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Connection between Local Commutativity and Regularity of Wightman Functions

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It is proved that a Wightman function (vacuum expectation value of a product of field operators) will be analytic and one valued at a real set of space-time points if and only if the fields possess a property of weak local commutativity at the same points. This statement assumes the validity of *TCP* invariance. A similar but more complicated statement is proved for theories without *TCP* invariance.

1. STATEMENT OF RESULTS

JOST[1] has discovered the equivalence between the *TCP* theorem[2] and a property of field operators which we shall call WLC (weak local commutativity). We say that a set of field operators $(\psi_0,\psi_1,\cdots,\psi_n)$ has WLC at a set of real 4-vectors $(\xi_1,\cdots,\xi_n)$ if the relation

$$\langle\psi_0(y_0)\psi_1(y_1)\cdots\psi_n(y_n)\rangle_0 = \epsilon\langle\psi_n(y_n)\cdots\psi_1(y_1)\psi_0(y_0)\rangle_0 \quad (1)$$

holds for all $(y_0,y_1,\cdots,y_n)$ such that the differences

$$\eta_j=y_{j-1}-y_j \quad (2)$$

lie within some real neighborhood of the ξ_j. Here ϵ is ±1, the sign depending on whether the permutation of fermion fields between the left and right sides of Eq. (1) is even or odd. The vacuum expectation values may be written for brevity

$$W(\eta)=W(\eta_1,\cdots,\eta_n)=\langle\psi_0(y_0)\cdots\psi_n(y_n)\rangle_0, \quad (3)$$

$$V(\eta)=V(\eta_1,\cdots,\eta_n)=\langle\psi_n(-y_n)\cdots\psi_0(-y_0)\rangle_0. \quad (4)$$

The notation (η) will always denote a set of n 4-vectors $(\eta_1,\cdots,\eta_n)$, and $(-\eta)$ will denote $(-\eta_1,\cdots,-\eta_n)$. Thus Eq. (1) becomes

$$W(\eta)=V(-\eta). \quad (5)$$

The theorem of Jost states that for *TCP* invariance of the W function it is necessary and sufficient that the fields $(\psi_0,\cdots,\psi_n)$ have WLC at *one* set of vectors (ξ) in a special domain D. The domain D consists of those real (ξ) for which

$$\left(\sum_1^n \lambda_j\xi_j\right)^2<0 \quad (6)$$

when the λ_j are any set of real non-negative numbers not all zero.

The theory is assumed invariant under the restricted real Lorentz group without space or time reflection, and all states are assumed to have non-negative energy. Then the functions $W(\eta)$ and $V(\eta)$ have the properties of Wightman functions.[3] According to a theorem of Bargmann, Hall, and Wightman,[4] they are boundary values of functions $W(\zeta)$ and $V(\zeta)$, regular in the $4n$ complex variables $\zeta_{j\mu}$, $j=1,\cdots,n$; $\mu=0,1,2,3$; so long as these variables lie in a certain domain R_n'. Let the forward tube R_n be defined as the set of (ζ) for which

$$[\mathrm{Im}\,\zeta_j]^2>0,\quad \mathrm{Im}\,\zeta_{j0}>0,\quad j=1,\cdots,n, \quad (7)$$

and the backward tube $\bar{R}_n$ as the set for which

$$[\mathrm{Im}\,\zeta_j]^2>0,\quad \mathrm{Im}\,\zeta_{j0}<0,\quad j=1,\cdots,n. \quad (8)$$

Then R_n' is the set of points $(\Lambda\zeta)$ with (ζ) in R_n and Λ a complex Lorentz transformation of determinant $+1$. The value of $W(\Lambda\zeta)$ is independent of Λ, and this implies in particular, as Jost[1] observed,

$$W(\zeta)=W(-\zeta),\quad \zeta \text{ in } R_n'. \quad (9)$$

The domain D of Jost's theorem consists precisely of the real points in R_n'. The theorem states that for *TCP* invariance it is necessary and sufficient that the fields $(\psi_0,\cdots,\psi_n)$ have WLC at one real point in R_n'.

