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Interaction of Spin Waves and Ultrasonic Waves in Ferromagnetic Crystals*

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A field-theoretical treatment is given of the magnetoelastic coupling of magnons and phonons in a ferromagnetic crystal. The effects of the coupling are large when the wavelengths and frequencies of the two fields are equal. If the two transverse phonon states of a given wave vector are degenerate, then the rotatory dispersion of the phonons will be large. The possibility exists of creating nonreciprocal acoustic elements, such as acoustic gyrators. At simultaneous resonance the phonon attenuation is expected to be large. The possibility of magnetostrictive transducers at microwave frequencies is discussed. A calculation is given of the damping by eddy currents of spin waves in a metal.

I. INTRODUCTION

THIS paper is concerned with the phenomena expected to occur when spin waves (magnons) are coupled to lattice vibrations (phonons) in a magnetic crystal. We are interested particularly in the resonance behavior exhibited when both the magnon and phonon frequencies and wavelengths are equal. The effects of the coupling of energy in the magnetic and acoustic modes are most pronounced near resonance. At high phonon frequencies an observation of the resonance condition may provide a good measure of the value of the exchange-energy constant. It is also shown that a magnetic crystal has nonreciprocal acoustic properties. The acoustogyric effect, as it may be called, causes circularly-polarized elastic shear waves of different sense to have different velocities when propagated along the magnetization axis. The possibility thus exists of creating acoustic gyrators, isolators, and other nonreciprocal acoustic elements.

The dispersion relation for phonons is

$$\omega = vk, \quad (k = 2\pi/\lambda) \tag{1}$$

for wave vectors much smaller than a vector in the reciprocal lattice. When $\mathbf{k}$ is directed along a suitable symmetry axis of a nonmagnetic crystal, two of the three roots of the secular equation determining the velocity v coincide. These two roots are associated with shear waves. The third root is associated with a longitudinal wave. The phonon velocities may be of the order of 3×10^5 cm/sec.

The dispersion relation for magnons in a ferromagnet in the absence of magnetic fields and magnetocrystalline anisotropy is, in the long-wavelength limit,

$$\omega = (2\gamma A/M_s)k^2, \tag{2}$$

where γ is the magnetogyric ratio, M_s is the saturation magnetization, and A is the exchange-energy constant associated with the exchange-energy density f_{ex} in the Landau relation[1,2]

$$f_{\mathrm{ex}} = A\{(\nabla\alpha_x)^2 + (\nabla\alpha_y)^2 + (\nabla\alpha_z)^2\}. \tag{3}$$

Here $\boldsymbol{\alpha}$ is the unit vector in the direction of the magnetization. On an atomic model, the exchange-energy constant A may be expressed in terms of the exchange integral J by, for a body-centered cubic lattice,

$$A = 2JS^2/a, \tag{4}$$

if there are only nearest-neighbor interactions. Here S is the spin in units of $\hbar$, and a is the lattice constant. The value of A may be of the order of 10^{-6} erg/cm for a normal ferromagnet, M may be of the order of 5×10^2, and γ is approximately 2×10^7 (oersted-sec)$^{-1}$. Thus, roughly, $\omega \approx 10^{-1}k^2$.

If the wavelengths of magnons and phonons are equal, the frequencies will be in the ratio

$$\frac{\omega_m}{\omega_p} = \frac{2\gamma Ak}{vM_s} \approx \frac{k}{3\times10^6}, \tag{5}$$

so that the angular frequencies are equal when $k \approx 3\times10^6$ cm^{-1}, corresponding to $\omega \approx 10^{12}$ sec^{-1}. It is, fortunately, possible to increase ω_m by applying a dc magnetic field. A magnon with $\mathbf{k}\|\mathbf{H}$ will have its frequency increased by γH, and the dispersion relation becomes

$$\omega = \gamma H + (2\gamma A/M_s)k^2, \tag{6}$$

where we have assumed that demagnetizing effects on the magnon may be neglected. An anisotropy field will, if suitably oriented, act in the same manner as an external magnetic field. For $H = 100$ oersteds, we have $\omega \approx 2\times10^9$ radians/sec, which is about 300 Mc/sec. The corresponding acoustic wave vector is about 10^4 cm^{-1}. The contribution of the exchange term on the right-hand side of Eq. (6) for this value of the wave vector is about 10^7 radians/sec, only about one percent of the Zeeman contribution.

