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Connection between the Nuclear Shell Model and the Unified Model*

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The main point of this paper is to discuss the connection between the unified model and a modified form of the nuclear shell model. In the latter, the effective interactions between nucleons, i.e., those not included in the central average potential, are assumed to be factorable. It can be argued that inside nuclear matter the Pauli principle greatly suppresses the off-diagonal effects of the interactions, i.e. it prohibits most inelastic collisions; so that the nucleons move nearly freely within the nucleus, at least as far as low-energy phenomena are concerned. Short-range correlations due to the interactions are not suppressed, but these are expected to manifest themselves at rather high energies.

According to this view, the effective interactions between nucleons occur mainly at the nuclear surface and give rise to surface oscillations. While quadrupole oscillations predominate, higher modes also arise in a natural way. Under certain conditions, the problem of particles subject to mutual interactions may be solved by introduction of additional collective variables, which is the method of the unified model. Physically, the nuclear motion separates, at least approximately, into intrinsic and collective motions. The resulting wave functions are very similar to the ones used by Bohr and Mottelson, except that they are integrated over the collective variables. Also, the energy spectra are of the form obtained by Bohr and Mottelson, but the collective excitation spectrum is cut off, i.e., only a finite number of states occur.

This method is applied to a simplified two-dimensional nuclear model. The particles are assumed to move in an isotropic harmonic oscillator potential, and to be in addition subject to (a) one-body spin-orbit forces, and (b) mutual quadrupole-quadrupole interactions. In the absence of spin-orbit coupling, the spectrum separates into a series of rotational bands, while in the absence of mutual interactions we have an independent-particle spectrum. The intermediate coupling problem is also treated in the hope that it may provide some insight into the competition between independent-particle and collective motions in nuclei. In the present example, the transition between the two limiting schemes occurs quite suddenly, in agreement with the experimental evidence.

Another case of interest is the situation at the beginning of the nuclear $1p$ shell. In the limit of pure LS coupling, the states of maximum spatial symmetry form a rotational band, though only very few members can appear, and collective effects are not pronounced, because of the small number of particles involved. The effect of spin-orbit coupling can also be described in the language of the unified model.

INTRODUCTION

In recent years two nuclear models have had considerable success in accounting for a large variety of low-energy data. The shell model,[1,2] in which the nucleons are treated as moving independently in a central average potential, except for some residual two-body interactions, has generally worked well for nuclei containing only a few nucleons outside of (or missing from) closed shell configurations. On the other hand, in nuclei far removed from closed shell configurations, some of the low-lying excitations are most easily described as collective, i.e., they involve a sizable number of nucleons. Rotational states[3–5] are the best known examples of this phenomenon. Collective excitations as well as those involving single nucleons can both be treated by the unified model,[5–9] in which the particles are assumed to move essentially independently in the nuclear potential, but this potential is noncentral and variable in time. A special version of the unified model is the rotational model, in which the shape of the potential is regarded as fixed and only its orientation is allowed to vary. A rather basic assumption of the unified model is that the shape and orientation of the nuclear potential change slowly in comparison to the characteristic frequencies of individual nucleons.

While the shell model and unified model may appear, on first sight, to be quite different, there are good reasons to believe that these models are basically equivalent, being merely different ways of describing the same phenomena. A striking case in point is the fact that many features of the F^{19} level scheme can be accounted for by the shell model[10] and by the rotational model.[11]

The purpose of the present paper is to discuss the general connection between these models. The view is toward development of a single nuclear model which can describe the low-energy features of nuclei near to, as well as far from, magic numbers. In Sec. 1 we discuss the qualitative features of the effective two-body interactions, i.e., that part of the nuclear interactions which is not included in the average central potential. These effective interactions can be described either in the language of the shell model or of the unified model and a connection between the two approaches is dis-

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[1] M. G. Mayer and J. H. D. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley and Sons, Inc., New York, 1955); E. Feenberg, *Shell Theory of the Nucleus* (Princeton University Press, Princeton, 1955).

[2] J. P. Elliott and A. M. Lane, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39.

[3] A. Bohr and B. R. Mottelson, in *Beta- and Gamma-Ray Spectroscopy*, edited by K. Siegbahn (North-Holland Publishing Company, Amsterdam, 1955), Chap. 17.

[4] N. P. Heydenburg and G. M. Temmer, Ann. Rev. Nuclear Sci. **6**, 77 (1956).

[5] Alder, Bohr, Huus, Mottelson, and Winther, Revs. Modern Phys. **28**, 432 (1956).

[6] A. Bohr, Kgl. Danske-Videnskab. Selskab, Mat.-fys. Medd, **26**, No. 14 (1952); A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd **27**, No. 16 (1953).

[7] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd **30**, No. 1 (1955).

[8] D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953).

[9] S. A. Moszkowski, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39.

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cussed in Sec. 2. In Secs. 3 and 4 we treat a simplified two-dimensional nuclear model which is meant to illustrate the competition between independent-particle motion and collective motion. The level spectra at the beginning of the nuclear $1p$ shell are briefly studied in Sec. 5. We conclude this article with some general remarks concerning coupling schemes in heavy nuclei.

1. QUALITATIVE CONSIDERATIONS

A large part of the interactions between nucleons in nuclear matter can be replaced by a central nuclear potential which is essentially the average interaction of each nucleon with the rest. If this were the only effect of the interactions, each nuclear wave function would be a single Slater determinant of one-particle wave functions, and all excitations would involve a single particle at a time. The success of the single particle version of the shell model in accounting for ground-state spins of many odd-A nuclei and for the magic numbers implies that such an independent-particle description is at least a fair approximation to the actual situation in nuclei.[1,2] Of course, even this kind of coupling scheme implies the existence of some correlations rather than strictly independent motions of nucleons. First of all, the Pauli principle acts effectively like a repulsive force between identical particles (if their spin wave functions are the same), even in the absence of any explicit interactions between them. Secondly, a part of the correlations due to the interactions is already included in the single-particle model: Since it is unlikely to find a nucleon outside the nuclear surface, it is improbable to find any two nucleons more than one nuclear diameter apart.

A strict independent-particle description would imply that the energy of any state depends only on the orbits which the nucleons occupy, and not at all on the way in which these nucleons couple their angular momenta. Thus most levels would be highly degenerate, contrary to the experimental evidence. Consequently, in order to account for the features of nuclear level schemes, it is necessary to consider the interactions between nucleons, i.e., the correlations, in more detail. For example, since the interactions are mainly attractive, they tend to maximize the spatial symmetry of the low-lying states. These additional correlations can be regarded as due to so-called "effective" interactions. The problem of relating these effective interactions to the basic two-nucleon interactions has not yet been completely solved, but considerable progress in this direction has been made.[12] In this paper, we will restrict ourselves to some qualitative remarks on this subject. First of all, it is very likely that the effective interactions are less singular than the basic interactions, since the fluctuations are at least partially averaged out in nuclear matter. Thus, on the basis of reasonable assumptions regarding the basic interactions, it is expected that the probability of finding any two nucleons separated by 2×10^{-13} cm or more in the interior of a nucleus is given nearly correctly by an independent-particle picture.[13] On the other hand, short-range correlations involving distances of 10^{-13} cm or less are very sensitive to the details of the interactions, e.g., the existence of a repulsive core.[14] However, these correlations should manifest themselves only at high energies and are probably not of great importance for most low-energy phenomena.

It is possible to fit the properties of the two-nucleon system, *viz.*, the deuteron and nucleon-nucleon scattering, with a variety of interactions, especially if we restrict ourselves to low and medium energies. Thus, below about 100 Mev, the two-body data can be accounted for quite well with static (except possibly for spin-orbit effects) and rather nonsingular (i.e., Yukawa or Gaussian) interactions.[15] In most shell-model calculations to date,[1,2] these have been used as effective interactions and regarded as essentially the complete interactions between extra-shell nucleons, rather than as fluctuations about the average. This approach has proven very satisfactory, especially for nuclei with very few nucleons outside of, or missing from, closed-shell configurations. However, in these cases, the results depend mainly on the coupling of the nucleon angular momenta, and to a much lesser extent on the detailed form of the interactions, e.g., the radial integrals.

We would like to argue that the effective interactions may be actually somewhat different from the above form. Fluctuations of the matter density tend to be suppressed by the action of the exclusion principle, especially in the interior of the nucleus where the density is largest. On the other hand, in the region of the nuclear surface, the matter density is lower, and the Pauli principle is less effective in cutting down the correlations. Thus, it appears plausible that the effective interactions, at least those which are of the most importance for low-energy phenomena, occur mainly at the nuclear surface.† This approach is closely related

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[12] Brueckner, Eden, and Francis, Phys. Rev. **99**, 76 (1955); H. A. Bethe, *ibid.* **103**, 1353 (1956); R. J. Eden, in *Nuclear Reactions*, edited by P. M. Endt and M. Demeur (North-Holland Publishing Company, Amsterdam, 1957).

[13] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958); Gomes, Walecka, and Weisskopf, Ann. Phys. **3**, 241 (1958). This is also in accord with the small values of the absorption (imaginary) potential felt by low-energy nucleons scattered from nuclei (see reference 2).

[14] Brueckner, Eden, and Francis, Phys. Rev. **98**, 1445 (1955).

