K-Meson Dispersion Relations. I. Theory

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Dispersion relations for K-meson scattering are considered, particularly for the theoretical difficulties of contributions from a continuum in the unphysical region. Some general results relating to threshold values of matrix elements for creation processes in K^-+N collisions are proved.

1. INTRODUCTION

DISPERSION relations for K-meson—nucleon $(K-N)$ scattering have recently been written down by Sakurai and by Amati and Vitali.[1] Like the $\pi-N$ relations,[2] these provide important clues for the interpretation of K-meson data. In this paper we wish to consider some of the peculiar theoretical difficulties which arise from the unphysical ranges in these relations, while in the accompanying paper we make a first attempt from the existing experimental data towards fixing the parity of the K-meson and the K-coupling constants. No rigorous derivation of the relations is attempted in these papers.

2. DISPERSION RELATIONS

Write T_p^+, T_n^+, T_p^-, T_n^- for the elastic scattering amplitudes for K^+-p, K^+-n, K^--p, and K^--n scattering. The initial meson and nucleon 4-momenta are k, p, respectively, and final momenta k', p'. On the energy shell,[3]

$$k_0,\ k_0',\ p_0,\ p_0'>0, \tag{1}$$

$$k^2=k'^2=K^2,\quad p^2=p'^2=N^2, \tag{2}$$

$$k+p=k'+p'. \tag{3}$$

Combining (2) and (3), one obtains

$$k\cdot(p-p')=p\cdot p'-N^2. \tag{4}$$

If one writes

$$T(k,p,p')=\bar{u}(p')[L+\boldsymbol{k}M]u(p),$$

then on the energy shell L and M are functions of the two independent covariants $(p+p')\cdot k$ and $p\cdot p'$.

In complete analogy with the π-meson case, we define the retarded causal amplitudes

$$M^\pm(k,p,p')=i\int d^4x\theta(-x)(p'|[j^\pm(0),j^{\pm *}(x)]|p)e^{-ikx}$$
$$-i\int d^4x\delta(x_0)(p'|[j^\pm(0),\phi_K{}^{\pm *}(x)]|p)e^{-ikx}. \tag{5}$$

The second term in (5) appears only if the K-interaction Lagrangian contains $\phi_K{}^2(x)$ or $\phi_K{}^4(x)$ terms explicitly. In this paper only 3-field interactions are considered so that this term is consistently disregarded. On account of the mass relations, e.g., $\Lambda^2>N^2+K^2$, the retarded amplitude $M^\pm$ equals $T^\pm$, so long as $k\cdot p'\geqslant 0$. This includes the entire physical region.

Before writing the dispersion relations, some properties of $M^\pm$ may be noted:

$$\text{(a)}\quad M^{\pm *}(k,p,p')=M^\mp(-k,\,p',\,p). \tag{6}$$

This follows from the Bose-character of the K meson.

(b) ImM

$$=\tfrac{1}{2}(2\pi)^4\sum_n[(p'|j(0)|n)(n|j^*(0)|p)\delta^4(p+k-n)$$
$$-(p'|j^*(0)|n)(n|j(0)|p)\delta(p'-k-n)]. \tag{7}$$

Set

$$M^\pm=\bar{u}(p')[F^\pm+\boldsymbol{k}G^\pm]u(p), \tag{8}$$

$$x=k\cdot(p+p'),\quad y=p\cdot p'. \tag{9}$$

Upon using (6), the dispersion relations are

$$\mathrm{Re}F^+(x,y)=\frac{P}{\pi}\int_0^\infty\frac{\mathrm{Im}F^+(x',y)}{x'-x}dx'+\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}F^-(x',y)}{x'+x}dx', \tag{10}$$

$$\mathrm{Re}G^+(x,y)=\frac{P}{\pi}\int_0^\infty\frac{\mathrm{Im}G^+(x',y)}{x'-x}dx'+\frac{1}{\pi}\int_0^\infty\frac{\mathrm{Im}G^-(x',y)}{x'+x}dx', \tag{11}$$

with similar relations for ReF^- and ReG^-.