It may be necessary to carry out experiments with ultrasonic waves at microwave frequencies to obtain accurate measurements of the exchange constant A. Ultrasonic experiments in this range may become possible in the near future if paramagnetic relaxation effects are utilized successfully as detectors of microwave phonons. Eddy-current damping of spin waves is

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[1] L. Landau and E. Lifshitz, Physik. Z. Sowjet. 8, 153 (1935).
[2] C. Kittel, Revs. Modern Phys. 21, 541 (1949).

considered in Appendix A; such damping may be important in metals.

II. CLASSICAL CALCULATION

We first carry out a derivation of the coupled equations of motion of magnons and phonons treated as classical fields.[3] The Lagrangian density and Hamiltonian density will be denoted by $\mathfrak{L}$ and $\mathfrak{H}$, respectively,

$$\mathfrak{L}=\mathfrak{L}_Z+\mathfrak{L}_e+\mathfrak{L}_i+\mathfrak{L}_p; \quad (7)$$

$$\mathfrak{H}=\mathfrak{H}_Z+\mathfrak{H}_e+\mathfrak{H}_i+\mathfrak{H}_p; \quad (8)$$

where the subscripts Z, e, i, and p denote, respectively, the Zeeman, exchange, magnetoelastic interaction, and phonon terms.

Zeeman Term

The Lagrangian density may be written for M_x, $M_y \ll M_S$ in the form

$$\mathfrak{L}_Z=\frac{M_y\dot{M}_x}{\omega_s}-\gamma(M_x^2+M_y^2)\frac{H_0}{2\omega_s}, \quad (9)$$

with H_0 as the static field along the z direction, and

$$\omega_s=\gamma M_s. \quad (10)$$

We have expanded

$$M_z=[M_s^2-M_x^2-M_y^2]^{\frac{1}{2}} =M_s\left[1-\frac{M_x^2+M_y^2}{2M_S^2}+\cdots\right], \quad (11)$$

and preserved terms[4] not higher than the second degree in M_x, M_y. We show that Eq. (9), when substituted in the Lagrangian equations of motion for a field,

$$\frac{\partial\mathfrak{L}}{\partial\psi_j}-\frac{d\partial\mathfrak{L}}{dt\partial\dot{\psi}_j}-\sum_\alpha\frac{\partial}{\partial x_\alpha}\frac{\partial\mathfrak{L}}{\partial(\partial\psi_j/\partial x_\alpha)}=0, \quad (12)$$

gives the usual equations of motion of a spin system. Letting $\psi_j=M_x$, we have

$$dM_y/dt=-\gamma H_0M_x. \quad (13)$$

Similarly, letting $\psi_j=M_y$,

$$dM_x/dt=\gamma H_0M_y. \quad (14)$$

Equations (13) and (14) are the usual spin resonance equations.

The Hamiltonian is obtained by using the momentum density,

$$\pi=\frac{\partial\mathfrak{L}}{\partial\dot{M}_x}=\frac{\partial\mathfrak{L}_Z}{\partial\dot{M}_x}=\frac{M_y}{\omega_s}. \quad (15)$$

We find

$$\mathfrak{H}_Z=\pi\dot{M}_x-\mathfrak{L}_Z=(\omega_0/2\omega_s)(\omega_s^2\pi^2+\psi^2), \quad (16)$$

where $\omega_0=\gamma H_0$, and we have defined the field displacement ψ by

$$\psi\equiv M_x. \quad (17)$$

Exchange Term

The exchange Hamiltonian density (3) may be written as

$$\mathfrak{H}_e=(A/M_s^2)[\omega_s^2(\nabla\pi)^2+(\nabla\psi)^2], \quad (18)$$

if we neglect terms in M_x and M_y above the second degree. If π and ψ have a wave-like dependence of the for $e^{i(\omega t-kz)}$, then

$$\mathfrak{H}_e=-(Ak^2/M_s^2)(\omega_s^2\pi^2+\psi^2), \quad (19)$$

where now π, ψ are written as the amplitudes of the field momentum and displacement.

