Similarity between Shell Model and Deformed Nucleus Wave Functions*

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A body coordinate system $x'y'z'$ is defined in which the density of nuclear matter is assumed axially symmetric about the z' axis, the angular momentum $j_{z'}$ of each of the A nucleons in the nucleus about this axis being quantized. The wave function of a nucleon is φ_κ, with $j_{z'}\varphi_\kappa=\kappa\varphi_\kappa$. The system $x'y'z'$ is rotated relative to a system xyz fixed in space in such a way that the nucleus is in a state of total angular momentum J, with z component M, while the z' component remains $K=\kappa_1+\cdots+\kappa_A$. The nuclear wave function is $\Psi_{MK}{}^J$. The φ_κ are so defined that the $\Psi_{MK}{}^J$ are similar to wave functions obtained for shell-theoretical calculations. In order to represent at least 13 low states with $A=18$ and 19, three φ_κ are needed. These φ_κ resemble χ_κ, the wave functions of particles in a slightly deformed harmonic-oscillator well. Calculated ft values and magnetic moments are also similar to those for shell theory.

1. INTRODUCTION

THE existence of many very large quadrupole moments has led to deformed core or collective models of the nucleus.[1] These models have had remarkable success in predicting the rotational states and the large transition probabilities both for γ emission and Coulomb excitation of these states. In spite of the great flexibility of these models,[2] the concepts of nonspherical shape and hence rotation of the nucleus postulated by them appear to be well established.

In contrast to models involving collective motion of the nucleons of the core, there is the shell or individual-particle model. In recent calculations based on the shell model (the double closed shell-core model), interactions between configurations of all states of a given main shell were considered. They led both to substantial configuration interaction and, generally, to intermediate coupling. Detailed calculations have been made for the nuclei with mass number $A=18$ and 19[3-5] and for Pb^{206}.[6] While there does not exist agreement between theory and all experimental data already available, the extent of agreement for states with the parity predicted by the shell model seems to indicate that this model does have some validity at least in the vicinity of the double closed shells.

It has been pointed out that a tensor force interaction between each pair of nucleons may lead to an individual-particle model with deformed shape[7] and that the wave function for this model can be specified in configuration space,[8] that is, in the space of the $3A$ space and A spin coordinates of the nucleons of a nucleus with mass number A. In the present paper it will be seen that the wave functions for $A=18$ and 19 of the individual-particle model for a deformed nucleus are very similar to those previously obtained for the double closed shell-core model. Some details concerning the filling of the shells and coupling rules will be discussed and comparisons with experimental data given.

2. DESCRIPTION OF THE MODEL

Consider a box with one symmetry axis (for instance in the shape of an ellipsoid), containing A nucleons. Let the coordinate vectors of the box be $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$, with the axis of symmetry in the direction of $\mathbf{k}'$. The Hamiltonian for one particle in this box then commutes with $l_{z'}$, the operator for the z' component of the angular momentum of the particle, and with $j_{z'}=l_{z'}+s_{z'}$, where $s=$spin. The eigenvalue of $j_{z'}$ will be denoted κ.

For an assembly of A particles with quantum numbers $\kappa_1, \cdots, \kappa_A$, the eigenvalue of the z' component of total angular momentum J is

$$K=\sum_{i=1}^{A}\kappa_i. \tag{1}$$

The wave function for the assembly, antisymmetric under simultaneous exchange of coordinates, spins, and isotopic spins of any two nucleons will be denoted

$$\Phi_K(X_1',\cdots,X_A'), \tag{2}$$

where X_i' stands for the spin and space coordinates of the ith particle in the body system.

The wave function (2) has the expected nonspherical shape; however, it does not exhibit the other nuclear property so significantly demonstrated by the collective models—the rotation of the nucleus. This difficulty can be formulated in another way: Since (2) is not an eigenfunction of J (or, more precisely, of J^2), we do not

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[6] M. J. Kearsley, Nuclear Phys. **4**, 157 (1957); W. W. True and K. W. Ford, Phys. Rev. **109**, 1675 (1958).
[7] E. P. Wigner, *Symposium on New Research Techniques in Physics, 1952* (Academia Brasileira de Ciencias, Rio de Janeiro, 1954), p. 257.
[8] M. G. Redlich and E. P. Wigner, Phys. Rev. **95**, 122 (1954).

expect it to be an accurate approximation to the solution of the actual Schrödinger equation for a nucleus,

$$H\Psi = E\Psi,$$

with a rotationally invariant H.

It is possible, however, to form states which are superpositions of states of the A particles [like the state described by (2)] in spheroidal boxes which are identical except for the orientations of their coordinate systems. Certain of these superpositions of states will be eigenstates of J and of J_z, its component along the z axis of the laboratory system.

The coordinates and spin of the ith particle in the laboratory system are denoted X_i and the transformation R takes X_i into X_i'. This will be written

$$X_i' = RX_i. \tag{3}$$

For particles without spin, (3) has the following simple interpretation: X_i and X_i' stand for 3-component vectors of the Cartesian coordinates and R is a 3×3 rotation matrix. Inclusion of spin (which will be assumed in what follows) is entirely straightforward, but notationally more involved.

The wave function then is[8]

$$\Psi_{MK}{}^J(X_1,\cdots,X_A) = \mathfrak{N}\int \mathfrak{D}^J(R)_{KM}\,\Phi_K(RX_1,\cdots,RX_A)dR, \tag{4}$$

where $\mathfrak{D}^J(R)_{KM}$ is a representation coefficient,[9] M is an eigenvalue of J_z, $\mathfrak{N}$ is a normalization constant, and the integration is over the parameters specifying the rotation R (e.g., the Euler angles α, β, and γ). It is readily verified that $\Psi_{MK}{}^J$ is actually an eigenfunction of J^2 and J_z.

The $\mathfrak{D}^J(R)_{KM}$ play the rôle of coefficients in the expansion of $\Psi_{MK}{}^J$ in terms of a set of functions

$$\Phi_K(RX_1,\cdots,RX_A),$$

with all possible rotations R. It should be mentioned that two such functions with the same K but different R's are in general not orthogonal in the space of the coordinates and spins $X_1, \cdots, X_A$.

3. SINGLE-PARTICLE WAVE FUNCTIONS

The wave functions of a single particle in a deformed well have been calculated in recent years.[10] Results of such calculations for a deformed harmonic-oscillator well and the $1d$, $2s$ shell will be given here. The Hamiltonian is

$$H' = (\hbar^2/2m)\{-\Delta+\nu^2[(x')^2+(y')^2+(1+\alpha)(z')^2]\}, \tag{5}$$

[9] E. P. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (F. Vieweg und Sohn, Braunschweig, 1931), p. 180.

[10] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 16 (1955); S. A. Moszkowski, Phys. Rev. **99**, 803 (1955); K. Gottfried, *ibid.* **103**, 1017 (1956).

