small compared with the radius of a large core nucleus. Caution must be exercised in the use of δ-function interactions since there is then no (nonzero) absolute minimum for $\bar{U}$ corresponding to given B_Λ, the value $\bar{U}=0$ being attained for complete collapse of the system. There is, however, still the possibility that a local minimum may exist provided the distortions permitted for the core nucleus are sufficiently limited. The existence of such a local minimum will depend on the stiffness of the nuclear core in resisting deformation. This point is illustrated by Fig. 1, which shows a plot of $\bar{U}(\alpha)$ as function of a compression of the core nucleus by a radial

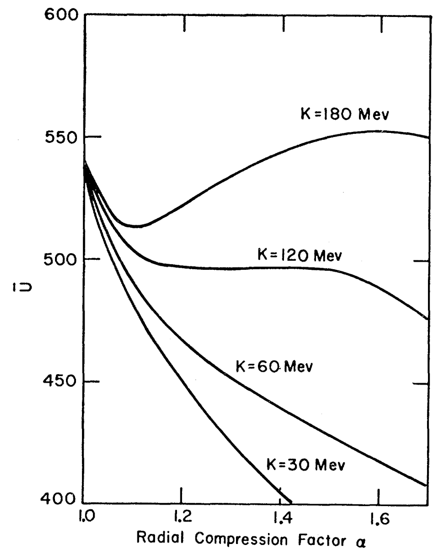

FIG. 1. The volume integral $\bar{U}(\alpha)$ [in units of Mev $(10^{-13}\text{ cm})^3$] of the total Λ-nucleon interaction for given binding ($B_\Lambda=1.44$ Mev) of a Λ particle to a nuclear core (radius $R_0=1.37\times10^{-13}$ cm) is plotted as a function of $\alpha=R_0/R$ for compression of the core to radius R for four values of K, the stiffness of the nuclear core.

factor α. For this case the core nucleus was chosen to be of Gaussian form with radius appropriate to H^3 for $\alpha=1$; the Schrödinger equation for $g(s)$ was solved numerically to compute $\bar{U}(\alpha)$ for each α, the distortion energy of the nuclear core being represented by $E(\alpha)=E(1)+\frac{1}{2}K(\alpha-1)^2$. It is apparent that there is a well-defined minimum near $\alpha=1$ for sufficiently large stiffness K, but that this disappears as K is reduced in strength.

Reasons why this product wave function should give a poor approximation to $\bar{U}$ for the hypertriton when ψ_{nucleus} is taken to be the normal deuteron wave function have already been discussed in Sec. III. In view of the remarks above, it is reasonable to expect that, on account of the small energy required to deform the deuteron, not even a local minimum in $\bar{U}$ will be found if radial distortion of the deuteron wave function is

TABLE III. Values of $\bar{U}_2$ [Mev$\times(10^{-13}$ cm$)^3$] as a function of the parameters γ, λ for a Hulthén $n-p$ function, using a product wave function and δ-function potentials.

$\lambda\diagdown\gamma$	0.20	0.25	0.30	0.35
2.0	742	540	441	390
2.5	652	487	415	372
3.0	600	459	400	360
3.5	547	442	389	350

allowed with the use of δ-function potentials. In order to show this, a calculation has been carried through using a Hulthén form $e^{-\gamma r}(1-e^{-2\lambda\gamma r})/r$ for the deuteron core, for various values of γ and λ. The n-p potential was chosen to be of Yukawa form with the parameters previously given. The Λ-function $g(s)$ was obtained by direct integration of the Schrödinger equation; for this it was most convenient to consider the equation

$$u''+Be^{-x}(1-e^{-\lambda x})^2u/x^2=\eta^2u,$$

the eigenvalue $B(\eta,\lambda)$ being computed[23] as a function of η and λ. The value of the volume integral $\bar{U}_2(\gamma,\lambda)$ appropriate to $B_\Lambda=0.3$ Mev is readily obtained from this as a function of γ and λ. Table III gives values of $\bar{U}_2(\gamma,\lambda)$ in the relevant region and shows that $\bar{U}_2(\gamma,\lambda)$ decreases monotonically with increasing γ or λ. No local minimum appears (this can be shown analytically for the case $\lambda=0$); this is in contradiction with the result of the hypertriton calculation reported by Brown and Peshkin.[24]

In conclusion, it should be emphasized here that, although the use of δ-function potentials (without distortion of the nuclear core) allowed a convenient discussion of the qualitative features of the Λ-nucleon interaction from the binding energies of light hypernuclei,[6] the use of potentials with physically appropriate ranges is essential not only for the case of the hypertriton but generally for complex hypernuclei if quantitatively reliable estimates of the interaction strength are to be obtained.

[23] These numerical computations were performed at the Computing Center of Brookhaven National Laboratory, and we are pleased to thank Dr. M. Rose and Mr. P. Mumford for their assistance.

[24] L. M. Brown and M. Peshkin, Phys. Rev. **107**, 272 (1957). (*Note added in proof.*—Dr. Peshkin has informed us that he has recently carried out more accurate calculations with the product wave function, which agree with the above remarks in giving no local minimum for the case of δ-function potentials.)

PHYSICAL REVIEW VOLUME 110, NUMBER 4 MAY 15, 1958

Past-Future Asymmetry of the Gravitational Field of a Point Particle

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(Received January 9, 1958)

The analytic extension of the Schwarzschild exterior solution is given in a closed form valid throughout empty space-time and possessing no irregularities except that at the origin. The gravitational field of a spherical point particle is then seen not to be invariant under time reversal for any admissible choice of time coordinate. The Schwarzschild surface $r=2m$ is not a singularity but acts as a perfect unidirectional membrane: causal influences can cross it but only in one direction. The apparent violation of the principle of sufficient reason seems similar to that which is associated with instabilities in other nonlinear phenomena.

I. INTRODUCTION

To define a gravitational universe we must give an analytic manifold and an analytic quadratic form $g_{\mu\nu}$ of correct signature.

For the manifold $\mathfrak{M}$ associated with a universe containing one point particle, one takes all of 4-space $\{x^\mu\}$ less the line $x^i=0$. (Greek indices$=0, 1, 2, 3$; Latin$=1, 2, 3$.) We might subject the gravitational field $g_{\mu\nu}(x)$ to the following requirements:

(a) The free space equation of Einstein:

$$R_{\mu\nu}(x)=0. \tag{1.1}$$

(b) Invariance under the one-parameter group

$$T_t:\begin{cases} x^0\rightarrow\bar{x}^0=x^0-t, \\ x^i\rightarrow\bar{x}^i=x^i. \end{cases} \tag{1.2}$$

(c) Invariance under the connected three-parameter group

$$R_r:\left\{\begin{matrix} x^0\rightarrow\bar{x}^0=x^0 \\ x^i\rightarrow\bar{x}^i=r_j{}^ix^j \end{matrix}\right\},\quad r^Tr=1,\quad \det r=1. \tag{1.3}$$

(d) Asymptotic to the Lorentz metric

$$g_{\mu\nu}(x)\rightarrow\eta_{\mu\nu},\quad x^ix^i\rightarrow\infty. \tag{1.4}$$

(e) Not extendable to the line $x^i=0$ (true singularity). This excludes the trivial case $g_{\mu\nu}(x)\equiv\eta_{\mu\nu}$.

(f) Invariant under the discrete group generated by

$$P:\begin{cases} x^0\rightarrow\bar{x}^0=x^0 \\ x^i\rightarrow\bar{x}^i=-x^i. \end{cases}$$