Magnon-Phonon Interaction

The interaction between magnetization direction and elastic strain (as observed as magnetostriction) is described by the magnetoelastic coupling.[5] To the first order in the strain components S_{ij} and to the second order in the direction cosines, the coupling in a *cubic* crystal is described by the magnetoelastic energy density

$$f_{\mathrm{me}}=b_1(\alpha_x^2S_{xx}+\alpha_y^2S_{yy}+\alpha_z^2S_{zz})+2b_2(\alpha_x\alpha_yS_{xy} +\alpha_y\alpha_zS_{yz}+\alpha_z\alpha_xS_{zx}). \quad (20)$$

If the static magnetization is along the z direction, we shall want to retain in a linear theory only those terms $\alpha_y\alpha_zS_{yz}$ and $\alpha_z\alpha_xS_{zx}$ linear in M_x and M_y. The Hamiltonian density $\mathfrak{H}_i$ associated with these interaction terms is

$$\mathfrak{H}_i=(2b_2/M_s)(\omega_s\pi S_{yz}+\psi S_{zx}), \quad (21)$$

where the shear components are defined by

$$S_{yz}\equiv\tfrac{1}{2}\left(\frac{\partial R_y}{\partial z}+\frac{\partial R_z}{\partial y}\right);\quad S_{zx}\equiv\tfrac{1}{2}\left(\frac{\partial R_z}{\partial x}+\frac{\partial R_x}{\partial z}\right). \quad (22)$$

Here $\mathbf{R}$ is the displacement vector of a point in the solid from its original position in the unstrained solid.

[3] The general techniques employed for the magnons are extensions of the methods used in a somewhat different connection by J. M. Luttinger and C. Kittel (unpublished) and C. Kittel and E. Abrahams, Revs. Modern Phys. 25, 233 (1953). E. Kondorsky (private communication) has remarked that some consequences of the magnon-phonon interaction have been discussed in unpublished work by S. A. Altschuler and by A. I. Akhiezer and coworkers in the U.S.S.R. S. Friedberg has observed effects of a magnetic field on thermal conductivity in ferrites and interpreted the results in terms of phonon-magnon scattering (private communication).

[4] S. Rodriguez has pointed out that the Lagrangian

$$\mathfrak{L}_Z=(1/2\omega_s)\left[\frac{d\mathbf{M}}{dt}\times\mathbf{M}\cdot\mathbf{H}-\gamma(\mathbf{M}\times\mathbf{H})^2\right]$$

gives the full torque relation $d\mathbf{M}/dt=\gamma\mathbf{M}\times\mathbf{H}$. The form (9) above is somewhat more convenient for our present purpose, as fewer variables are introduced.

[5] R. Becker and W. Döring, *Ferromagnetismus* (Verlag Julius Springer, Berlin, 1939), p. 136; see also reference 2.

Phonon Term

For convenience we make the assumption of elastic isotropy.[6] Then the Hamiltonian density $\mathcal{H}_p$ associated with the phonons is, as is well known,

$$\mathcal{H}_p=\frac{1}{2\rho}(\Pi_x^2+\Pi_y^2+\Pi_z^2)+\alpha\sum_{i,j}S_{ij}^2+\beta(\sum_i S_{ii})^2, \quad (23)$$

where

$$\mathbf{\Pi}=\rho d\mathbf{R}/dt \quad (24)$$

is the momentum density conjugate to the coordinate $\mathbf{R}$. Here α and β are elastic constants; ρ is the density.