[15] J. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), Chaps. 2 and 4; R. G. Sachs, *Nuclear Theory* (Addison Wesley Publishing Company, Cambridge, 1953), Chaps. 4 and 6; L. Hulthén and M. Sugawara, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39.

† *Note added in proof.*—Some effective low-energy interactions occur also between nucleons in the interior of the nucleus. More detailed theoretical investigations indicate that these residual interactions act essentially only between nucleons on top of the Fermi sea and with approximately opposite momenta. As has been pointed out by Bohr, Mottelson, and Pines (to be published) such

to the idea of the unified model. For example, it is known[6,16] that a deformation of the nucleus gives rise to an effective interaction between nucleons at the surface.

To study this relation a little more closely, let us consider the following effective interaction:

$$v_{ik}=-U_0\delta(r_i-R_0)\delta(r_k-R_0)\exp[-(r_{ik}/r_0)^2], \quad (1.1)$$

i.e., two nucleons interact (in addition to their interaction via the average potential) only if they are both on the nuclear surface, which is assumed to be sharp. The quantity r_{ik} is the distance between the two nucleons, R_0 denotes the nuclear radius, while r_0 and U_0 refer respectively to the range and strength of the effective interactions. The specific Gaussian dependence is chosen here only for convenience of illustration. Of course, the above form is oversimplified; more realistic effective interactions will certainly depend not only on spatial coordinates but also on spins, and probably also on the nucleon momenta.†

Note that (1.1) implies that the interactions are also suppressed outside the nucleus, while in actual fact, the interactions are not suppressed there at all. However, this should not introduce any significant errors, at least for calculations of bound states, since bound nucleons only rarely penetrate outside the surface region.

The above effective interaction can be decomposed into angular components as follows:

$$v_{ik}=-\sum_{\lambda=0}^{\infty} c_\lambda\delta(r_i-R_0)\delta(r_k-R_0)P_\lambda(\cos\theta_{ik}), \quad (1.2)$$

where θ_{ik} denotes angular distance between the two nucleons:

$$\theta_{ik}=r_{ik}/R_0. \quad (1.3)$$

In this expansion, terms for which

$$\lambda>R_0/r_0 \quad (1.4)$$

are expected to be very small.

Incidentally, for each λ there is only one radial term, and each radial integral can be factored into a product of two integrals each involving one particle only. Instead, if conventional interactions are used, the radial terms depend on the coordinates of the two particles in a more complicated way, i.e., they are not factorable. Thus the use of surface interactions may actually simplify the calculations. While the validity of form (1.1) depends on the assumption that the surface thickness is small compared to the nuclear radius, the factorability of the effective interactions may hold more generally.

By means of the well-known addition theorem for spherical harmonics, the effective interactions can be rewritten in the form

$$v_{ik}=-\sum_{\lambda}\sum_{\mu=-\lambda}^{\lambda}[4\pi(2\lambda+1)^{-1}c_\lambda\delta(r_k-R_0)Y_{\lambda\mu}{}^*(\theta_k,\varphi_k)$$
$$\times\delta(r_i-R_0)Y_{\lambda\mu}(\theta_i,\varphi_i)]. \quad (1.5)$$

The above interactions may be readily described in the language of the unified model. To begin with, suppose that at a given time, the nucleons are in specific single-particle states. Then the interactions give rise to an extra potential, in addition to the central average potential. The extra potential seen by particle i at a given time is

$$-\sum_\lambda\sum_\mu[4\pi(2\lambda+1)^{-1}c_\lambda\langle\sum_k\delta(r_k-R_0)Y_{\lambda\mu}{}^*(\theta_k,\varphi_k)\rangle$$
$$\times\delta(r_i-R_0)Y_{\lambda\mu}(\theta_i,\varphi_i)] \quad (1.6)$$

in first order in the c_λ. The angular brackets indicate the expectation value of the enclosed term. An extra potential can also be generated by having the potential well shape be nonspherical. Suppose, for example, that the nuclear surface is given by

$$R(\theta,\varphi)=R_0[1+\sum_\lambda\sum_\mu\alpha_{\lambda\mu}Y_{\lambda\mu}(\theta,\varphi)], \quad (1.7)$$

to first order in the deformation parameters $\alpha_{\lambda\mu}$, and that the depth of the potential is V_0 inside and zero outside. Then the extra potential is[6]

$$-\sum_\lambda\sum_\mu V_0R_0\alpha_{\lambda\mu}\delta(r_i-R_0)Y_{\lambda\mu}(\theta_i,\varphi_i), \quad (1.8)$$

to first order in the $\alpha_{\lambda\mu}$.

It appears from these remarks that the two descriptions are at least qualitatively equivalent, provided

$$\alpha_{\lambda\mu}=4\pi(2\lambda+1)^{-1}(c_\lambda/V_0R_0)$$
$$\times\langle\sum_k\delta(r_k-R_0)Y_{\lambda\mu}{}^*(\theta_k,\varphi_k)\rangle. \quad (1.9)$$

The connection between the two approaches is discussed more fully in the next section.

2. DESCRIPTION OF INTERACTIONS IN THE LANGUAGE OF THE UNIFIED MODEL

We shall assume, in view of the remarks in the last section, that the nuclear Hamiltonian can be expressed as a sum of one-body and two-body terms. Accordingly, we write the Hamiltonian in the form

$$H=\sum_i\mathcal{H}_i+\tfrac{1}{2}\sum_i\sum_k v_{ik}, \quad (2.1)$$

where $\mathcal{H}_i$ denotes the one-body Hamiltonian for each particle and v_{ik} represents the effective two-body interaction. The single-particle Hamiltonian is given by

$$\mathcal{H}=T+V, \quad (2.2)$$

where T denotes the kinetic energy operator and the potential V is spherically symmetric.

As written here, the two-body part of the Hamiltonian also contains self-interaction terms, since we

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interactions give rise to a pairing energy in even-even nuclei analogous to the energy gap in superconducting metals [Bardeen, Cooper, and Schrieffer, Phys. Rev. **108**, 1175 (1957)].

[16] C. Levinson and K. W. Ford, Phys. Rev. **100**, 1 (1955).

have

$$\tfrac{1}{2}\sum_{i}\sum_{k} v_{ik}=\sum_{i<k}\sum v_{ik}+\tfrac{1}{2}\sum_{i} v_{ii}. \tag{2.3}$$

These self-interaction terms (the v_{ii}) have the same effect as a slight change in the one-body potential, but it is mathematically convenient to include these terms in the effective interactions, rather than in the potential. This point is illustrated in Appendix B.

In line with the remarks of the last section, we also assume that the effective two-body interactions are attractive and factorable, i.e., that they can be expressed as follows:

$$v_{ik}=-\sum_{\nu} c_\nu g_\nu^*(k)g_\nu(i), \tag{2.4}$$

where the c_ν are positive constants, while the g_ν represent Hermitian one-body operators, the g_ν^* their complex conjugates, and the sum is assumed to contain a relatively small number of terms. [Each term may, for example, correspond to one of the terms in the multipole expansion (1.5) of the surface interaction.] This kind of decomposition is also used in conventional shell model calculations, but only for the angular part of the interaction. Incidentally, factorable interactions of the form (2.4) have been used previously for the purpose of classifying nuclear states. Thus Racah has extensively treated such interactions, in which the g_ν are the components of the angular momentum vector, or generalizations thereof.[17] In that case, the interaction has the form of a Casimir operator, and group theoretical methods may be applied. Elliott[18] has also considered these interactions for the case that the g_ν are the components of the quadrupole tensor. Note that the g_ν depend on the values chosen for the scale factors c_ν. If the c_ν have the dimensions of energy, then the g_ν are dimensionless variables. In summary, the nuclear Hamiltonian is assumed to be of the form

$$H=\sum_{i}\mathcal{H}_i-\tfrac{1}{2}\sum_{i}\sum_{k}\sum_{\nu} c_\nu g_\nu^*(k)g_\nu(i). \tag{2.5}$$

The instantaneous potential seen by particle i is given by

$$V_i'(\mathbf{r})=V_i(\mathbf{r})-\sum_k\sum_\nu c_\nu\langle g_\nu^*(k)\rangle g_\nu(i). \tag{2.6}$$

We can simulate the effect of the two-body interactions by letting the potential depend upon additional collective variables α_ν as follows:

$$V_i(\boldsymbol{\alpha},\mathbf{r})=V_i(\mathbf{r})-\sum_\nu c_\nu\alpha_\nu g_\nu(i), \tag{2.7}$$

where the symbol $\boldsymbol{\alpha}$ denotes the set of α_ν. The two descriptions are seen to be essentially equivalent, provided

$$\alpha_\nu=\langle\sum_k g_\nu^*(k)\rangle. \tag{2.8}$$

Next, we introduce a so-called intrinsic Hamiltonian:

$$h(\boldsymbol{\alpha},\mathbf{r})=\sum_i\mathcal{H}_i-\sum_i\sum_\nu c_\nu\alpha_\nu g_\nu(i)+\tfrac{1}{2}\sum_\nu c_\nu\alpha_\nu^2 \tag{2.9}$$

which depends upon the α_ν. In this paper the α_ν are assumed to be real, but this is no essential restriction. The set of particle coordinates is denoted by $\mathbf{r}$.