For forward scattering, $p=p'$; $y=N^2$. If ω is the K-meson energy in the laboratory system $[\omega=k$

[1] J. J. Sakurai, Bull. Am. Phys. Soc. Ser. II, 2, 177 (1957); D. Amati and B. Vitali (to be published).
[2] A. Salam, *Proceedings of the CERN Symposium on High-Energy Accelerators and Pion Physics, Geneva, 1956* (European Organization of Nuclear Research, Geneva, 1956), Vol. 2, p. 176; Low, Chew, Goldberger, and Nambu, Phys. Rev. **186**, 1337 (1957).
[3] The notation throughout this paper corresponds closely with that used for the π-N case in reference 2 by Salam. We use the metric $p\cdot q=p_0q_0-\mathbf{p}\cdot\mathbf{q}$, $\boldsymbol{k}=i\gamma k$, where the γ are Hermitian, $(\gamma^2=1)$. Thus $\boldsymbol{kp}+\boldsymbol{pk}=2k\cdot p$. All masses are written with their particle symbol. Thus N and K stand for the nucleon and K masses.

$\cdot(p+p')/2N]$, Eqs. (10) and (11) give[4]

$$\mathrm{Re}M^{\pm}(\omega)=\frac{P}{\pi}\int_0^\infty \frac{\mathrm{Im}M^{\pm}(\omega')}{\omega'-\omega}d\omega' +\frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}M^{\mp}(\omega')}{\omega'+\omega}d\omega'. \quad (12)$$

For $\omega>K$, the "optical theorem" takes the form

$$\mathrm{Im}M(\omega)=(\omega^2-K^2)^{\frac{1}{2}}\sigma_T(\omega), \quad (13)$$

where σ_T is the total cross section.

3. UNPHYSICAL RANGE

In order to make practical use of relation (12), one must have information about $\mathrm{Im}M(\omega)$ below the physical threshold $\omega=K$. In this region, $p=p'$; k_0, $p_0>0$; $k^2=K^2$; $p^2=N^2$. However, $k\cdot p<KN$; that is to say, the 3-momenta $\mathbf{k}$, $\mathbf{p}$ can be complex in such a way that the scalar products $\mathbf{k}^2$, $\mathbf{p}^2$, $\mathbf{k}\cdot\mathbf{p}$ are still real. (For the nonforward scattering case $\mathbf{p}'$, $\mathbf{k}'$ are also complex with $\mathbf{k}\cdot\mathbf{p}'$, $\mathbf{k}'\cdot\mathbf{p}$, etc., all real.) We now consider the case of forward scattering.

(1) From Eq. (7), we have

$$\mathrm{Im}M^{+}(\omega)=0, \quad 0\leqslant\omega\leqslant K.$$

This is because there is no physical state $|n)$ with strangeness +1 and rest mass $n^2=(p+k)^2<(K+N)^2$.

(2) Like the $\pi-N$ case, $\mathrm{Im}M_p^-$ has bound state contributions. These arise when

$$n^2=(p+k)^2=\Lambda^2 \quad\text{or}\quad \omega_\Lambda=(1/2N)(\Lambda^2-N^2-K^2), \quad (14)$$

and

$$n^2=(p+k)^2=\Sigma^2 \quad\text{or}\quad \omega_\Sigma=(1/2N)(\Sigma^2-N^2-K^2). \quad (15)$$

For M_n^-, only the Σ state contributes. If we assign (by convention) parity[5] (+) to Λ, then $\mathrm{Im}M_p^-(\omega_\Lambda)$ and $\mathrm{Im}M_p^-(\omega_\Sigma)$ depend on the parities of the K meson and the Σ particles. For the general nonforward case these so-called bound-state contributions are[6]

$$\mathrm{Im}M_p^-(k,p,p')$$
$$=-\frac{g_\Lambda^2}{4\pi}2\pi^2\bar{u}(p')[k-N+\Lambda]u(p)\delta[(p+k)^2-\Lambda^2], \quad (K\text{ pseudoscalar}), \quad (16)$$