The total Hamiltonian density is found, on combining Eqs. (16), (18), (21), and (23),

$$\begin{aligned}\mathcal{H}=&(\omega_0/2\omega_s)(\omega_s^2\pi^2+\psi^2)+(A/M_s^2)[\omega_s^2(\nabla\pi)^2+(\nabla\psi)^2]\\&+(2b_2/M_s)(\omega_s\pi S_{yz}+\psi S_{zx})+(1/2\rho)(\Pi_x^2+\\&+\Pi_y^2+\Pi_z^2)+\alpha\sum_{i,j}S_{ij}^2+\beta(\sum_i S_{ii})^2. \quad (25)\end{aligned}$$

We proceed to determine the equations of motion, which in Hamiltonian form are

$$\dot{\pi}=-\frac{\partial\mathcal{H}}{\partial\psi}+\sum_\alpha\frac{\partial}{\partial x_\alpha}\frac{\partial\mathcal{H}}{\partial(\partial\psi/\partial x_\alpha)}; \quad (26)$$

$$\dot{\psi}=\frac{\partial\mathcal{H}}{\partial\pi}-\sum_\alpha\frac{\partial}{\partial x_\alpha}\frac{\partial\mathcal{H}}{\partial(\partial\pi/\partial x_\alpha)}; \quad (27)$$

similar equations obtain for the components of $d\mathbf{\Pi}/dt$ and $d\mathbf{R}/dt$. It will shorten the equations considerably if we assume that all field quantities are independent of x and y. With this restriction we have, from Eq. (25),

$$\dot{\psi}=\omega_0\omega_s\pi+(2\omega_s b_2/M_s)S_{yz}-2(A\omega_s^2/M_s^2)\times(\partial^2\pi/\partial z^2); \quad (28)$$

$$\dot{\pi}=-(\omega_0/\omega_s)\psi-(2b_2/M_s)S_{zx}+2(A/M_s^2)\times(\partial^2\psi/\partial z^2); \quad (29)$$

$$d\Pi_x/dt=(b_2/M_s)(\partial\psi/\partial z)+\alpha(\partial^2R_x/\partial z^2); \quad (30)$$

$$d\Pi_y/dt=(b_2\omega_s/M_s)(\partial\pi/\partial z)+\alpha(\partial^2R_y/\partial z^2). \quad (31)$$

We omit the equation for $d\Pi_z/dt$, as longitudinal phonons along the z axis do not couple in first order with magnons when the static magnetization is along the z axis. Using the relations defining the coordinates and momenta, and assuming that all variables have the time and space dependence $e^{i(\omega t-kz)}$, we may rewrite Eqs. (28) to (31) in the form

$$i\omega M_x=[\omega_0+(2A\gamma k^2/M_s)]M_y-i\gamma b_2kR_y; \quad (32)$$

$$i\omega M_y=-[\omega_0+(2A\gamma k^2/M_s)]M_x+i\gamma b_2kR_x; \quad (33)$$

$$\omega^2\rho R_x=k^2\alpha R_x+i(b_2k/M_s)M_x; \quad (34)$$

$$\omega^2\rho R_y=k^2\alpha R_y+i(b_2k/M_s)M_y. \quad (35)$$

[6] In carrying over our results to actual crystals, we are restricted by this assumption to waves propagating along symmetry axes having two degenerate transverse waves.

It is convenient to form the combinations

$$M^\pm=M_x\pm iM_y;\quad R^\pm=R_x\pm iR_y. \quad (36)$$

Then, writing

$$\omega'\equiv\omega_0+(2A\gamma k^2/M_s), \quad (37)$$

we have

$$i(\omega\pm\omega')M^\pm=\mp\gamma b_2kR^\pm; \quad (38)$$

$$(\omega^2\rho-k^2\alpha)R^\pm=i(b_2k/M_s)M^\pm. \quad (39)$$

We may attempt to take account of spin relaxation by other mechanisms with relaxation time τ by writing (38) as

$$(\omega-i\tau^{-1}\pm\omega')M^\pm=\pm i\gamma b_2kR^\pm, \quad (40)$$

which may be compared with the usual equation for spin resonance in an rf magnetic field $H^\pm=H_x\pm iH_y$,