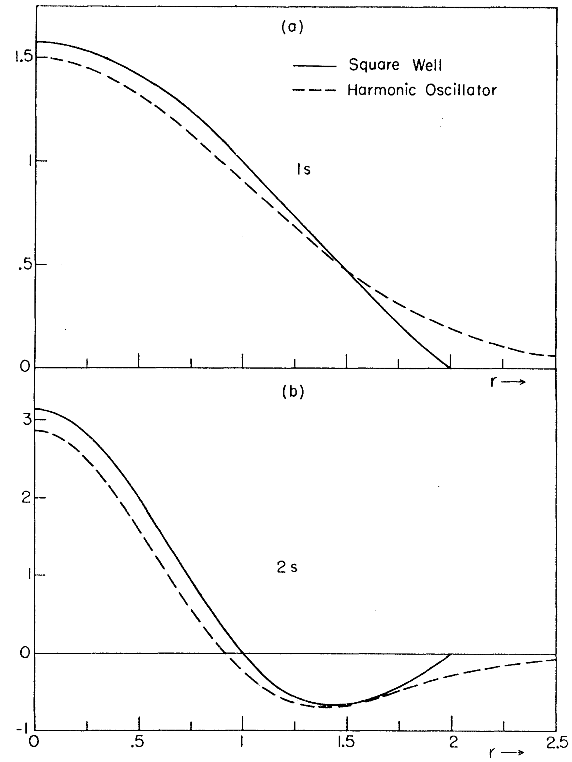

FIG. 1. The radial wave function for a harmonic-oscillator potential is compared with that of a square-well potential with infinitely high walls. In (a) the $1s$ functions are given and the parameter $\nu=1\times 10^{26}$ cm^{-2}, in (b) the $2s$ functions, with $\nu=1.8\times 10^{26}$ cm^{-2}. The radial coordinate is in 10^{-13} cm and the ordinate in $(10^{13}\text{ cm})^{-\frac{3}{2}}$.

where Δ is the Laplacian operator, m is the nucleon mass, ν is another constant, α determines the deformation, and the coordinate vector is $\mathbf{r}'$ in the body system.

It seems noteworthy that the wave functions for a harmonic-oscillator potential belonging to a given level are very similar to those of the corresponding level for a square-well potential. This is illustrated in Fig. 1. The $1s$ and $2s$ radial wave functions are given there for for both a spherical square well with infinitely high walls, i.e., a spherical box, and the harmonic oscillator (5) with $\alpha=0$. The radius of the box is 2×10^{-13} cm and the harmonic-oscillator parameter ν equals 1×10^{26} cm^{-2} for the $1s$ state and 1.8×10^{26} cm^{-2} for the $2s$ state. It is clearly necessary in this comparison to use different ν's for the different shells, since for a given ν the expectation value of $(r')^2$ for the qth level increases in direct proportion to the energy $(\hbar^2\nu/m)(q+\frac{3}{2})$ of that level.

The energies and wave functions of a single particle in the deformed well of (5) may be obtained (a) from an examination of the wave functions in rectangular or cylindrical coordinates, or (b) by using perturbation theory with a set of spherical wave functions and $(\hbar^2/2m)\alpha\nu^2(z')^2$ as perturbation operator.

We consider examples from the $1d$, $2s$ shell. The

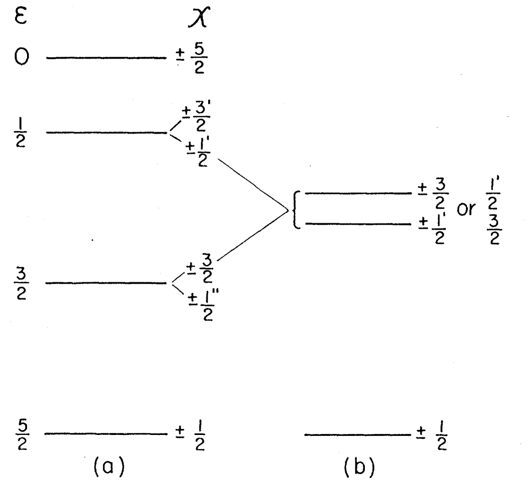

FIG. 2. Part (a) gives the deformation energies calculated for states of a single particle in the $1d$, $2s$ shell of a slightly deformed harmonic-oscillator potential. The energies are in units of $\alpha\hbar^2\nu/2m$, where $\alpha < 0$. Each level is labelled by the z' component κ of total angular momentum; the corresponding wave function is χ_κ.

In part (b) the level order of the states of the corresponding modified wave functions, φ_κ, is given, but only schematically.

spherical wave functions are $\psi_{1d\mu}$ and $\psi_{2s\mu}$, with μ an eigenvalue of $l_{z'}$. The wave functions in rectangular coordinates are u_{abc}, corresponding to energy

$$E=(\hbar^2\nu/m)[a+b+(1+\alpha)^{\frac{1}{2}}c+\tfrac{3}{2}],$$

where a, b, and c take on values 0, 1, 2, 3, $\cdots$ and label successive states of a particle in a harmonic-oscillator potential along x, y, and z directions. The spherical wave functions may be expanded in terms of the rectangular ones, e.g.,

$$\psi_{1d0}=6^{-\frac{1}{2}}(2u_{002}-u_{020}-u_{200}), \tag{6a}$$

$$\psi_{2s0}=-3^{-\frac{1}{2}}(u_{002}+u_{020}+u_{200}). \tag{6b}$$

As soon as the potential is deformed, a new set of eigenfunctions appears. These are $\bar{u}_{002}$, $\bar{u}_{020}$, and $\bar{u}_{200}$. From them, two eigenfunctions of $l_{z'}$ with $\mu=0$ may be formed:

$$\bar{u}_{002}=3^{-\frac{1}{2}}(\sqrt{2}\psi_{1d0}-\psi_{2s0})+\cdots, \tag{7a}$$

$$2^{-\frac{1}{2}}(\bar{u}_{200}+\bar{u}_{020})=-3^{-\frac{1}{2}}(\psi_{1d0}+\sqrt{2}\psi_{2s0})+\cdots. \tag{7b}$$

The dots indicate admixtures of wave functions of higher states. The amplitudes of these admixtures are in first order proportional to α, as may be seen from perturbation theory; the amplitudes of ψ_{1d0} and ψ_{2s0} in (7), however, are independent of α, since these states are degenerate for the spherical potential. The sign of α is related to shape as follows:

$\alpha<0$, prolate spheroid, $\langle z^2\rangle>\langle x^2\rangle$, (7a) is lower;

$\alpha>0$, oblate spheroid, $\langle z^2\rangle<\langle x^2\rangle$, (7b) is lower.

Here $\langle z^2\rangle$ is the expectation value of z^2.

The order of states and their deformation energies $\mathcal{E}$ in units of $\alpha\hbar^2\nu/2m$ in the limit of very small negative α are given in Fig. 2(a). The states are classified by $\kappa=\mu+\lambda$, where μ and λ are the components of orbital angular momentum and of spin in the z' direction. States $\pm\kappa$ are degenerate because of the reflection symmetry of the potential about the $x'y'$ plane. There are three states with $\kappa=\frac{1}{2}$; in order of increasing energy they are denoted by $\frac{1}{2}$, $\frac{1}{2}''$, $\frac{1}{2}'$, and two states with $\kappa=\frac{3}{2}$, the lower denoted by $\frac{3}{2}$, the higher by $\frac{3}{2}'$. In Table I(a) the wave functions χ_κ of the states which for $\alpha=0$ belong to the $1d$, $2s$ shell are given in terms of the spherical wave functions with spin, $\psi_{\frac{5}{2}\kappa}$, $\psi_{\frac{1}{2}\kappa}$, and $\psi_{\frac{3}{2}\kappa}$. The amplitudes for $\chi_{-\kappa}$, expanded in terms of ψ's with $-\kappa$ are equal in magnitude and change in sign for $1d_{\frac{3}{2}}$ only. The amplitudes do not depend upon ν.

