TABLE IV. Correction terms for the extrapolated ratio.

	$\Lambda_S=28$ g/cm² $\Lambda_L=33.5$ g/cm² δ_c	δ_a	$\Lambda_S=29$ g/cm² $\Lambda_L=32.5$ g/cm² δ_c	δ_a	$\Lambda_S=30$ g/cm² $\Lambda_L=31.5$ g/cm² δ_c	δ_a
$x=\bar{x}=11.2$ g/cm²	-0.40	2.05	-0.25	2.15	-0.10	2.25
$x=\langle x\rangle=14.6$ g/cm²	-0.60	1.80	-0.40	2.0	-0.15	2.05

$\delta=\delta_c+\delta_a$, where δ_c arises from the fact that the growth curve deviates from a straight line and δ_a arises from the finite rate of ascent. δ_c can be calculated by letting β in Eqs. (9a) and (9b) go to infinity. One finds $\delta_c=\Lambda[1-(x/\Lambda)-e^{-x/\Lambda}]$, where $1/\Lambda=(1/\Lambda_S)-(1/\Lambda_L)$. δ_a is then given by $\delta_a=\delta-\delta_c$. In Table IV, we give the correction term for various values of assumed absorption mean free paths and both values, $\bar{x}$ and $\langle x\rangle$. We see that, as expected, δ is not very sensitive to the choice of the exact value x at which the straight line is fitted to the growth curve. However, the median value $\bar{x}$ seems to be the more reasonable choice and we therefore shall use

$$\delta=\delta_c+\delta_a=1.90\pm0.25 \text{ g/cm}^2. \qquad (11)$$

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Multiple Production of Pions in Nuclear Collisions*

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The statistical theory of meson production in nuclear collisions is given a fully covariant formulation. A single parameter of the dimensions of a mass appears in the theory, which is normalized by matching a single experimental number. Numerical results for various processes are presented. A simple recurrence relation between the covariant phase-space integrals greatly facilitates the computations.

WE have attempted to calculate the relative transition probabilities of multiple meson processes by using a covariant statistical theory. The starting point is a reduction of the transition probability into a form in which the kinematic factors are separated out and the matrix element of an ordered product of field operators occurs as an unknown function. The statistical assumption[1] now permits an evaluation of the relative transition probabilities. The relative probabilities thus obtained are in much better agreement with the experimental findings than the predictions from various versions of the statistical theory already discussed by several authors.[2] Comprehensive calculations have been made of multiple meson production in nucleon-nucleon collisions, nucleon-pion collisions, and nucleon-antinucleon annihilation. A short summary of some of the results is presented below.

The relative transition probability for a final state involving two nucleons of four-momenta p_1 and p_2 and m pions of four-momenta q_1, $q_2\cdots q_m$ starting from an initial state of two nucleons of four-momenta p_1' and p_2' can be written in the form[3]

$$W_m=\int d^4p_1d^4p_2d^4q_1\cdots d^4q_m\,\delta^4(p_1+p_2+q_1+\cdots+q_m-p_1'-p_2')\delta(p_1^2-m^2)\delta(p_2^2-m^2)\times\delta(q_1^2-\mu^2)\cdots\delta(q_m^2-\mu^2)f(p,p',q).$$

Here $f(p,p',q)$ is an invariant function of the nucleon and pion four-momenta, and is obtained by averaging over initial spins and summing over the final spins the absolute square of the matrix element. Since all the external lines considered correspond to real noninteracting particles, we have the relations

$$p'^2=p^2=m^2,\quad q^2=\mu^2.$$

The statistical theory is obtained by replacing $f(p,p',q)$ by a constant quantity independent of the four-momenta but depending on the number of meson (and nucleon) lines in the diagram. Dimensional considerations lead to the form

$$f(p,p',q)=AS(m)\kappa^{-2m},$$

where κ is a quantity of the dimensions of a mass and A is a numerical constant. Since we are interested in the relative transition probability only, the constant A

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TABLE I. Comparison of meson multiplicities in proton-proton collisions.