$$(\omega-i\tau^{-1}\pm\omega_0)M^\pm=\pm\omega_sH^\pm. \quad (41)$$

Recalling that $-ikR_x=2S_{zx}$, we see that $2b_2S_{zx}/M_s$ is in some respects equivalent to a transverse magnetic field in its effects on the spin system. The difference lies in the use of ω' in Eq. (40) and ω_0 in Eq. (41). The equivalence of $2b_2S_{zx}/M_s$ and also $2b_2S_{yz}/M_s$ to a magnetic field is equally apparent from the original expression for the magnetoelastic coupling, Eq. (20). With b_2 of the order of 10^8 ergs/cm^3, the equivalent field intensity is of the order of 10^5 S oersteds, where S is the shear strain.

Production of Microwave Phonons by Ferromagnetic Resonance

At resonance ($\omega=\omega'$), we have from Eq. (40) that $|M^\pm|=\gamma b_2\tau k|R^\pm|$, so that for $\tau\approx10^{-8}$ sec we have $S\approx10^{-7}|M^\pm|$. For an rf magnetic field of 1 oersted it is reasonable to expect $|M^\pm|\approx10$ gauss at resonance, so that $S\approx10^{-6}$, corresponding to a phonon flux of ≈0.01 watt/cm^2. We have neglected demagnetizing effects on the resonance in making this estimate; we have also neglected the loss of energy from the phonon system which might arise in the event the ferromagnetic element is used as a transducer coupled mechanically to another elastic system. It appears not unlikely, however, that a thin ferrite crystal section could be driven as a magnetostrictive oscillator. The static shear strain accompanying a magnetization M_x is $S\approx(b_2/\alpha)(M_x/M_s)\approx10^{-7}M_x$, while the resonant effect just calculated is $S\approx(M_s^2/b_2)(\omega_s\tau)^{-1}(M_x/M_s)$. The two quantities are of the same order of magnitude for our values of the physical parameters, but the static effect will not be excited if $\omega\tau\gg1$. At microwave frequencies the resonance effect will probably have to be employed. It may be that piezoelectric transducers will be a simpler source of microwave phonons. We note that in quartz the strain is given by $S\approx10^{-7}E$, where E is the electric field intensity in esu.

We now discuss the acoustical properties of the medium. Combining Eqs. (39) and (40), we have

$$\{(\omega-i\tau^{-1}\pm\omega')(\omega^2\rho-k^2\alpha)\pm\omega_s k^2(b_2/M_s)^2\}R^{\pm}=0. \quad (42)$$

It would be simple to generalize Eqs. (39) and (42) to include an intrinsic relaxation time for the phonons, but we shall suppose that this time is larger than the spin relaxation time.

Damping of Phonons

We first estimate the damping of the acoustic wave for the lower choice of signs in Eq. (42) when the frequency ω is equal to ω', the resonance frequency of the spin system. Then, from Eq. (42),[7]

$$k^2=\frac{\omega^2\rho/\alpha}{1+i(\omega_s\tau/\alpha)(b_2/M_s)^2}. \quad (43)$$

Writing

$$k=k_1-ik_2,$$

we have

$$k^2\approx k_1^2-2ik_1k_2, \quad (44)$$

on the assumption that $k_2\ll k_1$; that is, we assume that the attenuation per wavelength is small. If the imaginary term in the denominator is small in comparison with unity,

$$k_1^2-2ik_1k_2\cong\omega^2(\rho/\alpha)\{1-i(\omega_s\tau/\alpha)(b_2/M_s)^2\}, \quad (45)$$

and we find for the attenuation per wavelength

$$k_2/k_1\cong\omega_s\tau b_2^2/(2\alpha M_s^2)=\gamma\tau b_2^2/(2\alpha M_s). \quad (46)$$

Now with $\gamma\approx 2\times10^7$ (oersted/sec)$^{-1}$, $\tau\approx10^{-8}$ sec as for a typical ferrite, $b_2\approx10^8$ ergs/cm^3 as for nickel, $\alpha\approx10^{12}$ dynes/cm^2, and $M_s\approx500$, we have $k_2/k_1\approx2$. This means that the absorption of sound is very large indeed at the resonance frequency of the spin system. Our assumption $k_2\ll k_1$ is violated, but we may safely conclude that at resonance the phonons are highly damped and hardly propagate at all. We may remark that Eq. (46) follows also from a quantum-mechanical calculation of the transition probability on the assumption that the individual spin wave modes are resolved, so that the density of states at resonance is $\tau/\hbar$. Raman transitions of phonons will give additional attenuation.