Clearly, h generates independent motions of all the particles in the same nonspherical potential. As will be seen, this intrinsic Hamiltonian can be regarded as a starting point in a perturbation calculation. Note that the magnitude of α_ν depends upon the value chosen for c_ν. If c_ν has the dimension of energy, then α_ν is dimensionless. The intrinsic wave functions and energies are given by

$$h(\boldsymbol{\alpha},\mathbf{r})\chi_K(\boldsymbol{\alpha},\mathbf{r})=E_K(\boldsymbol{\alpha})\chi_K(\boldsymbol{\alpha},\mathbf{r}). \tag{2.10}$$

The intrinsic wave functions are antisymmetrized products of single-particle functions. Note that the second term on the right-hand side of (2.9) essentially counts all the two-body interactions twice, but this error is effectively compensated by the last term.

The main point of this paper is to show the close relation between the original Hamiltonian H and the intrinsic Hamiltonian h. Thus approximate solutions (energy levels and wave functions) of the many-body problem (2.5) can sometimes be obtained in terms of solutions of the intrinsic Hamiltonian, without having to explicitly diagonalize the complete energy matrix. In order to investigate this relation in more detail, we begin by noting that, in

$$H=h-\tfrac{1}{2}\sum_{\nu}\frac{1}{c_\nu}\left(\frac{\partial h}{\partial\alpha_\nu}\right)^2. \tag{2.11}$$

While both h and $\partial h/\partial\alpha_\nu$ depend on α_ν, the α_ν-dependent terms which appear on the right-hand side of (2.11) must cancel, since H does not involve $\boldsymbol{\alpha}$.

Incidentally, it may be convenient to include a part of the two-body interactions explicitly [i.e., in the form (2.4)] in the intrinsic Hamiltonian.† Then the intrinsic wave functions are no longer products of single-particle wave functions. In this case, (2.11) still holds except that the sum ν now extends only over those indices which stand for the remaining interactions, i.e., those not already included in h.

In general, the intrinsic wave functions are not eigenfunctions of the original Hamiltonian. However, let us suppose that operation of H on the intrinsic function χ_K gives the following:

$$H(\mathbf{r})\chi_K(\mathbf{r},\boldsymbol{\alpha})=\mathcal{O}_K(\boldsymbol{\alpha},\partial/\partial\boldsymbol{\alpha})\chi_K(\mathbf{r},\boldsymbol{\alpha}), \tag{2.12}$$

where $\mathcal{O}_K$ denotes an arbitrary Hermitian differential operator which involves α_ν and $\partial/\partial\alpha_\nu$ but not particle coordinates or momenta. We shall show that a wave

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[17] G. Racah, "Group Theory and Spectroscopy," mimeographed lecture notes, Princeton, 1951 (unpublished). G. Racah, *Farkas Memorial Volume* (Research Council of Israel, Jerusalem, 1952), Vol. 1, p. 29. This approach was applied to calculations of nuclear binding energies by I. Talmi and R. Thieberger, Phys. Rev. **103**, 718 (1956); see also R. Thieberger and A. De Shalit, Phys. Rev. **108**, 378 (1957).

[18] J. P. Elliott, Proceedings of the Pittsburgh Conference on Nuclear Structure, June, 1957. Elliott has also shown that the wave functions generated by these quadrupole-quadrupole interactions are surprisingly close to those resulting from conventional shell model calculations, e.g., using Yukawa interactions of the usual range.

function of the form[19,20]

$$\psi_{K\eta}(\mathbf{r})=\int \chi_K(\mathbf{r},\boldsymbol{\alpha})\Phi_{K\eta}(\boldsymbol{\alpha})d\boldsymbol{\alpha} \tag{2.13}$$

is actually a solution of the many-body problem:

$$H\psi_{K\eta}=E_{K\eta}\psi_{K\eta}. \tag{2.14}$$

The quantity $\Phi_{K\eta}$ is a collective wave function. The first subscript K describes the corresponding intrinsic wave function, while η refers to the particular quantum state of the collective motion. In this paper, we shall be concerned mainly with the collective spectrum based on the lowest intrinsic state $\chi_L(\boldsymbol{\alpha})$. For the sake of simplicity we shall, for the moment, restrict ourselves to consideration of a single collective variable α which may be any function of the α_ν. It is readily shown by partial integration that

$$\int [\mathcal{O}_K(\alpha,\partial/\partial\alpha)\chi_K(\mathbf{r},\alpha)]\Phi_{K\eta}(\alpha)d\alpha \\ =\int \chi_K(\mathbf{r},\alpha)[\mathcal{O}_K^*(\alpha,\partial/\partial\alpha)\Phi_{K\eta}(\alpha)]d\alpha. \tag{2.15}$$

Thus, if (2.12) holds, (2.14) takes the form:

$$\int \chi_K(\mathbf{r},\alpha)[\mathcal{O}_K^*(\alpha,\partial/\partial\alpha)\Phi_{K\eta}(\alpha)]d\alpha \\ =E_{K\eta}\int \chi_K(\mathbf{r},\alpha)\Phi_{K\eta}(\alpha)d\alpha. \tag{2.16}$$

The previous equation must be satisfied for every value of $\mathbf{r}$. This requires that the collective wave function obeys the following equation:

$$\mathcal{O}_K^*(\alpha,\partial/\partial\alpha)\Phi_{K\eta}(\alpha)=E_{K\eta}\Phi_{K\eta}(\alpha). \tag{2.17}$$

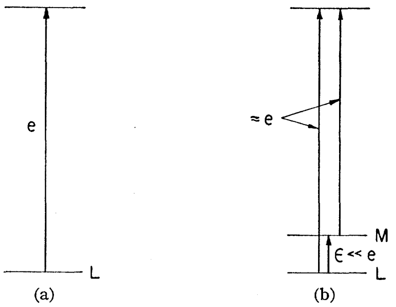
These considerations are readily generalized to take into account several collective variables. We shall show below that under certain conditions of interest an equation of the form (2.12) is in fact approximately satisfied. In particular, it will be found that

$$H\chi_L=\epsilon_0(\boldsymbol{\alpha})\chi_L+\textstyle\sum_\nu \epsilon_{2\nu}(\partial^2\chi_L/\partial\alpha_\nu^2), \tag{2.18}$$

where the ϵ_0 are functions of $\boldsymbol{\alpha}$ but the $\epsilon_{2\nu}$ are all constants.

Under these conditions, the differential equation governing the collective motion is of the form

$$\epsilon_0(\boldsymbol{\alpha})\Phi_{L\eta}+\sum_\nu \epsilon_{2\nu}\frac{\partial^2}{\partial\alpha_\nu}\Phi_{L\eta}=E\Phi_{L\eta}. \tag{2.19}$$



(a) (b)

FIG. 1. Schematized intrinsic level spectra. (a) Uniform spacing of levels. Collective displacement operator $\partial/\partial\alpha$ couples only states in adjacent groups. (See Sec. 2.) (b) Essentially uniform spacing of levels, but collective displacement operator couples close-lying states as well as states in adjacent groups. (See Appendix A.) Arrows connect pairs of states coupled by $\partial/\partial\alpha$. The total wave function can be expressed in the form (2.46). In case (a), we have a collective spectrum built on the lowest intrinsic state. In case (b), the wave function is a sum of two terms corresponding to states L and M.

Before studying the collective excitation spectrum in more detail, we give a few general relations which will prove useful in this connection. Consider any collective variable α (say, a real function of the α_ν). Starting with the commutation relation:

$$[\partial/\partial\alpha,h]=\partial h/\partial\alpha, \tag{2.20}$$

it is readily shown that

$$\frac{\partial h}{\partial\alpha}\chi_L=-\sum_K(E_K-E_L)\left(\frac{\partial}{\partial\alpha}\right)_{KL}\chi_K+\frac{\partial E_L}{\partial\alpha}\chi_L, \tag{2.21}$$

where the matrix element is given by:

$$\left(\frac{\partial}{\partial\alpha}\right)_{KL}=-\left(\frac{\partial}{\partial\alpha}\right)_{LK}=\int \chi_K^*\frac{\partial\chi_L}{\partial\alpha}d\mathbf{r}. \tag{2.22}$$

In particular, the diagonal element of $\partial h/\partial\alpha$ for the intrinsic ground state is

$$(\partial h/\partial\alpha)_{LL}=\partial E_L/\partial\alpha. \tag{2.23}$$

If α refers to any one of the α_ν, it follows that from the above and (2.9) that the self-consistency relation (2.8) must be satisfied whenever $\partial E_L/\partial\alpha$ vanishes, i.e., when we are at an equilibrium deformation.[21]

Let us now make some specific assumptions regarding the intrinsic energy level spectrum. The simplest case we will treat is illustrated in part (a) of Fig. 1. It is assumed that the collective displacement operator $\partial/\partial\alpha$ couples the lowest intrinsic state χ_L only to states at a specific energy, denoted here by e, above it. Thus we have

$$(\partial/\partial\alpha)_{KL}=0, \quad \text{unless} \quad E_K-E_L=e. \tag{2.24}$$

In this case we may replace every term (E_K-E_L) in Eq. (2.21) by e. No error is made by this substitution,

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[19] Wave functions of this form have also been used as starting point for a variational treatment of the collective motion. See R. E. Peierls and J. Yoccoz, Proc. Phys. Soc. (London) **70**, 381 (1957). T. H. R. Skyrme, Proc. Roy. Soc. (London) **239**, 399 (1957); and reference 20.