The purpose of this note is to draw a further deduction from the foregoing ideas of Jost and Wightman. We find that there is a close correspondence between those real points at which WLC holds and those at which the functions W and V are analytic. More precisely, we have the following theorem.

Theorem

Let S be the set of real points (ξ) at which the field-operators $(\psi_0,\psi_1,\cdots,\psi_n)$ have WLC. Then one of the three following alternatives holds.

(*a*) S is null.

(*b*) S includes D. In this case the functions $W(\zeta)$ and $V(\zeta)$ are identical. They are one valued and analytic in a complex domain R including R_n'. A real point (ξ) can be included in R if and only if it belongs to S.

(*c*) S is not null and is disjoint from D. In this case the functions $W(\zeta)$ and $V(\zeta)$ are two branches of a function analytic and one valued on a two-sheeted Riemann surface R covering R_n' twice. A real point (ξ) belongs to S if and only if it lies within R at a place

[1] R. Jost, Helv. Phys. Acta **30**, 409 (1957).
[2] G. Lüders, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **28**, No. 5 (1954).
[3] A. S. Wightman, Phys. Rev. **101**, 860 (1956).
[4] D. Hall and A. Wightman, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **31**, No. 5 (1957).

where the W branch over R_n crosses over into the V branch over $\bar{R}_n$.

The physically interesting case of this theorem is alternative (*b*), since this is the only case consistent with *TCP* invariance.

Corollary 1.—If *TCP* invariance holds, then the set of real points (ξ) at which the field operators $(\psi_0,\cdots,\psi_n)$ have WLC is identical with the maximal set of real points in a complex domain R including R_n' within which the Wightman function $W(\zeta)$ is regular and one-valued.

Corollary 2.—For the fields $(\psi_0,\cdots,\psi_n)$ to have WLC at every point of a connected real domain D' including D, it is necessary and sufficient that *TCP*-invariance hold and that the Wightman function $W(\zeta)$ be analytic at each point of D'.

Note the essential fact that W must be one valued in the domain R of Corollary 1. In general it is not true that the set of points S at which WLC holds is identical with the set at which $W(\zeta)$ is analytic. It may well happen that $W(\zeta)$ is analytic at a real point but that the values obtained by continuing from R_n and from $\bar{R}_n$ do not agree.[5] In this case $W(\zeta)$ is analytic but double valued, and the point in question cannot belong to S. In Corollary 2, however, the condition of one-valuedness is omitted and simple analyticity suffices.

Corollary 2 provides an answer to the question with which this investigation started, namely, under what conditions will $W(\xi)$ be analytic at all real points $(x_0,x_1,\cdots,x_n)$ for which the vectors (x_i-x_j) are space-like? This will be so if WLC holds at the same points. The author is indebted to Dr. Jost for raising this question, and for many helpful discussions.

2. PROOFS

The main tool in the proofs will be the "edge of the wedge" theorem of Oehme, Taylor, and Bremermann.[6] The form of the theorem which we shall use is as follows.

Lemma (Edge of the Wedge)

Let $F(\zeta)$ be regular in the forward tube R_n, and $G(\zeta)$ in the backward tube $\bar{R}_n$. For real (ξ), we define

$$F(\xi)=\operatorname*{Lim}_{\zeta\to\xi}F(\zeta),\quad (\zeta)\text{ in }R_n, \tag{10}$$

$$\bar{G}(\xi)=\operatorname*{Lim}_{\zeta\to\xi}G(\zeta),\quad (\zeta)\text{ in }\bar{R}_n, \tag{11}$$

supposing these limits to exist as distributions. If

$$F(\xi)=\bar{G}(\xi) \tag{12}$$

for (ξ) in a real neighborhood of zero, say for

$$\textstyle\sum_\mu|\xi_{j\mu}|<a,\quad j=1,\ \cdots,\ n, \tag{13}$$

then $F(\zeta)$ and $G(\zeta)$ are a single analytic function regular in the complex neighborhood[7]

$$\textstyle\sum_\mu|\zeta_{j\mu}|<\frac{1}{2}a,\quad j=1,\ \cdots,\ n, \tag{14}$$

as well as in R_n and $\bar{R}_n$.