$$=-\frac{g_\Lambda^2}{4\pi}2\pi^2\bar{u}(p')[k-N-\Lambda]u(p)\delta[(p+k)^2-\Lambda^2], \quad (K\text{ scalar}), \quad (17)$$

and

$$\mathrm{Im}M_p^-(k,p,p')$$
$$=-\frac{g_\Sigma^2}{4\pi}2\pi^2\bar{u}(p')[k-N+\Sigma]u(p)\delta[(p+k)^2-\Sigma^2], \quad (K\text{ pseudoscalar, }\Lambda,\Sigma\text{ same parity}), \quad (18)$$

$$=-\frac{g_\Sigma^2}{4\pi}2\pi^2\bar{u}(p')[k-N-\Sigma]u(p)\delta[(p+k)^2-\Sigma^2], \quad (K\text{ scalar, }\Lambda,\Sigma\text{ opposite parity}). \quad (19)$$

(3) Unlike the $\pi-N$ case, $\mathrm{Im}M_p^-$ has contributions below the physical threshold, from the continuum of (π,Λ), (π,Σ), and $(2\pi,\Lambda)$ states—more precisely, when[7]

$$n^2=(p+k)^2\geqslant(\Lambda+\Pi)^2,$$
$$n^2=(p+k)^2\geqslant(\Sigma+\Pi)^2,$$
$$n^2=(p+k)^2\geqslant(\Lambda+2\Pi)^2.$$

As an example, consider the Λ, Π contribution, to $\mathrm{Im}M^-$. If q and s are the Λ and π momenta in the real physical intermediate state, then for scalar K, one has

$$\mathrm{Im}M_{\Lambda\pi}^-(p,p',k)=-\frac{1}{4(2\pi)^2}\int\bar{u}(p')[X+kY]i\gamma_5[q-\Lambda] \times i\gamma_5[X^*+kY^*]u(p)\theta(q)\theta(s)\delta(q^2-\Lambda^2)\delta(s^2-\Pi^2) \times\delta(p+k-q-s)d^4q d^4s, \quad (20)$$

where $i\gamma_5$ is replaced by 1 if K is pseudoscalar. X and Y in the first square bracket depend on $p'\cdot q$ and $p'\cdot s$; in the second on $p\cdot q$ and $p\cdot s$.

Specializing to forward scattering, we have

$$\mathrm{Im}M_{\Lambda\pi}^-(\omega)=\frac{1}{2^5(2\pi)^2N}\int\mathrm{Tr}\{(p-N)[X+kY](q\pm\Lambda) \times[X^*+kY^*]\}\delta(q^2-\Lambda^2)\delta(s^2-\Pi^2)\delta(p+k-q-s) \times\theta(p)\theta(q)\theta(s)d^4q d^4s. \quad (21)$$

The expression in the integrand in curly brackets

[4] We would like to restress the important role in dispersion theory of the use of covariant variables like $k\cdot p$ and $p\cdot p'$. The commonly used center-of-mass quantities (c.m. momentum $|\mathbf{p}^c|$ or c.m. angle θ^c) have fairly awkward expressions in terms of $k\cdot p$, $p\cdot p'$. It has come to be recognized commonly, that momentum transfer $2|\mathbf{p}^c|\sin(\theta^c/2)$ (or better still its square) is the more significant variable for plotting experimental results. This is a recognition of the fact that momentum transfer (and not $\cos\theta^c$) is simply related to the covariant variable $p\cdot p'$; $[p\cdot p'-N^2=|\mathbf{p}^c|^2\sin^2(\theta^c/2)]$. We wish to point out that, likewise, for exhibiting the energy-dependence of total cross sections, the theoretically significant variable in the present context is $p\cdot k$ ($=N\omega$, where ω is the meson laboratory energy) or equivalently $(p+k)^2=(E^c)^2$ (where E^c is the c.m. energy), and not the variable $|\mathbf{p}^c|$. A simple relation between lab momentum $|\mathbf{p}^l|$ and $|\mathbf{p}^c|$ may be noted in passing;

$$[(p\cdot k)^2-K^2N^2]^{\frac{1}{2}}=N|\mathbf{p}^l|=|\mathbf{p}^c|E^c.$$

[5] P. T. Matthews, Nuovo cimento 5, 642 (1957).