[20] J. J. Griffin and J. A. Wheeler, Phys. Rev. **108**, 311 (1957).

[21] S. A. Moszkowski, Phys. Rev. **103**, 1328 (1956).

since, in view of (2.24), the only states which contribute to the sum over K are those for which $E_K - E_L$ equals e. Using the closure relation, we then obtain

$$\frac{\partial h}{\partial \alpha}\chi_L = \left[-e\frac{\partial}{\partial \alpha} + \frac{\partial E_L}{\partial \alpha}\right]\chi_L. \tag{2.25}$$

We shall assume, in addition, that e is independent of α and that E_L depends only weakly, if at all, on α, Then we can write

$$(\partial h/\partial \alpha)\chi_L = -e(\partial \chi_L/\partial \alpha). \tag{2.26}$$

If α is an angle variable, this assumption is obviously satisfied as a result of rotational invariance. However, if α denotes some other collective variable, it is not, in general, valid. Nevertheless, even if the intrinsic energies depend strongly on α, relatively little error is made by using (2.26) as we shall see in the following sections.

Next, we turn to the evaluation of $H\chi_L$. From (2.9) it follows that

$$\partial^2 h/\partial \alpha_\nu \partial \alpha_{\nu'} = c_\nu \delta_{\nu\nu'}. \tag{2.27}$$

We can verify the commutation relation:

$$[\partial/\partial \alpha_\nu, \partial h/\partial \alpha_\nu] = c_\nu. \tag{2.28}$$

With the help of (2.26) it then is easy to show that

$$(\partial h/\partial \alpha_\nu)^2 \chi_L = e^2(\partial^2 \chi_L/\partial \alpha_\nu^2) + e c_\nu \chi_L. \tag{2.29}$$

Substituting this result into (2.11) and using (2.10), we obtain

$$H\chi_L = -\tfrac{1}{2}e^2 \sum_\nu \frac{1}{c_\nu}\frac{\partial^2 \chi_L}{\partial \alpha_\nu^2} + (E_L - \tfrac{1}{2}ne)\chi_L, \tag{2.30}$$

where n is the number of collective variables. This equation is just of the form (2.18) with

$$\epsilon_0 = E_L - \tfrac{1}{2}ne, \tag{2.31a}$$

$$\epsilon_2 = -e^2/(2c_\nu). \tag{2.31b}$$

It is often convenient to use as collective variables not the α_ν defined by (2.9), but certain real functions of the α_ν. These quantities which we denote by β_μ, may for example, describe the deformation and orientation of the system. We shall assume that the transformation between the α_ν and β_μ satisfies

$$\partial \beta_\mu/\partial \alpha_\nu = (c_\nu/\gamma_\mu)^{\frac{1}{2}} u_{\mu\nu}, \tag{2.32}$$

where the c_ν have been defined previously, the γ_μ and $u_{\mu\nu}$ are functions of the β variables, and the $u_{\mu\nu}$ form the components of a real orthogonal matrix. An explicit expression for the γ_μ can be obtained as follows: We assume that the intrinsic energy E_L is essentially independent of all collective variables. Then according to (2.23) we have:

$$(\partial h/\partial \beta_\mu)_{LL} = 0 \tag{2.33}$$

for every β_μ. Using (2.32), we readily obtain

$$\partial \alpha_\nu/\partial \beta_\mu = (\gamma_\mu/c_\nu)^{\frac{1}{2}} u_{\mu\nu}. \tag{2.34}$$

With the help of the above relations and (2.27) it can be shown that

$$\gamma_\mu \equiv \left(\frac{\partial^2 h}{\partial \beta_\mu^2}\right)_{LL} = \left[\frac{\partial}{\partial \beta_\mu}, \frac{\partial h}{\partial \beta_\mu}\right]_{LL}. \tag{2.35}$$

It also follows that

$$\sum_\nu \frac{1}{c_\nu}\frac{\partial^2}{\partial \alpha_\nu^2} = \sum_\mu \frac{1}{F}\frac{\partial}{\partial \beta_\mu} F \frac{\partial}{\partial \beta_\mu}, \tag{2.36a}$$

where

$$F = (\gamma_1 \cdot \gamma_2 \cdots \gamma_n)^{\frac{1}{2}}. \tag{2.36b}$$

Substitution of this result into (2.30) gives

$$H\chi_L = -\tfrac{1}{2}e^2\left(\sum_\mu \frac{1}{F}\frac{\partial}{\partial \beta_\mu}\frac{F}{\gamma_\mu}\frac{\partial}{\partial \beta_\mu}\right)\chi_L + (E_L - \tfrac{1}{2}ne)\chi_L. \tag{2.37}$$

Let us attempt to take into account the dependence of E_L on the β_μ. For simplicity, we shall, as before, regard e as a constant. Using (2.25), we find

$$H\chi_L = -\tfrac{1}{2}\sum_\nu \frac{1}{c_\nu}\left[-e\frac{\partial}{\partial \alpha_\nu} + \frac{\partial E_L}{\partial \alpha_\nu}\right]^2 \chi_L + (E_L - \tfrac{1}{2}ne)\chi_L. \tag{2.38}$$

Now suppose that the intrinsic wave function χ_L, but not the intrinsic Hamiltonian h, is independent of one or more of the collective variables say, β_1, i.e., all matrix elements of the operator $\partial/\partial\beta_1$ vanish. In this case, the intrinsic energy *must* depend on β_1. In particular, with the help of (2.23) and (2.35) it is seen that

$$\partial^2 E_L/\partial \beta_1^2 = \gamma_1. \tag{2.39}$$

Let us, however, neglect any dependence of E_L on the *other* collective variables, and assume that β_1 is fixed at some value for which all $\partial E_L/\partial \alpha_\nu$ vanish, i.e., at an equilibrium deformation. Making the transformation of (2.30) to β variables, we obtain Eq. (2.37) except that the sum is now to be taken over all variables except β_1 and n is to be replaced by $n-1$.

In all the cases treated above, the total wave function is of the form (2.13) except that we have several collective variables. If the intrinsic structure satisfies (2.37), the corresponding collective wave functions $\Phi_{L\eta}$ obey the equation

$$-\tfrac{1}{2}e^2 \sum_\mu \frac{1}{F}\frac{\partial}{\partial \beta_\mu}\frac{F}{\gamma_\mu}\frac{\partial}{\partial \beta_\mu}\Phi_{L\eta}(\boldsymbol{\beta}) + \left(E_L - \frac{n}{2}e\right)\Phi_{L\eta}(\boldsymbol{\beta}) = E\Phi_{L\eta}(\boldsymbol{\beta}). \tag{2.40}$$

Thus, we have a collective excitation spectrum built upon the lowest intrinsic state. The first term in (2.40) can be regarded as a collective kinetic energy, essentially of the form

$$T_{\text{coll}}=-\sum_{\mu}\frac{\hbar^2}{2B_\mu}\frac{\partial^2}{\partial\beta_\mu{}^2}, \tag{2.41}$$

where the inertial parameter B_μ is given by

$$B_\mu=\hbar^2\gamma_\mu/e^2. \tag{2.42}$$

If β_μ denotes an angle variable, then the corresponding parameter is a moment of inertia. Let us compare (2.42) with the value obtained by use of time-dependent perturbation theory[22,23]:

$$B_\mu=2\hbar^2\sum_K\frac{1}{E_K-E_L}\left(\frac{\partial}{\partial\beta_\mu}\right)_{KL}^2. \tag{2.43}$$

By combining Eqs. (2.21) and (2.35) (again under the assumption that E_L is independent of β_μ), we may readily show that

$$\sum_K(E_K-E_L)\left(\frac{\partial}{\partial\beta_\mu}\right)_{KL}^2=\tfrac{1}{2}\gamma_\mu. \tag{2.44}$$

If all energy differences appearing in the sum are equal to e, we immediately obtain (2.42). The quantity $E_L-\frac{1}{2}ne$ represents the potential energy of the collective motion. If the variation of e with β_μ can be ignored, then the intrinsic energy E_L and the collective potential energy differ only by an additive constant.

Next suppose that the intrinsic level structure is more complicated so that instead of (2.12) we have the more general relation:

$$H\chi_K=\sum\nolimits_{K'}\mathcal{O}_{KK'}\chi_{K'}, \tag{2.45}$$

where each of the $\mathcal{O}_{KK'}$ is again a differential operator involving α_μ and $\partial/\partial\alpha_\mu$. Then each eigenfunction of H is no longer of the simple form (2.13) but is a linear combination of such functions:

$$\psi(\mathbf{r})=\sum\nolimits_K\int\chi_K(\mathbf{r},\alpha)\Phi_{K\eta}(\alpha)d\alpha, \tag{2.46}$$

where the collective wave functions $\Phi_i(\alpha)$ obey the coupled set of differential equations:

$$\sum\nolimits_{K'}\mathcal{O}_{K'K}{}^*(\alpha,\partial/\partial\alpha)\Phi_{K'\eta}(\alpha)=E_{K\eta}\Phi_{K\eta}(\alpha). \tag{2.47}$$

In the example illustrated in part (b) of Fig. 1, each of the wave functions is a sum of two terms. This case is discussed in more detail in Appendix A.