Far from resonance the damping is reduced considerably. If we consider the "wrong" sense of circular polarization of the transverse phonons, we have for $\omega=\omega'$, on the same assumptions as above, the attenuation

$$k_2/k_1\approx\gamma b_2^2/(8\omega^2\tau\alpha M_s), \quad (47)$$

which is lower than above by the factor $1/(4\omega\tau)^2$. For $\omega\approx10^9$ sec^{-1} and $\tau\approx10^{-8}$ sec, the attenuation in the "wrong" sense is 1/200 of the attenuation for phonons in the "right" sense of circular polarization.

[7] It should be pointed out that we are assuming for convenience (1) that the exchange contribution to ω' is entirely real, and (2) that the magnons will take up the same value of k as the phonons.

Rotatory Dispersion of Phonons

We now consider the rotatory dispersion. We suppose that the resonance frequency ω' of the spin system is considerably less than the phonon frequency, so that

$$\omega'=\pm\eta\omega,\quad \eta\ll1. \quad (48)$$

We suppose that we may neglect $1/\tau$ in comparison with ω; this means that the phonon attenuation at ω arising from coupling with the spin system is taken as negligible. Then, making the usual expansions, we have

$$k^2=\frac{\omega^2\rho}{\alpha}\left(1\pm\frac{\gamma b_2^2}{\alpha\omega M_s}(1\mp\eta)\right). \quad (49)$$

Now for the two senses of circular polarization of the phonons we write

$$k^{\pm}=k_0(1\pm\epsilon), \quad (50)$$

where ϵ is a constant supposed small in comparison with unity. Then

$$(k^{\pm})^2\cong k_0^2(1\pm2\epsilon), \quad (51)$$

and, by comparison with Eq. (49),

$$\epsilon\cong\gamma b_2^2/(2\alpha\omega M_s), \quad (52)$$

independent of η. The rotation of the plane of polarization per wavelength is measured by

$$(k^+-k^-)/k_0\cong\gamma b_2^2/(\alpha\omega M_s)\approx4\times10^9/\omega. \quad (53)$$

For $\omega\approx10^9$ sec^{-1}, the fractional rotation is about $\frac{1}{2}$, which is a very substantial rotation. Our assumption $\omega\tau\gg1$ must be remembered. It does not seem as if the experimental effects predicted by Eqs. (46) and (53) should be difficult to observe, but one must bear in mind that other phonon attenuation mechanisms may be present. The predicted rotation of the plane of polarization of degenerate transverse elastic waves is the basis for application as a nonreciprocal mechanical circuit element, and a whole class of new devices becomes possible in principle.

It is perhaps not always relevant to ask how much the spin system is damped by the magnetoelastic coupling with a phonon system having an independent relaxation mechanism. When the spin system of a ferromagnetic insulator is excited by electromagnetic radiation the magnons usually have a very low k, entirely too low to match the k required of phonons at the same frequency. If by some device the spin system could be driven in such a way that the k's of magnons and phonons were matched, then at the spin resonance frequency the coupling contribution τ' to the spin relaxation time could be estimated in appropriate limits by the result (46), the phonon attenuation k_2 now being taken as arising from an independent mechanism.