[6] This is a simple illustration of the general result stated by A. Salam [Nuclear Phys. 5, 687 (1958)], that matrix elements for a pseudoscalar meson theory can be obtained from those for a scalar meson theory by changing $\Lambda\to-\Lambda$.

[7] Here Π stands for the π-meson mass.

equals[8]

$$4F(p\cdot q, p\cdot k, k\cdot q)=4\{XX^*(p\cdot q\mp N\Lambda) - (XY^*+X^*Y)(Nq\cdot k\mp\Lambda p\cdot k) + YY^*[2(p\cdot k)(q\cdot k)-k^2(p\cdot q\pm\Lambda N)]\}, \quad (22)$$

where the upper signs are for scalar K mesons and the lower signs are for pseudoscalar K mesons. Insofar as $\mathrm{Im}M(\omega)\sim\sum|(p|j|n)|^2$, one may naively expect that $\mathrm{Im}M(\omega)\geqslant 0$ both in the physical and the unphysical region. If this were so, the sign of the right-hand side in relation (12) (the relation for $\mathrm{Re}M^+$) would be fully determinate and the repulsive or attractive character of the K^+-N "potential" could be unambiguously stated. Unfortunately it turns out that although $\mathrm{Im}M_{\Lambda\pi}^-\geqslant 0$ for the physical region, its continuation below physical energies can become negative. It is perhaps instructive to reconstruct the proof for positive definiteness of $\mathrm{Im}M_{\Lambda\pi}^-$ above physical threshold.

Consider (22) as a quadratic form. Since $p\cdot q\geqslant \Lambda N$ above threshold, the coefficient of XX^* is positive. Thus it is sufficient to prove that

$$[(Nq\mp\Lambda p)\cdot k]^2\leqslant (p\cdot q\mp\Lambda N)[2(p\cdot k)(q\cdot k) - k^2(p\cdot q\pm\Lambda N)], \quad (23)$$

or

$$k^2[p^2q^2-(p\cdot q)^2]-[p(q\cdot k)-q(p\cdot k)]^2\geqslant 0. \quad (24)$$

In the laboratory frame $[p=(N,0)]$, Eq. (24) reduces to

$$\mathbf{k}^2\mathbf{q}^2-(\mathbf{k}\cdot\mathbf{q})^2\geqslant 0. \quad (25)$$

For real $\mathbf{k}$ and $\mathbf{q}$, (25) is obviously true. No statement, however, can be made in the unphysical region. In the next section, by considering a special example, we shall show that $\mathrm{Im}M_{\Lambda\pi}^-$ can indeed be negative in the unphysical region.

4. UNPHYSICAL CONTINUUM

Rewrite (21) as

$$\mathrm{Im}M_{\Lambda\pi}^-(p\cdot k)=\frac{1}{4(2\pi)^2}\int G(p\cdot k, p\cdot q)\delta(q^2-\Lambda^2) \times\delta[(p+k)^2+\Lambda^2-\Pi^2-2q\cdot(p+k)] \times\theta(q_0)\theta(p_0+k_0-q_0)d^4q. \quad (26)$$

To perform the (covariant) integration in (26), it is convenient to specialize to the c.m. frame. If $\mathbf{p}^c$ and $\mathbf{q}^c$ be the initial and final c.m. momenta (both $|\mathbf{p}^c|$ and $|\mathbf{q}^c|$ are functions of $p\cdot k$[9]) the integration in (26) yields

$$\mathrm{Im}M_{\Lambda\pi}^-(p\cdot k)=\frac{|q^c|}{8(2\pi)^4E^c}\sum_{n=0}\frac{(|p^c||q^c|)^{2n}}{2n+1!} \times G^{2n}\left(p\cdot k, \frac{p\cdot(p+k)}{2(p+k)^2}[(p+k)^2+\Lambda^2-\Pi^2]\right), \quad (27)$$

where

$$G^{2n}=\partial^{2n}G/\partial(p\cdot q)^{2n}.$$

By using (27), it is possible to draw general conclusions regarding the behavior of the matrix elements at (physical or unphysical) creation thresholds ($q^c\to 0$).