It should be noted that the integration over α introduces correlations into the total wave function, even if the intrinsic wave function is a single Slater determinant. Also, though the χ and Φ are separately normalized to unity, the total wave function as given by (2.13) is not, since it is not a simple product.

Now consider the matrix element of a one-body operator, i.e.,

$$\mathcal{M}=\sum\nolimits_i\mathcal{M}(\mathbf{r}_i) \tag{2.48}$$

between two states denoted here by the indices $K\eta$ and $K'\eta'$, respectively. Even if we restrict ourselves to a single collective variable, such matrix elements have a rather complicated form. Thus it is seen that

$$\int\psi_{K'\eta'}{}^*\mathcal{M}\psi_{K\eta}d\mathbf{r}=\frac{\int\int\Phi_{K'\eta'}{}^*(\alpha')\mathcal{M}_{K'K}(\alpha',\alpha)\Phi_{K\eta}(\alpha)d\alpha'd\alpha}{(\int\int\Phi_{K'\eta'}{}^*(\alpha')N_{K'K'}(\alpha',\alpha)\Phi_{K'\eta'}(\alpha)d\alpha'd\alpha)^{\frac{1}{2}}(\int\int\Phi_{K\eta}{}^*(\alpha')N_{KK}(\alpha',\alpha)\Phi_{K\eta}(\alpha)d\alpha'd\alpha)^{\frac{1}{2}}}, \tag{2.49}$$

where

$$N_{KK}(\alpha',\alpha)=\int\chi_K{}^*(\mathbf{r},\alpha')\chi_K(\mathbf{r},\alpha)d\mathbf{r}, \tag{2.50}$$

and

$$\mathcal{M}_{K'K}(\alpha',\alpha)=\int\chi_{K'}{}^*(\mathbf{r},\alpha')\mathcal{M}(\mathbf{r})\chi_K(\mathbf{r},\alpha)d\mathbf{r}. \tag{2.51}$$

However, if the number of particles is large, a simple approximate expression for the matrix element can be obtained, as we will now show.

Each intrinsic wave function is essentially a product of single particle functions, thus any normalization integral can be written as a product of overlap integrals, each involving one particle. If α and α' are only slightly different, each of the single-particle functions changes only a little between α and α'; thus each of the one-particle overlap integrals is only slightly less than unity. Still, if the number of particles is sufficiently large, the total overlap integral will be small.[20] We may represent this quantity as follows:

$$N(\alpha',\alpha)\xrightarrow[n\to\infty]{}S(\alpha'-\alpha) \tag{2.52}$$

where S is proportional to the Dirac delta function:

$$\begin{aligned}S(x)&=1\quad\text{if}\quad x=0\\&=0\quad\text{if}\quad x\neq0.\end{aligned} \tag{2.53}$$

By the same kind of argument, one can show that:

$$\mathcal{M}(\alpha',\alpha)\xrightarrow[n\to\infty]{}\mathcal{M}(\alpha,\alpha)S(\alpha'-\alpha). \tag{2.54}$$

Consequently, if the overlap integrals are proportional

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[22] D. R. Inglis, Phys. Rev. **96**, 1059 (1954); **97**, 701 (1955).

[23] A summary of other approaches to the calculation of inertial parameters especially moments of inertia, is given by T. Tamura, Fortschr. Physik (to be published).

to δ functions, the matrix element is given by:

$$\int \psi_{K'\eta'}{}^{*}\mathfrak{M}\psi_{K\eta}d\mathbf{r} = \int \Phi_{K'\eta'}{}^{*}(\alpha)\mathfrak{M}_{K'K}(\alpha,\alpha)\Phi_{K\eta}(\alpha)d\alpha. \tag{2.55}$$

This result is also obtained if we regard the α not as a generator coordinate as in (2.13) but as an extra variable, i.e., if we write each wave function in the product form used by Bohr and Mottelson[6]:

$$\psi_{K\eta}(\mathbf{r},\alpha)=\chi_K(\mathbf{r},\alpha)\Phi_{K\eta}(\alpha). \tag{2.56}$$

Although this wave function is not strictly correct, (it contains too many variables), we can use it for the approximate evaluation of one-particle matrix elements, at least if the overlap integrals are essentially δ functions. This will usually be the case if we are dealing with a large number of particles in unfilled shells.

3. SIMPLIFIED TWO-DIMENSIONAL NUCLEAR MODEL; ENERGY SPECTRA

An important aspect of nuclear structure is the competition between independent-particle motion and collective motion. If all interactions were of the one-body type, the nucleons would move independently in the average nuclear potential, and excitations would involve only single particles at a time. Closely associated with this kind of coupling scheme is the existence of a nuclear shell structure and also of spherical nuclear shapes.

While a large part of the basic internucleon interactions can be represented in the form of an effective one-body interactions, i.e., the self-consistent potential, there are always some remaining terms which give rise to correlations between the motions of nucleons. At least a part of the correlations involve polarization of the nucleus. If these effects are strong enough, they lead to stable nonspherical nuclear shapes and to rotational spectra. In nuclei, we have a situation intermediate between the two extremes mentioned above. It is expected that collective effects become relatively more important as the number of particles in unfilled shells increases. Thus nuclei which contain very few nucleons outside of closed shell configurations are usually well described by an independent-particle model, while nuclei which are far removed from closed-shell configurations usually possess rotational spectra.

In this section, we shall consider a very simplified two-dimensional nuclear model which, it is hoped, will provide some insight into the intermediate-coupling situation. All the particles are assumed to move in a two-dimensional harmonic oscillator potential. The single-particle states of this potential are characterized by the principal quantum number N and the orbital angular momentum λ. [In two dimensions (x,y) there is only one possible direction of angular momentum—along the z-axis.] The possible values of λ are $\pm N$, $\pm(N-2)\cdots\pm 1$ or 0, and the single-particle energies are given by

$$\epsilon_{N\lambda}=\hbar\omega_0(N+1), \tag{3.1}$$

independent of λ. The quantity ω_0 denotes the characteristic oscillator frequency. Each particle is assumed to have an intrinsic spin of magnitude $\frac{1}{2}$, i.e., the component σ of spin along the z axis is either $\frac{1}{2}$ or $-\frac{1}{2}$. Besides the oscillator potential, we postulate that each particle is subject to a spin-orbit coupling:

$$V_{\text{s.o.}}=-c\lambda_i\sigma_i. \tag{3.2}$$

Finally we will suppose that the particles are subject to two-body interactions of the form

$$v_{ik}=-G^2(m\omega_0/2\hbar)^2r_i^2r_k^2\cos 2(\varphi_i-\varphi_k), \tag{3.3}$$

i.e., two-dimensional attractive quadrupole-quadrupole interactions. We impose one additional condition: all matrix elements of the interaction which involve a change in the N of any particle are assumed to vanish. Altogether the Hamiltonian for the system of particles can be written as follows:

$$H=H_{\text{h.o.}}+H_{\text{s.o.}}+H_{Q-Q}, \tag{3.4}$$

where

$$H_{\text{h.o.}}=\textstyle\sum_i(N_i+1)\hbar\omega_0, \tag{3.5}$$

$$H_{\text{s.o.}}=-c\textstyle\sum_i\lambda_i\sigma_i, \tag{3.6}$$

and

$$H_{Q-Q}=-\tfrac{1}{2}G^2\textstyle\sum_i\sum_k\sum_\nu f_\nu{}^*(k)f_\nu(i). \tag{3.7}$$

The quantities f_ν are defined as follows:

$$f_1(i)=(m\omega_0/2\hbar)r_i^2\cos 2\varphi_i, \tag{3.8a}$$

$$f_2(i)=(m\omega_0/2\hbar)r_i^2\sin 2\varphi_i, \tag{3.8b}$$

but all matrix elements of the f's between single-particle states of different N vanish. Note that (3.7) contains so-called "self-interaction" terms. The significance of these is discussed in Appendix B.

A more accurate expression for the f's which automatically gives the same matrix elements as (3.8) between states with the same N and vanishes when taken between states of different N is

$$f_1=\frac{m\omega_0}{2\hbar}\left(\frac{x^2-y^2}{2}\right)+\frac{1}{2m\omega_0\hbar}\left(\frac{p_x{}^2-p_y{}^2}{2}\right), \tag{3.9a}$$

$$f_2=\frac{m\omega_0}{2\hbar}xy+\frac{1}{2m\omega_0\hbar}p_xp_y. \tag{3.9b}$$

We may also write H_{Q-Q} in the form (2.5), i.e.,

$$H_{Q-Q}=-\tfrac{1}{2}\sum_i\sum_k\sum_{\nu=1,2}c_\nu g_\nu{}^*(k)g_\nu(i), \tag{3.10}$$

where

$$g_\nu=Gf_\nu, \tag{3.11}$$

$$c_\nu=1. \tag{3.12}$$

For the present problem the calculation of the energy matrix may be simplified by the introduction of some new quantum numbers. We regard each single-particle state of principal quantum number N as having associated with it a fictitious three-dimensional angular momentum t given by

$$t=\tfrac{1}{2}N, \tag{3.13}$$

and with component along a fictitious z axis equal to:

$$t_z=\tfrac{1}{2}\lambda. \tag{3.14}$$

In the following we assume that there are n particles in the shell of principal quantum N. All other shells are assumed to be either completely filled or empty and thus they do not enter into the excitation spectrum. The first term in (3.4), namely $H_{\text{h.o.}}$ may be ignored, since it has the same value for all possible states.