Proof of lemma.—For a fully rigorous proof of the lemma we refer to the paper of Oehme, Taylor, and Bremermann. It seemed worthwhile to present here an alternative proof making no pretensions to rigor but having the virtue of brevity.

Let (ζ) be any complex point satisfying

$$\textstyle\sum_\mu|\zeta_{j\mu}|<\frac{1}{2}b,\quad b<a,\quad j=1,\cdots,n. \tag{15}$$

Let B be the constant vector $(0,0,0,b)$. Consider the surface formed by the points

$$Z_j(u)=Bu+\zeta_j(1-u^2) \tag{16}$$

as the complex variable $u=\rho e^{i\theta}$ moves in the circle $|u|\leq 1$. Writing $\zeta_j=\xi_j+i\eta_j$, we have

$$\begin{aligned}\operatorname{Im} Z_j(u)=&(1-\rho^2)\eta_j\\&+\rho\sin\theta[B-2\rho(\xi_j\cos\theta-\eta_j\sin\theta)].\end{aligned} \tag{17}$$

Equation (15) implies that $Z_j(u)$ lies in R_n for $0<\theta<\pi$ and in $\bar{R}_n$ for $\pi<\theta<2\pi$, provided *either* $\rho=1$ *or* $\eta_j=0$.

Suppose first that all $\eta_j=0$. Then $F(Z_j(u))$ and $G(Z_j(u))$ are functions regular in u in the upper and lower halves of the circle $|u|\leq 1$ and have equal limit-values on the real segment $-1\leq u\leq 1$. The two functions therefore define a single function of u regular in $|u|\leq 1$. The value of this function at $u=0$ is

$$F(\xi)=\bar{G}(\xi)=H(\xi), \tag{18}$$

where $H(\zeta)$ is defined by the Cauchy integral

$$H(\zeta)=\frac{1}{2\pi}\left[\int_0^\pi F(Z_j(e^{i\theta}))d\theta+\int_\pi^{2\pi}G(Z_j(e^{i\theta}))d\theta\right]. \tag{19}$$

We have proved Eq. (18) only for real $(\zeta)=(\xi)$ satisfying Eq. (15).

Next take (ζ) complex. The path of integration in $H(\zeta)$ still lies entirely in R_n and $\bar{R}_n$ except for the end-points $\theta=0,\ \pi$. The integral defines $H(\zeta)$ as a function of the complex variables $\zeta_{j\mu}$ regular in the domain (15) [it is at this point that the proof lacks rigor, however there is no difficulty if one assumes $F(\zeta)$ and $G(\zeta)$ to be bounded in the neighborhood of the end points $(\zeta)=Z_j(\pm 1)=\pm B$]. Now if (ζ) and u are complex and lie in a certain neighborhood of zero, the point $Z_j(u)$ satisfies Eq. (15) and so $H(Z_j(u))$ is regular in the $(4n+1)$ variables $\zeta_{j\mu}$ and u. When all the variables are

[5] The author was fortunate in being able to study some lecture notes of W. Pauli (unpublished). Pauli has greatly clarified the situation by enumerating the possibilities which arise in the simplest case $n=1$. All types of behavior which occur for any n can already be seen with $n=1$.

[6] Oehme, Taylor, and Bremermann (to be published). The essential idea of this theorem occurs already in the work of Bogoliubov, Medvedev, and Polivanov, Uspekhi Mat. Nauk (to be published).

[7] The constant $\frac{1}{2}$ is not best possible. For the best possible result in the case $n=1$, see R. Jost and H. Lehmann, Nuovo cimento **5**, 1598 (1957).

real, Eq. (18) gives

$$F(Z_j(u))=H(Z_j(u)). \tag{20}$$

By analytic continuation in u, Eq. (20) must also hold for real (ζ) and complex u in the upper semicircle. But for complex u the point $Z_j(u)$ is interior to R_n, where both sides of Eq. (20) are analytic in (ζ). Hence we may extend Eq. (20) by analytic continuation from real to complex (ζ). The final result is

$$F(\zeta)=H(\zeta) \tag{21}$$

for all complex (ζ) in R_n and satisfying Eq. (15). Similarly, $G(\zeta)=H(\zeta)$ in $\bar{R}_n$. Thus F, G, H are the same analytic function, regular in R_n, $\bar{R}_n$ and the domain (15).