(1) In the physical region

$$\mathrm{Im}M_{\Lambda\pi}^-=(\omega^2-K^2)^{\frac{1}{2}}\sigma_{K+N\to\pi+\Lambda} =(|\mathbf{p}^c|E^c/N)\sigma_{K+N\to\pi+\Lambda}. \quad (28)$$

Thus $\sigma_{K+N\to\pi+\Lambda}\sim(|\mathbf{q}^c|/|\mathbf{p}^c|)G$ for small p^c. This is the well-known $1/v$ law for exothermic reactions near physical threshold.

(2) For $q^c\to 0$, only the first term in the summation in (27) survives, so that statements about $\mathrm{Im}M_{\Lambda\pi}^-$ can be made simply by considering the integrand in (26). Thus

$$\lim_{q_c\to 0}\mathrm{Im}M_{\Lambda\pi}^-\to 0. \quad (29)$$

Also

$$\lim_{q_c\to 0}\frac{\partial}{\partial(p\cdot k)}\mathrm{Im}M_{\Lambda\pi}^-(p\cdot k)=\frac{1}{q_c}\lim G=\pm\infty. \quad (30)$$

For endothermic reactions, as shown in (25), G is always positive so that in (30) the limit is $+\infty$. *Thus matrix-elements for endothermic processes start from zero at creation-thresholds with a positive infinite slope.* This has been noted previously by Wigner and Breit.[10]

For exothermic reactions $\mathrm{Im}M_{\Lambda\pi}^-$ can approach zero at the unphysical creation threshold with $+\infty$ or $-\infty$ slope depending on the theory. As an example, consider a scalar K meson and a 4-field Lagrangian $\bar\psi_\Lambda i\gamma_5\psi_N\phi_\pi\phi_K$. In the lowest order perturbation calculation,

$$G(p\cdot k, p\cdot q)=(\text{positive constant})\times(p\cdot q-\Lambda N).$$

[8] The corresponding 2π, Λ contribution to $\mathrm{Im}M(k,p,p')$ equals

$$-\frac{1}{4(2\pi)^5}\int\bar u(p')[X+kY+sZ][q\mp\Lambda][X^*+kY^*+sZ^*]u(p) \times\theta(q)\theta(s)\theta(r)\delta(q^2-\Lambda^2)\delta(s^2-\Pi^2)\delta(r^2-\Pi^2) \times\delta(p+k-q-s-r)d^4q d^4s d^4r,$$

with the same convention for the signs. Here q is the Λ momentum and r and s are π momenta.

[9] $\mathbf{p}=-\mathbf{k}=\mathbf{p}^c$; $p_0^c+k_0^c=E^c=[\Lambda^2+(\mathbf{q}^c)^2]^{\frac{1}{2}}+[\Pi^2+(\mathbf{q}^c)^2]^{\frac{1}{2}}$.
Covariantly,

$$(\mathbf{p}^c)^2=\frac{(p\cdot k)^2-K^2N^2}{(p+k)^2} =\frac{1}{4(p+k)^2}[(p+k)^2+(N+K)^2][(p+k)^2-(N+K)^2],$$

$$(\mathbf{q}^c)^2=\frac{(q\cdot s)^2-\Lambda^2N^2}{(p+k)^2} =\frac{1}{4(p+k)^2}[(p+k)^2-(\Lambda+\Pi)^2][(p+k)^2-(\Lambda-\Pi)^2].$$

[10] E. P. Wigner, Phys. Rev. **73**, 1002 (1948); C. Breit, Phys. Rev. **107**, 1612 (1957). Also see R. J. Eden, Proc. Roy. Soc. (London) **A210**, 388 (1952).