(a) Only Quadrupole-Quadrupole Interactions Present

We will first treat the case of no spin-orbit coupling, i.e., that only quadrupole-quadrupole interactions are present.[24] These interactions can be conveniently expressed in terms of the t operators. Thus it is easily shown that f_1 has the same matrix elements as the operator t_x. These are given as follows:

$$(f_1)_{N'\lambda',N\lambda}=\tfrac{1}{4}[(N'-\lambda_<)(N'+\lambda_>)]^{\frac{1}{2}}\delta_{N'N}\delta_{\lambda',\lambda\pm2}, \tag{3.15}$$

and

$$(t_x)_{t't_z',tt_z}=\tfrac{1}{2}[(t'-t_{z<})(t'+t_{z>})]^{\frac{1}{2}}\delta_{t't}\delta_{t_z',t_z\pm2}, \tag{3.16}$$

where $\lambda_>$ and $\lambda_<$ denote, respectively, the larger and smaller of λ and λ', and the same for the t_z. Thus we can make the correspondence:

$$f_1\rightarrow t_x \tag{3.17a}$$

and similarly

$$f_2\rightarrow t_y. \tag{3.17b}$$

The total Hamiltonian can then be expressed as follows:

$$H=H_{Q-Q}=-\tfrac{1}{2}G^2[T_x^2+T_y^2]=-\tfrac{1}{2}G^2[T^2-T_z^2], \tag{3.18}$$

where

$$T_\nu=\textstyle\sum_i t_\nu(i). \tag{3.19}$$

Each resultant many-particle state is characterized by the two quantum numbers T and T_z, where

$$T_z=\tfrac{1}{2}\Lambda, \tag{3.20}$$

$$\Lambda=\textstyle\sum_i \lambda_i. \tag{3.21}$$

The T_z range from $-T$ to T in steps of unity.

In this limit, the energy levels are given as follows:

$$E_{T\Lambda}=\tfrac{1}{2}G^2[-T(T+1)+\tfrac{1}{4}\Lambda^2]. \tag{3.22}$$

Thus all states of the same T can be regarded as forming a rotational band with Λ having the values $\pm2T$, $\pm2T-2$, $\cdots\pm1$ or 0. (The physical justification for this description is discussed below.) Of course, the energy ratios are slightly different from the values appropriate to three-dimensional rotation. Thus, for an even number of particles, the sequence of Λ is 0, ±2, ±4, $\cdots$ with an energy ratio of second excited state equal to 4 rather than to 10/3. Note that all bands have the same moment of inertia:

$$\hbar^2/2\mathfrak{J}=G^2/8. \tag{3.23}$$

For any configuration, the band with the largest value of T, denoted by T_0, occurs lowest in energy. The quantity T_0 is equal to the maximum possible value of $\frac{1}{2}\Lambda$ for the particular configuration. It is easily shown that if there is an even number n of identical spin $\frac{1}{2}$ particles in a shell which can accommodate g particles, then

$$T_0=\tfrac{1}{4}n(g-n). \tag{3.24}$$

In particular for the shell of principal quantum number N, we have

$$g=2N+2. \tag{3.25}$$

As a simple example, for a configuration of two particles in the shell N the allowed values of T range from 0 to T_0 (which equals N), in steps of unity. Figure 10(b) shows the calculated energy level scheme for the configuration of two particles in the $N=2$ shell.

The present problem can also be treated very conveniently by the method of the unified model. The intrinsic Hamiltonian corresponding to (3.7) is given by:

$$h=-G\textstyle\sum_i[\alpha_1f_1(i)+\alpha_2f_2(i)]+\tfrac{1}{2}(\alpha_1^2+\alpha_2^2). \tag{3.26}$$

We next define two new collective variables β, θ as follows:

$$\alpha_1=\beta\cos2\theta, \tag{3.27a}$$

$$\alpha_2=\beta\sin2\theta. \tag{3.27b}$$

Using definitions (3.8) for the f, we can rewrite (3.26) in the form

$$h=-\beta G(m\omega_0/2\hbar)\textstyle\sum_i[r_i^2\cos2(\varphi_i-\theta)]+\tfrac{1}{2}\beta^2, \tag{3.28}$$

but with the extra condition that all matrix elements involving a change in the N of any particle vanish. The first term in h represents a deviation of the potential from spherical. If this term is added to the central oscillator potential, it is found that the surfaces of constant potential are ellipses. The quantity β can be regarded as a deformation parameter. The angle between the major axis of each ellipse (i.e., the direction of maximum elongation) and the fixed axis is denoted by θ. The second term in h denotes a restoring term which tends to oppose the deformation.

In order to find the intrinsic energy levels and wave functions, it is convenient to express the h in terms of

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[24] This is the two-dimensional analog of the model considered by Elliott, reference 18.

the t operator. Thus we obtain

$$h=-\beta G[T_x\cos 2\theta+T_y\sin 2\theta]+\tfrac{1}{2}\beta^2. \tag{3.29}$$

It is seen that the eigenfunctions of h are also eigenfunctions of T^2, since this operator commutes with T_x and with T_y. However, since the potential depends on direction, the orbital angular momentum is not a good quantum number for these functions. The eigenfunctions of h can be expressed in the form $\chi(\mathbf{r},\varphi')$, where

$$\varphi_i'=\varphi_i-\theta \tag{3.30}$$

denotes the angular coordinate of particle i in the body reference frame, i.e., relative to the direction of maximum elongation.

It can be seen that the eigenvalues of h must be the same as the eigenvalues of

$$-\beta G T_z+\tfrac{1}{2}\beta^2. \tag{3.31}$$

In particular, the lowest eigenvalue, the "ground state intrinsic energy" is given by

$$E_L=-\beta G T+\tfrac{1}{2}\beta^2. \tag{3.32}$$

The corresponding intrinsic wave function is denoted by χ_L. Note that in the present case all intrinsic wave functions are independent of β. the equilibrium deformation, which gives minimum energy is

$$\beta_{\mathrm{eq}}=GT, \tag{3.33}$$

and the corresponding intrinsic energy equals

$$E_{\mathrm{eq}}=-\tfrac{1}{2}G^2T^2. \tag{3.34}$$

At this point, there are two alternative ways of proceeding. One method is to regard β as fixed and equal to GT, and to treat only variations of θ (i.e., rotations). This constitutes the rotational model. The other possibility is to treat variations of both θ (rotations and β (vibrations). The latter approach is a more general version of the unified model.

We shall first treat the problem by the rotational model. Note that the intrinsic levels are uniformly spaced in energy, with intervals between adjacent levels ($\Delta T=1$) given by

$$e=\beta G=G^2T. \tag{3.35}$$

It is easy to show, in addition, that the operator $\partial/\partial\theta$ couples only intrinsic states separated by e. Consequently, this problem can be treated by the method discussed in Sec. 2. The total wave functions are of the form (2.13), where the collective rotational wave functions $\Phi_{T\Lambda}$ obey an equation of the form (2.40). From (2.32) it is readily shown that the quantities γ_β and γ_θ are given by

$$\gamma_\beta=1, \tag{3.36a}$$

$$\gamma_\theta=4\beta^2. \tag{3.36b}$$

Thus, F defined by (2.36b) equals

$$F=2\beta. \tag{3.37}$$

In view of the remarks following Eq. (2.39), the sum μ in (2.40) extends only over the angle variable θ and the effective number n of collective modes is unity. Thus the equation for the collective motion is

$$-\tfrac{1}{8}G^2(d^2\Phi_{T\Lambda}/d\theta^2)-\tfrac{1}{2}G^2T(T+1)\Phi_{T\Lambda}=E\Phi_{T\Lambda}. \tag{3.38}$$

The collective (rotational) wave functions are

$$\Phi_{T\Lambda}=(2\pi)^{-\frac{1}{2}}e^{i\Lambda\theta}, \tag{3.39}$$

and the energy levels are given exactly by (3.22).

We see that the rotational model reproduces the correct energy levels at least for the present problem. Each total wave function is of the form

$$\psi_{T\Lambda}(\mathbf{r})=(2\pi)^{-\frac{1}{2}}\int\chi_L(\mathbf{r},\varphi',\theta)e^{i\Lambda\theta}d\theta \tag{3.40}$$

except for a normalization constant. Note that while χ_L is not an eigenfunction of Λ, the integration over angles picks out of it the component of given Λ. Thus if we express χ_L in the form

$$\chi_L(\mathbf{r},\varphi',\theta)=\textstyle\sum_\Lambda a_\Lambda\chi_{T\Lambda}(\mathbf{r},\varphi'), \tag{3.41}$$

then

$$\psi_{T\Lambda}(\mathbf{r})=\mathrm{const}\,a_\Lambda\chi_{T\Lambda}(\mathbf{r},\varphi). \tag{3.42}$$

The total wave function is nonvanishing only if the corresponding a_Λ is finite, even though the collective wave function (3.39) by itself is an eigenfunction of (3.38) regardless of the value of Λ. The rotational spectrum (3.40) goes only up to a certain value of Λ, and for each spectrum only even (or odd) values of Λ can occur, according to whether T is integer or half-integer.