III. QUANTUM THEORY

The classical Hamiltonian density is, according to Eq. (25),

$$\mathcal{H}=(\omega_0/2\omega_s)(\omega_s^2\pi^2+\psi^2)+(A/M_s^2)[\omega_s^2(\nabla\pi)^2+(\nabla\psi)^2] +(2b_2/M_s)(\omega_s\pi S_{yz}+\psi S_{zx})+(1/2\rho)(\Pi_x^2 +\Pi_y^2+\Pi_z^2)+\alpha\sum_{i,j}S_{ij}^2+\beta(\sum_i S_{ii})^2. \quad (54)$$

Here ψ is the magnon field displacement, and π is the conjugate magnon momentum density; R_i involved in S_{ij} is a phonon field displacement component; and Π_i is the conjugate phonon momentum density.

The Hamiltonian, as we have written it, refers to field quantities, but a parallel atomic treatment may be given starting from the usual lattice vibration theory for the phonons and from the spin wave theory with the Heisenberg exchange interaction for the magnons. The magnetoelastic coupling term arises on an atomic model directly from the strain dependence of the anisotropic exchange interaction and the quadrupole-quadrupole interaction.[8] The atomic and field-theoretical treatments are equivalent for wavelengths long in comparison with the lattice spacing, and this is the situation of interest to us. We will content ourselves with the quantization and indication of the solution of the problem associated with the Hamiltonian (54). The commutation relations are

$$[R_s(\mathbf{r},t),\Pi_{s'}(\mathbf{r}',t)]=i\hbar\delta(\mathbf{r}-\mathbf{r}')\delta_{ss'}, \quad (55)$$

for the phonon field and

$$[\psi(\mathbf{r},t),\pi(\mathbf{r}',t)]=i\hbar\delta(\mathbf{r}-\mathbf{r}') \quad (56)$$

for the magnon field. We are now of course interpreting the R_s, Π_s, ψ, and π as operators. The quantum equations of motion are

$$i\hbar\dot{R}_s=[R_s,H]; \quad i\hbar d\Pi_s/dt=[\Pi_s,H], \quad (57)$$

and similarly for ψ and π. Here

$$H=\int \mathcal{H}dV \quad (58)$$

is the total Hamiltonian. It is straightforward to verify that the quantum equations of motion are identical in form with the classical equations of motion (28)–(31). The Hamiltonian has been written in an approximation giving equations of motion linear in the field variables. It will, therefore, follow that expectation values satisfy the classical equations of motion, so that nothing essentially new results from quantizing the fields.

ACKNOWLEDGMENT

I am indebted to Professor Kei Yosida and Mr. Sergio Rodriguez for helpful discussions in connection with this paper.

[8] J. H. Van Vleck, Phys. Rev. 74, 1168 (1948). Other mechanisms may also be effective.

APPENDIX A. EDDY-CURRENT DAMPING OF SPIN WAVES

We may ask what effect eddy-current damping has on the motion of spin waves. The problem will be solved for pure exchange spin waves under conditions of the normal skin effect. The equation of motion of the spin waves is, from Eqs. (32) and (33),

$$\frac{d\mathbf{M}}{dt}=\frac{2\gamma A}{M_s^2}\mathbf{M}\times\nabla^2\mathbf{M}+\gamma\mathbf{M}\times\mathbf{H}. \quad (A.1)$$

From the Maxwell equations,

$$c\,\mathrm{curl}\mathbf{H}=4\pi\sigma\mathbf{E}, \quad (A.2)$$

$$c\,\mathrm{curl}\mathbf{E}=-\frac{d\mathbf{H}}{dt}-4\pi\frac{d\mathbf{M}}{dt}, \quad (A.3)$$

we have

$$4\pi\frac{d\mathbf{M}}{dt}=-\frac{d\mathbf{H}}{dt}+\frac{c^2}{4\pi\sigma}\nabla^2\mathbf{H}. \quad (A.4)$$

Assuming that all variables have the time and space dependence $e^{i(\omega t-kz)}$, where ω may be complex, we have from Eq. (A.4)

$$H_x=M_x\Big/\left\{-\frac{1}{4\pi}+\frac{ic^2k^2}{(4\pi)^2\sigma\omega}\right\}, \quad (A.5)$$

which we shall write as $H_x=CM_x$, thereby defining C. Then Eq. (A.1) becomes

$$i\omega M_x=\left[\frac{2\gamma A}{M_s}k^2-\gamma CM_s\right]M_y; \quad (A.6)$$

$$i\omega M_y=-\left[\frac{2\gamma A}{M_s}k^2-\gamma CM_s\right]M_x. \quad (A.7)$$