Proof of theorem.—We now return to the Wightman functions defined by Eqs. (3) and (4). For any real (ξ) we have according to Wightman[3]

$$W(\xi)=\operatorname*{Lim}_{\zeta\to\xi}W(\zeta),\quad \zeta \text{ in } R_n. \tag{22}$$

Since $\bar{R}_n$ is included in R_n', W is regular also in $\bar{R}_n$, and we may define for real (ξ)

$$\bar{W}(\xi)=\operatorname*{Lim}_{\zeta\to\xi}W(\zeta),\quad \zeta \text{ in } \bar{R}_n. \tag{23}$$

By Eq. (9)

$$\bar{W}(\xi)=W(-\xi). \tag{24}$$

Let (ξ) be a point of the set S defined in the theorem. Then Eq. (5) holds for real (η) in a neighborhood of (ξ). Combined with Eq. (24), this gives

$$W(\eta)=\bar{V}(\eta) \tag{25}$$

in a real neighborhood of (ξ). The conditions of the lemma are satisfied, and therefore $W(\zeta)$, $V(\zeta)$ are the same analytic function, regular in a simply-connected region consisting of R_n, $\bar{R}_n$ and a complex neighborhood of (ξ). Conversely, if $W(\zeta)$ in R_n and $V(\zeta)$ in $\bar{R}_n$ are connected by analytic continuation through a real point (ξ), then Eq. (25) holds in the neighborhood and so (ξ) belongs to S.

To complete the proof of the theorem it is only necessary to enumerate the possible relations between W and V. Either $W(\zeta)$ and $V(\zeta)$ are identical functions in R_n' or they are distinct. If they are identical, then Eq. (25) holds at real points interior to R_n', that is to say at all points of D. Then S includes D, and we are in case (b) of the theorem. We define the domain R to consist of R_n' together with complex neighborhoods of all points of S. The function W is analytic and one-valued in R. But R cannot be extended to include any real point (ξ) not in S. For every real point not in D is already on the boundary of R_n', being approachable both from R_n and from $\bar{R}_n$. If the boundary values of W from the two sides agree, then the point in question is in S and is interior to R. But if the boundary values disagree we cannot bring the point into any extension of R in which W remains one valued.

If $W(\zeta)$ and $V(\zeta)$ are distinct functions, then Eq. (25) cannot hold over any neighborhood in D and thus S is disjoint from D. We are either in case (a) or in case (c) of the theorem. In case (a), there is no analytic connection between the functions W and V, and there is nothing to prove. Suppose that we are in case (c). Equation (25) holds in the neighborhood of a non-null set of real points S. We define similarly S' to be the set of points in the neighborhood of which

$$V(\eta)=\bar{W}(\eta) \tag{26}$$

holds. Likewise T, U are the sets where

$$W(\eta)=\bar{W}(\eta), \tag{27}$$

$$V(\eta)=\bar{V}(\eta), \tag{28}$$

hold in a real neighborhood. Since the functions W, V are distinct, the pairs ST, SU, $S'T$, and $S'U$ are disjoint, but the pairs SS' and TU may overlap. The Riemann surface R is defined to be the domain R_n' covered by two sheets and with certain connecting linkages added. At all points of S a complex neighborhood is added, connecting the upper sheet over R_n to the lower sheet over $\bar{R}_n$. At all points of S' the upper sheet over $\bar{R}_n$ is connected to the lower over R_n. At points of T the two upper sheets are connected, and at points of U the two lower sheets. In particular, all points of D belong to both T and U. The Riemann surface R has the properties required by the theorem. For the same reasons as in case (b), it is impossible to extend R so as to include any real points not in one of the sets S, S', T, U, so long as the function (W,V) is restricted to be at most two valued. In particular, the set of points S is uniquely determined by the cross points of the two sheets of R. This ends the proof of the theorem.

Corollary 1 is a restatement of part of the theorem, and needs no further discussion. Corollary 2 requires only the observation that $W(\zeta)$ is automatically one-valued in $(R_n'+D')$ if it is analytic in D' and if $(D+D')$ is a connected set.