If $\theta=0$, then each of the single-particle wave functions in the deformed potential is an eigenfunction of t_x. All single-particle states in the major shell N are characterized by a fictitious angular momentum $N/2$. The lowest intrinsic state has component $\tfrac{1}{2}N$ along the x axis. For the next higher state, the x component equals $\tfrac{1}{2}N-1$, and so on. The intrinsic two-particle state of lowest energy is evidently obtained by putting two particles in the orbit with component $\tfrac{1}{2}N$. This two-particle state is characterized by a fictitious angular momentum $T_0=N$ with maximum possible component along the x direction, and it generates the lowest rotational band. The intrinsic state which generates the next higher rotational band ($T=T_0-1$) may be formed by putting two particles into the lowest two orbits, but with a spatially antisymmetric wave function. (The corresponding spatially symmetric wave function generates the ground-state band.)

As was mentioned earlier, the present problem may also be treated by regarding both θ and β as collective variables. Since the intrinsic wave functions are independent of β, all matrix elements of $\partial/\partial\beta$ vanish. The method of Sec. 2 is again applicable, since all that is required is that matrix elements between states of energy difference *other* than e vanish (2.24). On the other hand, since the intrinsic energy depends on β, and this dependence is only partially taken into account in this method, we cannot expect to reproduce the correct energy levels and wave functions exactly. This time, *all* excitations (i.e., changes in θ or in β) are regarded as collective, based on the intrinsic state of lowest energy (with $T=T_0$). The sum over μ in Eq. (2.40) now extends over both β and θ and $n=2$. We regard e as a constant, but the other quantities as functions of β. Substitution of (3.32) and (3.35–3.37) into (2.40) gives

$$-\frac{G^4T_0^2}{2}\left(\frac{\partial^2}{\partial\beta^2}+\frac{1}{\beta}\frac{\partial}{\partial\beta}+\frac{1}{4\beta^2}\frac{\partial^2}{\partial\theta^2}\right)\Phi \\ +[-\beta GT_0+\tfrac{1}{2}\beta^2-G^2T_0)\Phi=E\Phi. \quad (3.43)$$

According to (3.43) the collective motion consists of rotations and vibrations, the latter involving oscillations of β about the equilibrium value GT_0. If the zero-point amplitude of the vibrations is small compared to the equilibrium deformation, the previous equation can be approximated as follows:

$$-\frac{G^4T_0^2}{2}\frac{\partial^2\Phi}{\partial\beta^2}-\frac{G^2}{8}\frac{\partial^2\Phi}{\partial\theta^2}+\tfrac{1}{2}(\beta-GT_0)^2\Phi \\ -\tfrac{1}{2}G^2T_0(T_0+2)\Phi=E\Phi. \quad (3.44)$$

The characteristic vibrational energy is given by

$$\hbar\omega_{\mathrm{vib}}=G^2T_0. \quad (3.45)$$

In the limit of very large T_0, this is the same as the energy difference between bands whose T differs by 1, and in any case, it is equal to the excitation energy of the next to lowest band. The zero-point amplitude is given by

$$\Delta\beta=\left(\hbar\omega_{\mathrm{vib}}\Big/\frac{\partial^2E_L}{\partial\beta^2}\right)^{\frac{1}{2}}=G(T_0)^{\frac{1}{2}}, \quad (3.46)$$

which is, indeed, small compared to the equilibrium deformation (3.33) if T_0 is large. The resulting energy spectrum is:

$$E=G^2[-\tfrac{1}{2}T_0(T_0+1)+\mu T_0+\tfrac{1}{8}\Lambda^2]. \quad (3.47)$$

If T_0 is very large, this agrees with the previous result (3.22) provided we identify μ with the quantity T_0-T. The corresponding collective wave function is of the form

$$\Phi_{T\Lambda}=u_{T_0-T}(\beta-GT_0)e^{i\Lambda\theta}, \quad (3.48)$$

where the u are the solutions of the one-dimensional oscillator problem.

(b) Only Spin-Orbit Coupling Present

Up to now, we have considered only quadrupole-quadrupole interactions in this section and neglected the one-body spin-orbit coupling (3.6). We will now treat the opposite limiting case; i.e., only spin-orbit coupling present, besides the central oscillator potential and no mutual interactions between the particles. In this limit, the particles move independently.

Note that while the spin-orbit interaction is spin-dependent, it cannot give rise to any spin-flip. Thus the λ and σ of each particle separately remain good quantum numbers. (On the other hand, in the three-dimensional case, neither l_z nor s_z remain good quantum numbers, only their resultant does.) In the absence of any further interactions, each of the single-particle states is doubly degenerate; the states λ, σ and $-\lambda$, $-\sigma$ occur at the same energy. Thus, a configuration consisting of an even number of particles will have a ground state with total orbital angular momentum and total spin both equal to 0, while for an odd-A configuration, the net angular momentum is just that of the last odd particle.

Another special property of the spin-orbit coupling in two dimensions (but not in three) is that the levels are uniformly spaced in energy. It is evident from (3.6) that an energy of c is required to raise any particle to the next higher orbit. Let us restrict our consideration to states with $\Sigma=0$. Any single-particle excitation between two such states involves a change of two units in the orbital angular momentum λ of this particle, and therefore also in the total orbital angular momentum Λ. Thus the first excited state of an even-A configuration occurs at energy c above the ground state and has $\Lambda=\pm2$, while the second excited state (at $2c$) is degenerate with $\Lambda=0$, ±4.

Altogether, we can express the energy spectrum in the form

$$E=E_0+\nu c, \quad (3.49)$$

where E_0 is the ground-state energy, and ν is a non-negative integer, which may be regarded as the number of excitation quanta. The equal spacing of levels is suggestive of an alternative description of the spectrum, namely to regard the excitations not as jumps of individual particles, but as vibrational excitations.

Since there are no mutual interactions, the intrinsic Hamiltonian contains no terms which tend to deform the nucleus. For this case, it follows from (2.9) that

$$h=H+\tfrac{1}{2}\beta^2. \quad (3.50)$$

The intrinsic energy of the lowest state ($\nu=0$) is given by

$$E_L=E_0+\tfrac{1}{2}\beta^2. \quad (3.51)$$

The coefficient of the β^2 term is actually arbitrary for this case and we set it equal to unity. As before the method of Sec. 2 is applicable, since the intrinsic levels are uniformly spaced, at intervals given by:

$$e=c. \quad (3.52)$$

This time Eq. (2.40) has the form

$$-\tfrac{1}{2}c^2\left(\frac{\partial^2}{\partial\beta^2}+\frac{1}{\beta}\frac{\partial}{\partial\beta}+\frac{1}{4\beta^2}\frac{\partial^2}{\partial\theta^2}\right)\Phi$$
$$+(E_0-c+\tfrac{1}{2}\beta^2)\Phi=E\Phi, \quad (3.53)$$

which is just the Schrödinger equation for a two-dimensional isotropic harmonic oscillator. The characteristic vibrational energy is given by

$$\hbar\omega_{\text{vib}}=c, \quad (3.54)$$

just the same as the spacing between intrinsic levels and the collective energy spectrum is exactly given by (3.49). Note that the term $-c$ in (3.53), i.e., the last term on the left-hand side of (2.40), is just cancelled by the zero-point energy.

(c) Both Quadrupole-Quadrupole and Spin-Orbit Interactions Present

Having treated separately both the cases of quadrupole-quadrupole forces and spin-orbit coupling, we are now in a position to study the more general intermediate-coupling case that both interactions are present. This forms a highly simplified nuclear model, which exhibits the transition between independent-particle motion and collective motion. For this kind of system, the total orbital angular momentum Λ and also the intrinsic spin Σ are each constants of the motion. We will suppose that there are two particles with σ equal to $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively, so that the resultant Σ vanishes. Each particle is assumed to have a fictitious angular momentum $T_0/2$. The Hamiltonian is given by

$$H=-c\{[T_z(+)]-[T_z(-)]\}-\tfrac{1}{2}G^2[T_x^2+T_y^2]. \quad (3.55)$$

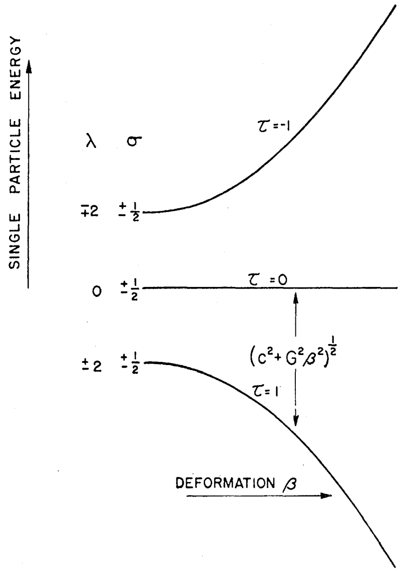

FIG. 2. Intrinsic single-particle levels in the $N=2$ shell as function of deformation. Each state is characterized by a quantum number τ defined by (3.58). Also given are the values of λ and σ in the limit of $\beta=0$. Each energy level is doubly degenerate.