In order to obtain Φ_K of (2), it is necessary only to take the product of A single-particle wave functions χ_κ, each multiplied by an isotopic spin function and to antisymmetrize this product. A state in the body system is then specified by

$$\kappa_1,\ \kappa_2,\ \cdots,\ \kappa_A. \tag{8}$$

$\Psi_{MK}{}^J$ is obtained by substitution of Φ_K in (4) and, strictly speaking, should also be labelled by (8).

The modified wave functions of Table I(b) and Fig. 2(b) will be discussed at the beginning of Sec. 5.

4. EXPANSION OF $\Psi_{MK}{}^J$

Let us substitute a two-particle wave function

$$\chi_\kappa(X_1)\cdot\chi_{\kappa'}(X_2)=[a\psi_{\frac{5}{2}\kappa}(X_1)+b\psi_{\frac{1}{2}\kappa}(X_1)+c\psi_{\frac{3}{2}\kappa}(X_1)][a'\psi_{\frac{5}{2}\kappa'}(X_2)+b'\psi_{\frac{1}{2}\kappa'}(X_2)+c'\psi_{\frac{3}{2}\kappa'}(X_2)], \tag{9}$$

into (4), instead of Φ_K. An actual antisymmetric Φ_K will consist of one or two terms like (9) multiplied by an isotopic spin function. The integral in (4) of individual terms, like $\psi_{j\kappa}(X_1)\cdot\psi_{j'\kappa'}(X_2)$, can be readily performed. It is:

$$\int \mathfrak{D}^J(R)_{KM}\cdot\psi_{j\kappa}(RX_1)\cdot\psi_{j'\kappa'}(RX_2)dR=\sum_{r,r'}\int \mathfrak{D}^J(R)_{KM}\mathfrak{D}^j(R)_{\kappa r}{}^*\mathfrak{D}^{j'}(R)_{\kappa' r'}{}^*dR\cdot\psi_{jr}(X_1)\cdot\psi_{j'r'}(X_2)$$

$$=\frac{h}{2J+1}\cdot C^{j\ j'\ J}_{\kappa\ \kappa'\ K}\cdot\sum_r C^{j\ \ j'\ \ J}_{r\ M-r\ M}\cdot\psi_{jr}(X_1)\cdot\psi_{j'M-r}(X_2)=\frac{h}{2J+1}\cdot C^{j\ j'\ J}_{\kappa\ \kappa'\ K}\cdot\psi^{jj'J}_{\ \ M}(X_1,X_2). \tag{10}$$

Here the expansion for $\psi_{j\kappa}(RX)$ [Wigner,[9] p. 117, Eq. (26)] and the integral for the product of three representation coefficients [Wigner,[9] p. 204, Eq. (22)] have been used. The C's are vector-addition coefficients, and $h=\int dR$. The wave function $\psi^{jj'J}{}_M$ is defined in (10); it is the wave function for two particles with angular momenta j and j' coupled to give J and M. It is plain from (10) that a general two-particle $\Psi_{MK}{}^J$ can be expanded in terms of the ordinary shell-model wave functions, and they will have the same symmetry under exchange of space, spin, and isotopic spin as Φ_K does.

A similar expansion of $\Psi_{MK}{}^J$ for the 16 particles of the double closed $1s$ and $1p$ shells, i.e., for O^{16}, will lead to the $(T,J)=(0,0)$ state of the double closed shells $1s_{\frac{1}{2}}{}^4\, 1p_{\frac{3}{2}}{}^8\, 1p_{\frac{1}{2}}{}^4$ plus small admixtures of states of higher shells.

For $A=18$, one expects an 18-particle wave function Φ_K. If, however, the expansion of $\Psi_{MK}{}^J$ in terms of shell-model wave functions is restricted to those with 16 states in the $1s$, $1p$ double closed shell, the problem is essentially reduced to the two-particle problem worked out above.

Similarly, a three-particle wave function for $A=19$ can be expanded in terms of shell-model wave functions. The wave functions for the state (T,J) may be obtained by vector addition of the wave function for the third particle to the anti-symmetric wave function of two particles in a state $[T',J']$ and subsequent antisymmetrization.[11] The expansion of $\Psi_{MK}{}^J$ follows directly from a double application of the just mentioned formula for the integral of the product of three representation coefficients, and from the theorem involving fractional parentage coefficients given in the Appendix.

It is readily seen from calculations like the above or from symmetry that $\Psi_{M-K}{}^J$, with κ for every particle multiplied by -1, is just equal, apart from a phase factor, to $\Psi_{MK}{}^J$.

The convention of reference 9 that the spherical harmonic $Y_m{}^l(\vartheta,\varphi)$ has dependence $\exp(-im\varphi)$ upon the azimuthal angle φ has been adopted in this section as well as in formula (4). The now more usual convention $\exp(+im\varphi)$, however, is used in the remainder of the present paper and in reference 5. Phase differences have been taken account of in all tables.

5. COMPARISON WITH THE SHELL MODEL

The results of the shell model for $A=18$ of reference 3 have been augmented by complete diagonalization for all (T,J) states of the $1d$, $2s$ shell of the matrices of the central Gaussian Serber two-nucleon interaction $+0.685\Delta$, where Δ is the operator which raises the $1d_{\frac{3}{2}}$ and $2s_{\frac{1}{2}}$ levels by the amounts observed in O^{17}. The introduction of the factor 0.685 instead of 1 is the only change; all other parameters are the same. This factor leads to essentially the interaction used[5] for $A=19$. It changes the wave functions only slightly. The states are denoted $(T,J)x$, where $x=a, b, c, \cdots$ denotes the order of the state, a being the lowest with these (T,J).

Modified Single-Particle Wave Functions

With the χ_κ of Table I(a), the wave functions for the deformed harmonic-oscillator well, one can form

$$\Phi_0=\frac{1}{\sqrt{2}}[\chi_{\frac{1}{2}}(1)\cdot\chi_{-\frac{1}{2}}(2)-\chi_{\frac{1}{2}}(2)\cdot\chi_{-\frac{1}{2}}(1)]\cdot\eta(1)\cdot\eta(2),$$

where η is an isotopic spin function denoting a neutron state. Let us now substitute Φ_0 into (4), and obtain, using (10), the following amplitudes for a (1,0) state:

$$(d_{5/2})^2\colon 0.51;\quad (d_{3/2})^2\colon 0.42;\quad (s_{1/2})^2\colon 0.75. \qquad (11)$$

These may be compared to the shell-model amplitudes of Table II; both sets are normalized to 1 in the $1d$, $2s$ shell. We see that the signs of the amplitudes of (11) are those of the shell-model state $(1,0)a$, but their magnitudes differ considerably. Let us next consider another single-particle wave function, φ_κ, also having eigenvalue κ of $j_{z'}$, but with different coefficients $a_{j\kappa}$:

$$\varphi_\kappa=a_{\frac{5}{2}\kappa}\psi_{\frac{5}{2}\kappa}+a_{\frac{1}{2}\kappa}\psi_{\frac{1}{2}\kappa}+a_{\frac{3}{2}\kappa}\psi_{\frac{3}{2}\kappa}. \qquad (12)$$

The coefficients for $\kappa=\frac{1}{2}$ and $-\frac{1}{2}$ may be determined so that the amplitudes (11) become precisely those of the shell-model wave function $(1,0)a$. It turns out that if states with other J's both for $A=18$ and 19 are formed from this $\varphi_{\frac{1}{2}}$ and the corresponding $\varphi_{-\frac{1}{2}}$, there is considerable similarity between the deformed-nucleus and shell-model wave functions for three other states with $A=18$ and three with $A=19$.