The secular equation is

$$\frac{\omega}{\gamma}=\frac{2Ak^2}{M_s}-CM_s=\frac{2Ak^2}{M_s}+\frac{4\pi M_s}{1-(ic^2k^2/4\pi\sigma\omega)}. \quad (A.8)$$

We note that

$$c^2k^2/(4\pi\sigma\omega)=\tfrac{1}{2}\delta^2k^2, \quad (A.9)$$

where δ is the classical skin depth for permeability unity. For purposes of an estimate, we may take from the usual dispersion relation

$$k^2/\omega\approx M_s/(2A\gamma), \quad (A.10)$$

so that

$$c^2k^2/(4\pi\sigma\omega)\approx c^2M_s/(8\pi A\sigma\gamma)\approx 10^4, \quad (A.11)$$

where we have taken $M_s\approx 10^3$, $A\approx 10^{-6}$ erg/cm, and $\sigma\approx 10^{17}$ esu as for iron at room temperature.

We may use the ratio of real to imaginary parts of ω as a measure of the Q of a spin wave; this ratio is approximately

$$Q\approx\frac{\omega/\gamma}{(4\pi)^2M_s\sigma\omega/c^2k^2}\approx\frac{c^2k^2}{16\pi^2M_s\gamma\sigma};\qquad\text{(A.12)}$$

or

$$Q\approx c^2\omega/(32\pi^2A\gamma^2\sigma),\qquad\text{(A.13)}$$

using (A.10). The Q is of the order

$$Q\approx10^{-7}\omega,\qquad\text{(A.14)}$$

with the foregoing constants. Thus the damping of a magnon is small above 100 Mc/sec; the damping is trivial for thermal magnons. This result agrees with the conclusion of Ament and Rado.†

The situation is rather less favorable for magnons whose energy arises largely from interaction with an external magnetic field (or with an anisotropy field) and which are such that ω and k are in resonance with a phonon. With $\omega=vk$, where v is the phonon velocity, we infer from the appropriate modification of (A.8) and (A.12) that

$$Q\approx\frac{\omega^2}{(4\pi\gamma M_s)(4\pi\sigma)}\left(\frac{c}{v}\right)^2,\qquad\text{(A.15)}$$

or, with our usual numerical values,

$$Q\approx10^{-19}\omega^2.\qquad\text{(A.16)}$$

Thus, the Zeeman magnons are strongly damped in a metal for $\omega<10^{10}$ radians/sec. The damping generally is proportional to $1/k^2$; the Zeeman magnons suffer from having a lower k for the same ω as compared with a pure exchange magnon.

We may understand the form of the result (A.12) by an elementary consideration. The energy per unit volume in an exchange spin wave of amplitude M_x, M_y and wave vector k is, from Eq. (3),

$$f\approx Ak^2M_x^2/M_s^2.\qquad\text{(A.17)}$$

The eddy-current energy loss per unit volume per cycle when the skin depth δ is large in comparison with a wavelength is of the order of

$$M_x^2/(\delta k)^2,$$

if we use the known result for eddy-current losses in slabs. Thus, by the definition of Q,

$$Q\approx Ak^4/(M_s^2\delta^2)\approx Ak^4c^2/(M_s^2\omega\sigma)\approx c^2k^2/(\gamma M_s\sigma),\qquad\text{(A.18)}$$

when we use (A.10). We have dropped numerical factors in making this estimate, but the result agrees in form with (A.12).

Our calculation is based on the assumption that the conduction electron mean free path is much less than δ, the skin depth. The calculation of magnon—conduction-electron relaxation by Abrahams[9] does not apply under these circumstances.

† W. S. Ament and G. T. Rado, Phys. Rev. **97**, 1558 (1955).

[9] E. Abrahams, Phys. Rev. **98**, 387 (1955).