By using (27), it is easy to see that

$$\frac{\partial\,\mathrm{Im}M_{\Lambda\pi}{}^{-}(p\cdot k)}{\partial(p\cdot k)}=-\infty.$$

In this case $\mathrm{Im}M_{\Lambda\pi}{}^{-}$ starts from a positive value at the scattering threshold, $(p+k)^2=(K+N)^2$. As $p\cdot k$ decreases, it goes through a (spurious) zero and approaches zero again infinitely fast from negative values when the beginning of the unphysical continuum is reached.

This behavior of $\mathrm{Im}M_{\Lambda\pi}{}^{-}$ makes it difficult to make precise numerical predictions from dispersion relations. One possible way out of the difficulty presents itself provided XX^*, XY^*+X^*Y, and YY^* in Eq. (22) are slowly varying functions of $p\cdot k$, etc. In this case $\sigma_{N+K\to\pi+\Lambda}$ at three energies near threshold would determine these three parameters, giving G in the unphysical region. Another possible means to extrapolate G into the unphysical region is provided by measurement of polarization near threshold, since the polarization function involves these same functions XX^*, etc.

5. CONVERGENCE OF DISPERSION RELATIONS

The integrals on the right in relations (12) and (13) as they stand do not converge, unless $\sigma(\omega)$ falls faster than $\omega^{-\frac{1}{2}}$. To secure better convergence one can do two things:

(1) Consider each relation at two energies:

$$\frac{\mathrm{Re}[M^{\pm}(\omega_1)-M^{\pm}(\omega_2)]}{\omega_1-\omega_2}=\frac{P}{\pi}\int\frac{\mathrm{Im}M^{\pm}(\omega)}{(\omega-\omega_1)(\omega-\omega_2)}d\omega-\frac{1}{\pi}\int\frac{\mathrm{Im}M^{\mp}(\omega)}{(\omega+\omega_1)(\omega+\omega_2)}d\omega. \quad (31)$$

(2) Alternatively subtract the relations for M^+ and M^-:

$$\frac{\mathrm{Re}[M^+(\omega)-M^-(\omega)]}{\omega}=\frac{2P}{\pi}\int\frac{\mathrm{Im}(M^+-M^-)}{\omega'^2-\omega^2}d\omega'. \quad (32)$$

Either of these subtractions would remove the extra terms from $\phi_K{}^2$ or $\phi_K{}^4$ interactions, referred to after Eq. (5). That a subtraction of the type (31) or (32) above is necessary is also shown if we consider the lowest order perturbation approximation to M^+ and M^-. As a theoretical example, consider the case $g_\Lambda\neq 0$ with scalar K mesons, while all other coupling constants vanish. Then to this order,

$$\mathrm{Re}M_p{}^{+}=\frac{g_\Lambda{}^2}{4\pi}\left(\frac{2\pi}{N}\right)\frac{\omega-N-\Lambda}{\omega+\omega_\Lambda}, \quad (33)$$

$$\mathrm{Re}M_p{}^{-}=\frac{g_\Lambda{}^2}{4\pi}\left(\frac{2\pi}{N}\right)\frac{\omega+N+\Lambda}{\omega-\omega_\Lambda}. \quad (34)$$

To this order, the only contributions to the right-hand side of (11) and (12) come from the bound-state terms. From (16) and (17), using (33) and (34), these are

$$\mathrm{Re}M_p{}^{+}+\frac{g_\Lambda{}^2}{4\pi}\left(\frac{2\pi}{N}\right), \quad (35)$$

and

$$\mathrm{Re}M_p{}^{-}+\frac{g_\Lambda{}^2}{4\pi}\left(\frac{2\pi}{N}\right). \quad (36)$$

Thus the relations (11) and (12) do not check in the lowest order perturbation calculation.[11] One may take the attitude that either a perturbation calculation is not valid, or the failure of (11) and (12) may be ascribed to the lack of convergence of the integrals involved on the right-hand side. Notice that relations (31) and (32) both check in the perturbation calculation. In the accompanying paper we shall use (32).

ACKNOWLEDGMENT

We would like to thank Dr. J. C. Taylor for numerous discussions.

[11] The same is true of the π-N dispersion relations. See R. Arnowitt and G. Feldman, Phys. Rev. **108**, 144 (1957).