The modified wave function φ_κ for $\kappa=\frac{1}{2}$ in Table I(b) is not precisely the one just determined, but has amplitudes which differ from it by only a few percent. It was chosen in such a way as to give the best fit to both the $(1,0)a$ and the $(0,1)a$ shell-model wave functions. No κ's other than $\pm\frac{1}{2}$ would have been suitable, since the $(1,0)a$ shell-model wave function has a considerable $(s_{\frac{1}{2}})^2$ amplitude.

TABLE I. In part (a) the wave functions χ_κ for a very slightly deformed harmonic-oscillator well are expanded in terms of wave functions $\psi_{j\kappa}$ of a particle in a spherical harmonic-oscillator well. The subscripts $j=\frac{5}{2}$, $\frac{3}{2}$, and $\frac{1}{2}$ denote $1d_{5/2}$, $1d_{3/2}$ and $2s_{1/2}$ states. In part (b) the modified wave functions, φ_κ, (see Sec. 5) are expanded in terms of $\psi_{j\kappa}$'s. All wave functions are normalized to 1 in the $1d$, $2s$ shell. The order of coupling is s first and then l to form j.

κ	(a) Deformed harmonic-oscillator wave function, χ_κ	(b) Modified wave function, φ_κ
$\frac{5}{2}$,	$\psi_{\frac{5}{2}\frac{5}{2}}$	
$\frac{3}{2}'$	$0.45\psi_{\frac{5}{2}\frac{3}{2}} \quad -0.90\psi_{\frac{3}{2}\frac{3}{2}}$	
$\frac{1}{2}'$	$0.45\psi_{\frac{5}{2}\frac{1}{2}}+0.82\psi_{\frac{1}{2}\frac{1}{2}}+0.37\psi_{\frac{3}{2}\frac{1}{2}}$	$0.44\psi_{\frac{5}{2}\frac{1}{2}}+0.89\psi_{\frac{1}{2}\frac{1}{2}}+0.10\psi_{\frac{3}{2}\frac{1}{2}}$
$\frac{3}{2}$	$0.90\psi_{\frac{5}{2}\frac{3}{2}} \quad +0.45\psi_{\frac{3}{2}\frac{3}{2}}$	$0.96\psi_{\frac{5}{2}\frac{3}{2}} \quad -0.29\psi_{\frac{3}{2}\frac{3}{2}}$
$\frac{1}{2}''$	$0.63\psi_{\frac{5}{2}\frac{1}{2}} \quad -0.78\psi_{\frac{3}{2}\frac{1}{2}}$	
$\frac{1}{2}$	$0.63\psi_{\frac{5}{2}\frac{1}{2}}-0.58\psi_{\frac{1}{2}\frac{1}{2}}+0.52\psi_{\frac{3}{2}\frac{1}{2}}$	$0.80\psi_{\frac{5}{2}\frac{1}{2}}-0.44\psi_{\frac{1}{2}\frac{1}{2}}+0.40\psi_{\frac{3}{2}\frac{1}{2}}$

[11] See, for example, reference 5, Sec. 2.

This holds *a fortiori* for (1,0)b, and one can choose configuration $\frac{1}{2}, -\frac{1}{2}'$ (denoting κ_1, κ_2) for this state and determine the modified wave function $\varphi_{\frac{1}{2}'}$ in a similar way. For $J=0$, its component K must also be 0. The $\varphi_{\frac{1}{2}'}$ of Table I(b) was determined as that wave function which is orthogonal to $\varphi_{\frac{1}{2}}$ and fits best (1,0)b as well as several other states. In a similar way it was necessary to introduce $\varphi_{\frac{3}{2}}$ to account for the states $(\frac{3}{2}, \frac{5}{2})$ and $(\frac{3}{2}, \frac{3}{2})$ at $A=19$. Configurations with $\frac{1}{2}$, $-\frac{1}{2}$, and $\frac{1}{2}'$ states lead to zero amplitudes for $(d_{5/2})^3$ for both $(\frac{3}{2}, \frac{5}{2})$ and $(\frac{3}{2}, \frac{3}{2})$. This is in sharp contradiction to the shell model results in which this is the dominant configuration for both states (Table IV).

All three modified wave functions φ_κ resemble deformed harmonic-oscillator-well functions χ_κ. For each φ_κ the $d_{3/2}$ amplitude is smaller than for the similar χ_κ. This is surely an indication of $d_{5/2}-d_{3/2}$ splitting. In $\varphi_{\frac{3}{2}}$, the small $d_{3/2}$ amplitude has opposite sign from that in $\chi_{\frac{3}{2}}$. No way to represent all the changes by a change in the single-particle Hamiltonian H' is apparent.

The empirical level order is given schematically in Fig. 2(b). It appears to be, for modified states:

$$\pm\tfrac{1}{2}, \pm\tfrac{1}{2}' \quad \text{for} \quad A=18; \quad \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \pm\tfrac{1}{2}' \quad \text{for} \quad A=19.$$

It is surprising that the $\varphi_{\frac{1}{2}}$ resembles most the $\chi_{\frac{1}{2}}$ for the lowest state of a prolate (cigar-shaped) spheroid, while an oblate spheroid would be expected at the beginning of a new shell. Possibly the deformation is oblate at $A=17$, but changes with the addition of another particle.

The shell-theoretical wave function for (1,0)a could also be obtained from rotation of particles with spheroidal or modified spheroidal wave functions in L-S coupling, like those of (7) with $\mu=0$. Since (1,0)a contains both ^1S and ^3P states, however, it would be necessary to use two configurations with both $\mu=0$ and $\mu=1$ states, so that the description becomes more complicated and less natural to the problem.

$A=18$

The single-particle wave functions for a slightly deformed $1d$, $2s$ shell lead, when substituted into (4), to more than 100 states for each of $T=0$ and 1. Actually there are only 14 independent states for each T in the $1d$, $2s$ shell. Therefore, there must be many linear relations between wave functions of these states. For instance, the shell-theoretical wave function for (0,3)b is given fairly well by either $\frac{1}{2}, \frac{1}{2}'$ or $\frac{3}{2}, -\frac{1}{2}$. Using the notation $|\kappa_1\kappa_2; (T,J)x\rangle$ for the normalized state vector of a deformed-nucleus state (with nonzero amplitudes assumed to be only in the $1d$, $2s$ shell), we find that the scalar product of the state vectors for these two (0,3)b states is

$$\langle\tfrac{3}{2}, -\tfrac{1}{2}; (0,3)b|\tfrac{1}{2}, \tfrac{1}{2}'; (0,3)b\rangle = 0.89.$$

The shell-theoretical wave functions are compared with those for the particles with modified deformed well wave functions in Table II. The amplitudes of the projections of $\Psi_{MK}{}^J$ on the shell-theoretical states of the $1d$, $2s$ shell are placed next to those of the shell-theoretical wave functions. Both types of wave functions are normalized to 1 in the $1d$, $2s$ shell. The numbers κ_1, κ_2 for the seven states of Table II, and some others for which their assignment is less certain are given in Table III, together with energies from shell-theoretical calculations, and the quantity δ^2. This is defined by

$$\begin{aligned}\delta^2 &= \big|\,|\kappa_1\kappa_2\rangle - \langle\text{S.M.}|\kappa_1\kappa_2\rangle|\text{S.M.}\rangle\big|^2 \\ &= 1 - |\langle\text{S.M.}|\kappa_1\kappa_2\rangle|^2.\end{aligned} \tag{13}$$

Here, as before, $|\kappa_1\kappa_2; (T,J)x\rangle = |\kappa_1,\kappa_2\rangle$ represents a state of the deformed nucleus, and $|\text{S.M.}\rangle$ the corresponding shell-model state. The first absolute value sign gives the length of the state vector.