	Ratio[a] single:double:triple (proton energy 1.5 Bev)	Ratio[a] single:double:triple (proton energy 2.75 Bev)
Present theory $\kappa^{-1}=1.5\times10^{-13}$ cm	77 : 28 : 3	39 : 29 : 5
Present theory $\kappa^{-1}=2.0\times10^{-13}$ cm	77 : 50 : 1	39 : 51 : 17
Experiment[b]	80 : 20 : 0	36 : 48 : 16
Fermi theory	94 : 6 : 0	78 : 20 : 2
Kovacs theory	55 : 45	28 : 78

[a] Statistical factors were taken from Milburn.[2]
[b] See reference 4.

can be dropped. The factor $S(m)$ takes account of the symmetry restrictions on the final state and the conservation of isotopic spin in the process. The mass κ is a parameter of the theory (the Compton wavelength corresponding to κ takes the place of the linear dimensions of the Fermi volume[1]), and can be chosen so as to fit the experimental data on a nucleon-nucleon process at any one energy.

For the purpose of evaluating the relative transition probability, it is worth noting that the invariant phase space factor

$$R_n(\mathbf{P},P_0)=\prod_{i=1}^{n}\left(\int d^4q_i\,\delta(q_i^2-m_i^2)\right)\delta^4\left(\sum_{i=1}^{n}q_i-P\right)$$

(where P is total energy and momentum), which on multiplication by $S(m)\kappa^{-2m}$ yields the relative probability, satisfies the following simple recurrence relation in the center-of-mass frame:

$$R_{n+1}(0,E)=\tfrac{1}{2}\int\frac{d^3\mathbf{q}_{n+1}}{(\mathbf{q}_{n+1}{}^2+m_{n+1}{}^2)^{\frac{1}{2}}}R_n(0,\mathcal{E}),$$

where

$$\mathcal{E}=[\{E-(\mathbf{q}_{n+1}{}^2+m_{n+1}{}^2)^{\frac{1}{2}}\}^2-\mathbf{q}_{n+1}{}^2]^{\frac{1}{2}}.$$

This relation was of considerable use in our calculations.

Within the frame-work of this statistical theory we find that the relative probabilities tend to have a maximum towards the higher multiplicities as the energy increases. The precise energy at which the cross-overs take place depend on the value chosen for the parameter κ. The expected values of the multiplicities in nucleon-nucleon collisions are compared with the experimental values[4] in Table I. We have also calculated and compared with experiment the branching ratios in pion-nucleon collisions[5] and the average multiplicity in antiproton-nucleon annihilation at rest[6] for various values of the parameter κ; these results are given in Tables II and III.

We might make some further remarks. Firstly, the statistical postulate in multiple meson production processes has not been exhausted of its consequences, and the above formulation has the virtue of being relatively simple. Secondly, the quantity κ (or rather κ^{-1}) takes the place of the linear dimension of the Fermi volume, but the manner in which the quantity enters the theory shows that it is clearly a covariant concept.

TABLE II. Comparison of meson multiplicities in pion-proton collisions at 1.3-Bev pion energy.

	Ratio[a] single:double:triple:quadruple
Present theory $\kappa^{-1}=1.5\times10^{-13}$ cm	1 : 2.02 : 0.79 : 0.33
Present theory $\kappa^{-1}=2.0\times10^{-13}$ cm	1 : 3.59 : 2.50 : 1.84
Experiment[b]	1 : 6.91 : 1.18
Fermi theory	1 : 3.7 : 0.64 : 0.02

[a] Statistical factors were taken from Milburn.[2]
[b] See reference 5.

TABLE III. Average multiplicity of π-meson production in antiproton-nucleon annihilation at rest.[a] Available energy for the process is 1.868 Bev.

Parameter κ^{-1} (10^{-13} cm)	1.5	2.0	3.0	4.0
Calculated average multiplicity[b]	3.36	3.8	4.58	5.2

[a] The observed average multiplicity was 5.3 ± 0.4 (see reference 6).
[b] Statistical factors were taken from G. Sudarshan, Phys. Rev. **103**, 777 (1956); see also C. Goebel, Phys. Rev. **103**, 258 (1956).

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