Quasi-Rotational Series

It has recently been suggested[12] on the basis of the collective model that even at $A=19$ there exist series of

TABLE II. $A=18$. Comparison of the wave functions of the model for a deformed nucleus (labelled D.N.), using the modified wave functions φ_κ of Table I(b), with those of shell-model calculations for a two-nucleon force between the outer nucleons (labelled S.M.). Serber (S) forces[a] and Rosenfeld (R) forces[b] were used for the shell-model wave functions. The letters $(T,J)x$ stand for isotopic spin, total angular momentum, and the order $(a,b,c,\cdots)$ starting with the lowest state. The configuration $\kappa_1\kappa_2$ of the D.N. states is also given.

	State (1,0)a			(1,0)b	
Configuration	S.M. S	S.M. R	D.N. $\frac{1}{2}, -\frac{1}{2}$	S.M. S	D.N. $\frac{1}{2}, -\frac{1}{2}'$
$(d_{5/2})^2$	0.86	0.89	0.85	−0.40	−0.46
$(d_{3/2})^2$	0.31	0.24	0.26	−0.06	−0.06
$(s_{1/2})^2$	0.40	0.39	0.45	0.91	0.88

	State (1,2a)	
Configuration	S.M. S	D.N. $\frac{1}{2}, -\frac{1}{2}$
$(d_{5/2})^2$	0.71	0.55
$(d_{3/2})^2$	0.14	0.16
$d_{5/2}d_{3/2}$	−0.20	−0.24
$d_{5/2}s_{1/2}$	0.64	0.70
$d_{3/2}s_{1/2}$	0.20	0.35

	State (0,1)a			(0,1)b	
Configuration	S.M. S	S.M. R	D.N. $(\frac{1}{2})^2$	S.M. S	D.N. $\frac{1}{2}, \frac{1}{2}'$
$(d_{5/2})^2$	0.67	0.58	0.73	−0.52	−0.40
$d_{5/2}d_{3/2}$	0.56	0.57	0.39	−0.04	−0.16
$(s_{1/2})^2$	0.46	0.55	0.43	0.84	0.87
$(d_{3/2})^2$	−0.16	−0.19	−0.23	0.10	0.06
$d_{3/2}s_{1/2}$	0.02	−0.02	0.28	−0.13	0.24

	State (0,3)a		(0,3)b	
Configuration	S.M. S	D.N. $(\frac{1}{2})^2$	S.M. S	D.N. $\frac{1}{2}, \frac{1}{2}'$
$(d_{5/2})^2$	0.62	0.61	0.76	0.52
$d_{5/2}d_{3/2}$	0.22	0.11	0.02	0.07
$(d_{3/2})^2$	−0.05	−0.23	−0.07	−0.09
$d_{5/2}s_{1/2}$	0.75	0.75	−0.64	−0.85

[a] M. G. Redlich, reference 3, extended to include higher states. The term Δ is changed to 0.685Δ.
[b] J. P. Elliott and B. H. Flowers, reference 4.

[12] E. B. Paul, Phil. Mag. 2, 311 (1957); G. Rakavy, Nuclear Phys. 4, 375 (1957).

TABLE III. $A=18$. Summary of all states with calculated interaction energy E negative. E follows from shell theory with internucleon interaction of Serber type. The numbers κ_1 and κ_2 are assigned from the D.N. model. $K=\kappa_1+\kappa_2$. The quantity δ^2 is defined by Eq. (13).

State (T,J)	Calculated energy E, in Mev	Calculated energy above ground state of O^{18}, in Mev	Observed energy[a] in Mev	κ_1	κ_2	K	δ^2
(1,0)*a*	−5.35	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0.02
(1,2)*a*	−2.38	2.97	1.98	$\frac{1}{2}$	$-\frac{1}{2}$	0	0.04
(1,0)*b*	−1.43	3.92		$\frac{1}{2}$	$-\frac{1}{2}'$	0	0.02
(1,4)*a*	−1.13	4.22	3.55	$\frac{1}{2}$	$-\frac{1}{2}$	0[b]	0.18
(1,2)*b*	−0.22	5.13		$\frac{1}{2}$	$-\frac{1}{2}'$	0	0.23
(0,1)*a*	−4.85	0.50	−1.3[c]	$\frac{1}{2}$	$\frac{1}{2}$	1	0.10
(0,3)*a*	−3.27	2.08		$\frac{1}{2}$	$\frac{1}{2}$	1	0.05
(0,5)	−2.84	2.51		$\frac{1}{2}$	$\frac{1}{2}$	1[b]	0
(0,1)*b*	−1.39	3.96		$\frac{1}{2}$	$\frac{1}{2}'$	1	0.17
(0,2)*a*	−1.22	4.13		$\frac{1}{2}$	$\frac{1}{2}$	1[b]	0.07
(0,3)*b*	−0.56	4.79		$\frac{1}{2}$	$\frac{1}{2}'$	1	0.11
(0,1)*c*	−0.16	5.19				(1)[b,d]	

[a] O. M. Bilaniuk and P. V. C. Hough, reference 15; F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1955).
[b] Assignment uncertain.
[c] This number = the experimental energy difference + (n-p) mass difference − Coulomb energy difference between ground states of F^{18} and O^{18}. It is thus the energy difference due to nuclear forces only.
[d] Possibly the first member of a third series with $K=1$.

levels similar to the rotational series which are well known in the rare earth and trans-Pb regions and were also discovered[13] at and around $A=25$.

The following general but not necessarily universal rules are suggested by the present model and the results for the oxygen region.

1. *A nuclear state may be considered as a member of the quasi-rotational series.*—That is, suppose that a state α may be obtained from A particles, with modified deformed-well wave functions, rotated to give the J of the state α. Then, using (4) one can also form from this A-particle configuration states with other values of J. The set of these states is called a quasi-rotational series. Under particular circumstances, we expect them to form an ordinary rotational series, with energies proportional to $J(J+1)$.

For low positive-parity states at $A=18$, a series is specified by κ_1, κ_2. The lowest ones for $T=1$ are

κ_1	κ_2	K	J
$\frac{1}{2}$	$-\frac{1}{2}$	0	0, 2, 4
$\frac{1}{2}$	$-\frac{1}{2}'$	0	0, 1, 2, 3, 4
$\frac{3}{2}$	$-\frac{1}{2}$	1	1, 2, 3, 4

Higher series occur for higher single-particle states. Also, there may be higher J values associated with each series, due to the higher states which occur in a single-particle wave function for a deformed well. The J's in the table just given can all be obtained from shell-model configurations of the $1d$, $2s$ shell alone. The series with two states which differ only in sign of κ do not lead to odd J, as may be seen by direct calculation, using (4) and (10). The amplitudes of states with odd J are then simply zero. Configurations like $\frac{1}{2}$, $-\frac{1}{2}'$, on the other hand, do lead to odd-J states.

For $T=0$, the low series at $A=18$ corresponding to configurations of the $1d$, $2s$ shell are

κ_1	κ_2	K	J
$\frac{1}{2}$	$\frac{1}{2}$	1	1, 2, 3, 4, 5
$\frac{1}{2}$	$\frac{1}{2}'$	1	1, 2, 3, 4, 5
$\frac{1}{2}$	$-\frac{1}{2}$	0	1, 3, 5

This time, even-J states are forbidden for $\frac{1}{2}$, $-\frac{1}{2}$, though not for $(\frac{1}{2})^2$.

2. *A single deformed-nucleus configuration describes many low-lying nuclear states fairly accurately.*—That is, configuration interaction is small near the ground state, as is seen from the columns of δ^2 in Table III. As the energy increases, however, discrepancies between shell-model and deformed-nucleus wave functions become larger. This may be an indication of inaccuracy of the shell-model wave functions, or of increased configuration interaction in the deformed-nucleus model.

It might be suspected that such configuration interaction would occur only between states of a given K. A calculation based on the collective model,[14] however, has suggested that mixing of states with different K's does occur for W^{183}.

The energies calculated from shell theory for the members of a given series and also the experimental energies[15] for the $\frac{1}{2}$, $-\frac{1}{2}$ series with $T=1$ in O^{18} are not proportional to $J(J+1)$. This might be due at least in part to interaction between different $\kappa_1\kappa_2$ configurations.

3. *The levels of a quasi-rotational series usually appear in order of ascending J.*—This seen for $A=18$ in Table III. For low levels at $A=19$, however, this rule is violated (Table V). It is known empirically and from the deformed-core models that $K=\frac{1}{2}$ series are anomalous.

$A=19$

The wave functions for the $J=\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$ states both for $T=\frac{3}{2}$ and $\frac{1}{2}$ and for both models are given in Table IV. Close correspondence again generally exists, even though the total number of amplitudes is very much larger than for $A=18$.

The results for the wave functions of Table IV, together with the calculated and observed energies, are summarized in Table V. We note a coupling rule, analogous to a Mayer-Jensen simple coupling rule:

4. *Like particles fill levels of the deformed nucleus in pairs for the low states. Thus, two like particles fill states κ and $-\kappa$ to give $K'=0$; a third particle then fills the lowest remaining state.*

One sees in Table V that the shell-theoretical calculation for $T=\frac{3}{2}$ leads to a ground state with $K=\frac{3}{2}$ and

[13] Litherland, Bartholomew, Paul, and Gove, Phys. Rev. **102**, 208 (1956).

[14] A. K. Kerman, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **30**, No. 15 (1956).

[15] O. M. Bilaniuk, University of Michigan dissertation, 1957 (unpublished); O. M. Bilaniuk and P. V. C. Hough, Phys. Rev. **108**, 305 (1957).

TABLE IV. $A=19$. Comparison of the deformed-nucleus (D.N.) wave functions[a] with those for the shell model (S.M.). The interaction is of Serber (S) type. The D.N. configuration $\kappa_1\kappa_2\kappa_3$ is given. The amplitudes of the lowest three states for $T=\frac{3}{2}$ and $T=\frac{1}{2}$ are listed in this table. When more than one state of a configuration $(j_1j_2)j_3$ exists, each state is specified by its parent state, $j_1j_2[T',J']$.

Configuration	$[T',J']$	State $(T,J)=(\frac{3}{2},\frac{5}{2})$ S.M. S	D.N. $\frac{1}{2},-\frac{1}{2},\frac{3}{2}$	$(\frac{3}{2},\frac{1}{2})$ S.M. S	D.N. $\frac{1}{2},-\frac{1}{2},\frac{1}{2}'$
$(d_{5/2})^3$		0.85	0.83	0.95	0.96
$(d_{5/2})^2s_{1/2}$		−0.04	0.08		
$(s_{1/2})^2d_{5/2}$		−0.43	−0.36		
$(d_{5/2})^2d_{3/2}$	[1,2]	−0.09	−0.21	0.06	0.05
	[1,4]	−0.06	−0.04		
$(d_{5/2}d_{3/2})s_{1/2}$	[1,2]	0.03	−0.11	−0.06	−0.06
	[1,3]	0.04	−0.22		
$(d_{3/2})^2d_{5/2}$	[1,0]	−0.28	−0.21	−0.05	−0.03
	[1,2]	0.04	0.01		
$(d_{3/2})^2s_{1/2}$		0.02	−0.13	0.30	0.26

Configuration	$[T',J']$	State $(\frac{3}{2},\frac{3}{2})$ S.M. S	D.N. $\frac{1}{2},-\frac{1}{2},\frac{3}{2}$	$[T',J']$	$(\frac{1}{2},\frac{1}{2})$ S.M. S	D.N. $\frac{1}{2},-\frac{1}{2},\frac{1}{2}$
$(d_{5/2})^3$		0.73	0.71		0.30	0.42
$(d_{5/2})^2s_{1/2}$		0.65	0.60	[0,1]	−0.37	−0.41
				[1,0]	0.52	0.60
$(s_{1/2})^3$					0.55	0.26
$(d_{5/2})^2d_{3/2}$	[1,0]	−0.01	−0.20	[1,2]	0.17	0.09
	[1,2]	0.07	0.19	[0,1]	−0.08	−0.10
$(d_{5/2}d_{3/2})s_{1/2}$	[1,1]	0.12	0.13	[1,1]	0.01	0
	[1,2]	0.08	0.08	[0,1]	−0.33	−0.38
$(s_{1/2})^2d_{3/2}$		0.10	−0.11		−0.00	0
$(d_{3/2})^2d_{5/2}$		−0.14	−0.09	[1,2]	0.12	0.15
				[0,3]	0.01	0.03
$(d_{3/2})^2s_{1/2}$		0.04	0.04	[1,0]	0.22	0.18
				[0,1]	0.09	0.08
$(d_{3/2})^3$		0.01	−0.09		0.03	0.06

Configuration	$[T',J']$	State $(\frac{1}{2},\frac{5}{2})$ S.M. S	D.N. $\frac{1}{2},-\frac{1}{2},\frac{1}{2}$	$[T',J']$	$(\frac{1}{2},\frac{3}{2})$ S.M. S	D.N. $\frac{1}{2},-\frac{1}{2},\frac{1}{2}$
$(d_{5/2})^3$	[1,0]	0.69	0.62		0.46	0.38
	⊥[b]	0.15	0.13			
$(d_{5/2})^2s_{1/2}$	[0,3]	−0.22	−0.31	[1,2]	0.22	0.36
	[1,2]	0.23	0.31	[0,1]	−0.11	−0.15
$(s_{1/2})^2d_{5/2}$	[0,1]	−0.25	−0.24		0.32	0.37
	[1,0]	0.36	0.35			
$(d_{5/2})^2d_{3/2}$	[1,2]	0.13	0.11	[1,0]	−0.54	−0.48
	[1,4]	0.20	0.17	[1,2]	−0.29	−0.24
	[0,1]	0.13	0.18	[0,1]	−0.28	−0.23
	[0,3]	0.10	0.09	[0,3]	−0.14	−0.10
$(d_{5/2}d_{3/2})s_{1/2}$	[1,2]	−0.08	−0.11	[1,1]	0.00	0
	[1,3]	0.01	0	[1,2]	−0.09	−0.16
	[0,2]	−0.07	−0.13	[0,1]	0.06	0.12
	[0,3]	−0.09	−0.14	[0,2]	0.13	0.23
$(s_{1/2})^2d_{3/2}$		0.08	0.16	[1,0]	−0.19	−0.25
				[0,1]	−0.09	−0.11
$(d_{3/2})^2d_{5/2}$	[1,0]	0.29	0.21	[1,2]	0.09	0.07
	[1,2]	0.07	0.10	[0,1]	−0.09	−0.05
	[0,1]	0.11	0.10	[0,3]	0.06	0.05
	[0,3]	0.04	0.04			
$(d_{3/2})^2s_{1/2}$	[1,2]	0.05	0.09	[1,2]	0.08	0.10
	[0,3]	0.01	0.02	[0,1]	0.07	0.10
$(d_{3/2})^3$		0.02	0.03		−0.20	−0.13

[a] Reference 5, extended to include states with $J=\frac{3}{2}$.
[b] The states [1,0] and [0,1] here are not orthogonal. The symbol ⊥ denotes the state orthogonal to [1,0]. [See reference 5, Eq. (14).]

$J=\frac{5}{2}$. This contradicts Rule 3, which leads to a ground state with $J=\frac{3}{2}$. The spin of this state is not known experimentally, but it must be either $\frac{3}{2}$ or $\frac{5}{2}$, since it β decays with allowed transitions to two states of F^{19} with spins $\frac{3}{2}$ and $\frac{5}{2}$.

There are five independent parameters for the three modified wave functions φ_κ; these account for six wave functions at $A=19$ with 75 amplitudes, in addition to seven wave functions at $A=18$ with 22 amplitudes.

Comparison with Experiment

a. Energies.—The wave functions for the shell-model calculation lead to the best energies for the double closed shell-core model. The wave functions of the present model differ only slightly from them. They undoubtedly correspond to a somewhat different Hamiltonian. It has been suggested[7] that nuclear deformation may be an effect of tensor forces. The present wave functions may very well be quite accurate for a Hamiltonian with a central+tensor interaction between each pair of particles. Tensor forces lead to a reduction in amplitudes for configurations $(2s_{1/2})^2$ and $(2s_{1/2})^3$ in low states; the $(2s_{1/2})^3$ amplitude is indeed low for the deformed-nucleus wave function, as is seen in Table IV. The discrepancy in $1d_{3/2}2s_{1/2}$ amplitudes for the states $(0,1)a$ and $(0,1)b$ may also be due to neglect of tensor forces in the shell-model calculation, since for central forces, there is no interaction between $d^2\,{}^3S$ and $ds\,{}^3D$.

The amplitudes of $(1d_{5/2})^2$ and $1d_{5/2}2s_{1/2}$ for state $(1,2)a$ have recently been derived from experimental data on the $O^{17}(d,p)$ cross sections.[15] They are 0.81 ±0.05 and 0.48±0.05. This compares with 0.55 and 0.70 for the D.N. wave function (Table II). The discrepancy may be an indication both of too high an amplitude for $2s_{1/2}$ in the $\kappa=\frac{1}{2}$ wave function and of interaction between D.N. configurations.

A central-force+spin-orbit splitting calculation generally leads to small but nonzero admixtures of $(1f_{7/2})^2$. This is not in accord with the deformed-nucleus model

TABLE V. $A=19$. Summary of the three lowest states of $T=\frac{3}{2}$ and $T=\frac{1}{2}$ types. The interaction energy E is calculated from shell theory. The numbers $\kappa_1, \kappa_2, \kappa_3$ specify the deformed-nucleus configuration and $K=\kappa_1+\kappa_2+\kappa_3$. All wave functions are given in Table IV. The quantity δ^2 is defined by Eq. (13). It is noteworthy that two single-particle states are $\frac{1}{2}, -\frac{1}{2}$ for each of these states.

State (T,J)	Calculated energy E in Mev	Calculated energy above ground state of F^{19} (Mev)	Observed energy[a] (Mev)	κ_1	κ_2	κ_3	K	δ^2
$(\frac{3}{2},\frac{5}{2})$	− 8.59	8.46	7.7[b,c]	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	0.15
$(\frac{3}{2},\frac{1}{2})$	− 8.02	9.03	9.2[c]	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}'$	$\frac{1}{2}$	0.00
$(\frac{3}{2},\frac{3}{2})$	− 6.58	10.47		$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	0.10
$(\frac{1}{2},\frac{1}{2})$	−17.05	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0.10
$(\frac{1}{2},\frac{5}{2})$	−16.89	0.16	0.20	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0.03
$(\frac{1}{2},\frac{3}{2})$	−15.40	1.65	1.57	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0.08

[a] F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1955).
[b] The spin of the ground state of O^{19} is either $\frac{5}{2}$ or $\frac{3}{2}$.
[c] This represents the energy difference due to nuclear forces. The Coulomb energy difference and (p-n) mass difference have been added to the experimental energy difference.

of the present paper. It would require the addition of a term for $\psi_{j\kappa}$ with $j=\frac{7}{2}$ in formula (12) for the modified wave function φ_κ; and orbital angular momenta $l=0$, 2, and 3 would all appear in φ_κ. Parity would not be conserved in φ_κ, and this would lead in general to states with different parities in the expansion of a two-particle $\Psi_{MK}{}^J$ also. If both the parity of $\Psi_{MK}{}^J$ and the reflection symmetry of Φ_K in the $x'y'$ plane should be preserved, these considerations suggest the possibility that the two-nucleon interaction be such a combination of central and tensor forces as to connect only states which on the unperturbed spherical harmonic-oscillator model belong to levels $2\hbar\omega$, $4\hbar\omega$, $\cdots$ apart ($\hbar\omega$ = energy interval between successive levels). On the other hand, it may be that the $(1f_{7/2})^2$ amplitude which surely does appear at least for most force combinations indicates a limit to the accuracy of the D.N. model.

The recent shell-model calculations[6] for Pb^{206}, which lead to excellent agreement with experiment, are consistent with tensor forces, since for the low states of the two neutron holes, singlet-even forces play the major rôle. For zero-range forces, more accurate here than in the oxygen region, odd states have zero interaction energy.

b. Other quantities.—A few experimental quantities have been calculated with the wave functions of the D.N. model. They are given in Table VI and compared with shell-model results and experiment. γ-ray lifetimes depend very sensitively upon higher configurations and have not been included. The high ft values for the two β transitions from the O^{19} ground state probably indicate that the D.N. wave function for this state is not very accurate.

TABLE VI. Comparison of the deformed-nucleus model (D.N.) with the shell model (S.M.) and experiment. μ is a magnetic moment.

	S.M.	D.N.	Experiment[a]
ft value for $F^{18}\rightarrow O^{18}$	2890	3430	4170±330
ft value for $Ne^{19}\rightarrow F^{19}$	1590	1760	1700±170
ft value for $O^{19}\rightarrow F^{19}(\frac{5}{2}+)$	373 000	511 000	335 000±100 000
ft value for $O^{19}\rightarrow F^{19}(\frac{3}{2}+)$	49 400	72 000	21 400±6500
$\mu(F^{19}, \frac{1}{2}+)$	2.94	2.91	2.628 nm
$\mu(F^{19}, \frac{5}{2}+)$	3.74	3.68	3.50±0.24 nm

[a] F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1955). $\mu(F^{19}, \frac{5}{2}+)$: W. R. Phillips and G. A. Jones, Phil. Mag. **1**, 576 (1956).

6. CONCLUSION

The results of Sec. 5 indicate that a wave function specified by $\kappa_1\cdots,\kappa_A$ with individual-particle states resembling those of a simple deformed well is a good approximation probably (1) to the wave function for the double closed shell-core model and possibly (2) to the actual wave function of some low-lying nuclear states. It may be that superpositions of states of two or more configurations of the deformed nucleus, perhaps all having the same K, would account for many other states.

The essential feature of the model is the existence of an axis in the nucleus about which the density of nuclear matter is cylindrically symmetric. The nuclear shape is surely not spherical. The single-particle wave functions are not precisely those for a very small ellipsoidal deformation; however, one characteristic of a spheroidal model, the symmetry of the quantum-mechanical probability about an axis fixed in the nucleus, is maintained.

From the model of particles moving in a deformed well, it is plain that the amplitudes for higher spherical well states in the expansion of the deformed-nucleus wave function increase with increased deformation. One expects this to be so also for the present model with modified single-particle wave functions. The small admixtures of higher shell-model configurations obtained for nuclear force calculations at $A=18$ and 19 then are an indication of low deformations.

This model can yield the observed rotational series with energies proportional to $J(J+1)$, for heavy nuclei.[16]

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APPENDIX: A THEOREM

Given: (1) A configuration $C=j_1j_2j_3$ with a complete set of p antisymmetric orthonormal wave functions, each with (T,J). They are $\psi(\alpha_1,TJ)$, $\cdots$, $\psi(\alpha_p,TJ)$. The α's specify all other parameters, including C. (2) An antisymmetrization operator A on any three-particle function $\chi(1,2,3)$, defined by

$$A\chi(1,2,3)=\chi(1,2,3)-\chi(3,2,1)-\chi(1,3,2). \tag{A1}$$

(3) Functions

$$\varphi[(C_kT'J')_a j_k(3),TJ] \tag{A2}$$

for particles 1 and 2 in an antisymmetric (a) state of configuration $C_k=Cj_k^{-1}$ with $[T',J']$, and particle 3 in a state $t=\frac{1}{2}$, j_k.

[16] R. E. Peierls and J. Yoccoz, Proc. Phys. Soc. (London) **A70**, 381 (1957); J. Yoccoz, *ibid.* **A70**, 338 (1957).

To prove: that

$$A\varphi[(C_kT'J')_a j_k(3),TJ]=3\sum_{i=1}^{p}(C_kT'J';j_k]\!]\alpha_iTJ)\cdot\psi(\alpha_i,TJ), \tag{A3}$$

where $(C_kT'J';j_k]\!]\alpha_iTJ)$ is a fractional parentage coefficient.

Proof.—The usual expansion of $\psi(\alpha_i,TJ)$ is

$$\psi(\alpha_i,TJ)=\sum_k{}'\sum_{T'J'}\varphi[(C_kT'J')_a j_k(3),TJ](C_kT'J';j_k]\!]\alpha_iTJ), \tag{A4}$$

where the first sum is over all distinct j_k. Since the set of $\psi(\alpha_i,TJ)$ is complete, the wave function (A3) can be expanded as

$$A\varphi[(C_kT'J')_a j_k(3),TJ]=\sum_{i=1}^{p}b(\alpha_i)\cdot\psi(\alpha_i,TJ), \tag{A5}$$

where

$$b(\alpha_i)=\int\psi(\alpha_i,TJ)^*\cdot A\varphi[(C_kT'J')_a j_k(3),TJ]d\tau_1d\tau_2d\tau_3. \tag{A6}$$

The integration in (A6) is over the coordinates and spins of the three particles. Introducing the symmetrization operator S, analogous to A of (A1), and recalling that $\psi(\alpha_i,TJ)$ is antisymmetric, we obtain

$$b(\alpha_i)=\int S\,\{\psi(\alpha_i,TJ)^*\varphi[(C_kT'J')_a j_k(3),TJ]\}d\tau_1d\tau_2d\tau_3, \tag{A7}$$

which, from (A4), equals just three times the same integral without S, and hence

$$b(\alpha_i)=3(C_kT'J';j_k]\!]\alpha_iTJ), \tag{A8}$$

which proves the theorem.

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Prompt Neutron Emission from Spontaneous-Fission Modes of Cf^{252}†

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The number of prompt neutrons emitted and the velocities of the fragment pairs have been measured for individual spontaneous fissions of Cf^{252}. The time-of-flight measurements of the fragment velocities have sufficient resolution to provide a good determination of the mode of fission as characterized by the total kinetic energy E_K and the mass ratio R_A of the fragments. The neutrons are detected with high efficiency in a large cadmium-loaded liquid scintillator. It is found that the dependence of the average number of neutrons per fission, $\bar{\nu}$, on the parameters E_K and R_A may be approximated by a plane $\bar{\nu}(E_K,R_A)$ over the region that includes the majority of the fission events. The slopes that specify the orientation of this plane are determined to be $\partial\bar{\nu}(E_K,R_A)/\partial E_K=-0.143\pm0.020$ (neutrons/fission)/Mev and $\partial\bar{\nu}(E_K,R_A)/\partial R_A=-6.3\pm1.1$ (neutrons/fission)/unit mass ratio. The value of the first slope indicates that the average total excitation energy of the fragments, required for the emission of one more neutron on the average, is 7.0 ± 1.0 Mev. From this number and the measured dependence of $\bar{\nu}$ on mass ratio, the average excitation energy $\bar{E}_X$ of the fragments is determined as a function of the mass ratio. This function $\bar{E}_X(R_A)$ and the measured dependence $\bar{E}_K(R_A)$ determine the average prompt energy of fission as a function of mass ratio. The widths of the neutron-number distributions have been obtained as functions of E_K and R_A. The data do not support the statistical theory of fission proposed by Fong.

I. INTRODUCTION

THE fission process, even for but one species of a spontaneously fissioning nucleus, yields a large variety of fragment nuclei and associated neutron and gamma radiations. Experimental technique has developed to the point where it has become feasible to investigate correlations that may exist between modes of fission and the associated radiations. Measurements of the velocity distribution of fission-fragment pairs from Cf^{252} and the coincident gamma-ray spectrum have recently been reported.[1] Studies of the neutron-emission probability as a function of the mode of fission

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[1] J. C. D. Milton and J. S. Fraser, Bull. Am. Phys. Soc. Ser. II, 2, 197 (1957).