In this way we are actually covering a more general case, as may be seen in the following way: Suppose that one or both of the particles (say the one with $\sigma=\frac{1}{2}$ and specified λ) is replaced by a *group* of particles each of which has $\sigma=\frac{1}{2}$ and which couple together to form a resultant T equal to $\frac{1}{2}T_0$ and orbital angular momentum Λ equal to λ. Since the Hamiltonian depends on the t's of the individual particles only through the operators

$$T_\nu(\pm)=\textstyle\sum_i t_\nu(i), \quad (\sigma_i=\pm\tfrac{1}{2}) \quad (3.56)$$

there will be no change in either of these quantities. Consequently the group of particles may be regarded as equivalent to the original single particle. For example, a system of two particles in the $N=1$ shell ($t=\frac{1}{2}$) which form a spatially symmetric wave function is equivalent to a single particle in the $N=2$ shell ($t=1$), and the problem of 4 particles in the $N=1$ shell is, in many respects, equivalent to that of two particles in the $N=2$ shell.

For the intermediate-coupling problem, the unified-model approach is simpler, though less accurate, than the explicit treatment of all the interactions, and will be presented first. We shall treat both vibrations and rotations. The intrinsic Hamiltonian corresponding to (3.55) is given by

$$h=-c[T_z(+)-T_z(-)]$$
$$-G\beta[T_x\cos2\theta+T_y\sin2\theta]+\tfrac{1}{2}\beta^2. \quad (3.57)$$

Apart from the effect of the restoring terms, the intrinsic single-particle energies are

$$-\tau(c^2+G^2\beta^2)^{\frac{1}{2}}, \quad (3.58)$$

where τ ranges from $-\frac{1}{2}T_0$ to $\frac{1}{2}T_0$ in steps of unity. For large values of β and $\theta=0$, τ is equal to the eigenvalue of t_x for the single particle state, while for small β and $\sigma=\pm\frac{1}{2}$ it approaches $\pm t_z$, i.e., $\pm\frac{1}{2}\lambda$. Figure 2 shows a sketch of the intrinsic energy levels for a particle in the $N=2$ shell as function of deformation.

The intrinsic energy of the two-particle system is obtained by adding the single-particle energies (3.58) and the restoring term ($\frac{1}{2}\beta^2$). For the lowest intrinsic state, we find

$$E_L(\beta)=-T_0(c^2+G^2\beta^2)^{\frac{1}{2}}+\tfrac{1}{2}\beta^2. \quad (3.59)$$

Higher states are uniformly spaced in energy at intervals of

$$e=(c^2+G^2\beta^2)^{\frac{1}{2}}. \quad (3.60)$$

We will find it convenient to define a coupling parameter as follows:

$$x=T_0G^2/c. \quad (3.61)$$

In Fig. 3 we have sketched the ground state intrinsic energy as function of β for several values of x. It is seen that if $x<1$, the equilibrium shape is spherical, while for $x>1$ it is nonspherical.[5]

Let us now consider the two coupling schemes in somewhat more detail. In the weak coupling case, an expansion of E_L in powers of β gives

$$E_L=-cT_0+\tfrac{1}{2}(1-x)\beta^2, \tag{3.62}$$

to second order in β. While the equilibrium shape is spherical, the energy curve becomes flatter as x increases toward unity. The spacing between adjacent intrinsic levels is equal to c at $\beta=0$, independent of x. The characteristic vibrational energy is quite generally given by

$$\hbar\omega_{\mathrm{vib}}=e(\partial^2E_L/\partial\beta^2)^{\frac{1}{2}}. \tag{3.63}$$

For the present case, we have

$$\hbar\omega_{\mathrm{vib}}=c(1-x)^{\frac{1}{2}}. \tag{3.64}$$

From (3.46) it follows that the zero-point vibration amplitude $\Delta\beta$ is given by

$$\Delta\beta=c^{\frac{1}{2}}(1-x)^{-\frac{1}{4}}, \tag{3.65}$$

According to (3.59) the effect of anharmonicity of E_L is expected to be proportional to the ratio

$$G^2(\Delta\beta)^2/c^2=x/[T_0(1-x)^{\frac{1}{2}}] \tag{3.66}$$

in lowest order.

As the coupling strength goes from 0 to 1, the energy levels move closer together, but without any change in their relative spacing. If T_0 is large, deviations from the harmonic law become appreciable only when x is close to unity. While the excitation spectrum can be qualitatively characterized as vibrational even in the limit $x=0$, this kind of description is completely accurate only if the frequency of the collective motion is small compared to the characteristics frequency of individual particles, i.e., if the collective motion occurs adiabatically. If the equilibrium shape is spherical, this occurs when the ratio

$$(\hbar\omega_{\mathrm{vib}}/e)=(1-x)^{\frac{1}{2}} \tag{3.67}$$

is small compared to unity, which requires that x is close to unity.

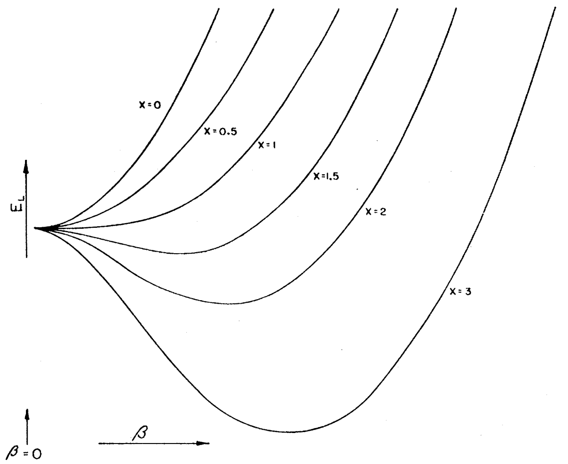

FIG. 3. Ground-state intrinsic energy E_L as function of deformation β for several values of the coupling parameter x. (Not to scale.)

When the coupling strength exceeds unity, a pronounced change in the spectrum occurs. The spherical shape becomes energetically unstable and the system requires an equilibrium deformation given by

$$\beta_{\mathrm{eq}}=GT_0(1-x^{-2})^{\frac{1}{2}}. \tag{3.68}$$

The ground-state intrinsic energy, expanded about the equilibrium deformation, is

$$E_L(\beta)=-\tfrac{1}{2}G^2T_0^2(1+x^{-2})+\tfrac{1}{2}(\beta-\beta_{\mathrm{eq}})^2(1-x^{-2}). \tag{3.69}$$

Thus, as x increases, the equilibrium deformation increases toward the limiting value GT_0, and the energy minimum becomes sharper. The energy difference between adjacent intrinsic states, evaluated at β_{eq} is given by

$$e=G^2T_0. \tag{3.70}$$

In this case, the deformation can oscillate about its equilibrium value, and we also have collective rotations about the z axis. The characteristic vibrational energy is

$$\hbar\omega_{\mathrm{vib}}=G^2T_0(1-x^{-2})^{\frac{1}{2}}, \tag{3.71}$$

while the characteristic rotational energy is given by

$$\frac{\hbar^2}{2\mathfrak{J}}=\frac{e^2}{8\beta_{\mathrm{eq}}{}^2}=\frac{G^2}{8}\,\frac{1}{(1-x^{-2})}. \tag{3.72}$$

Thus, as the coupling becomes stronger (e.g., if G and T_0 remain constant, while c decreases), the vibrational energies increase again, while the rotational energies continue to decrease, in agreement with the experimental evidence. With the help of (3.46), we can make the following rough estimate:

$$\frac{\Delta\beta}{\beta_{\mathrm{eq}}}=\frac{1}{T_0{}^{\frac{1}{2}}(1-x^{-2})^{\frac{3}{4}}}. \tag{3.73}$$

If the ratio (3.73) is small compared to 1, the rotational and vibrational degrees of freedom are uncoupled from each other. One would then expect to find relatively little perturbation of the rotational bands. Thus if T_0 is large, the energies in a rotational band should exhibit only slight deviations from the law, unless x is close to unity.

The transition between vibrational and rotational spectra is expected to occur when the effect of the nonspherical equilibrium shape is just "washed-out" by the zero-point oscillations. According to (3.73) this requires that

$$x=1+\gamma T_0{}^{-\frac{2}{3}}, \tag{3.74}$$

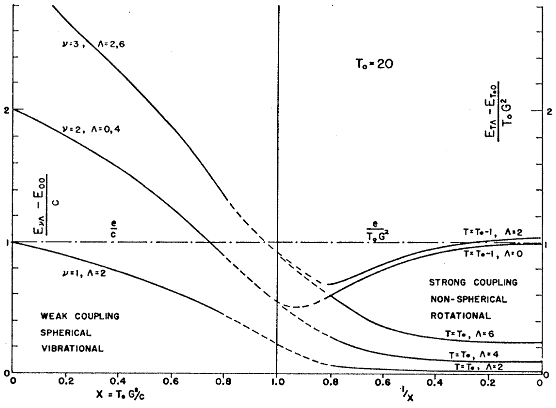

FIG. 4. Energy spectrum for two-dimensional intermediate-coupling problem according to unified model. Each particle is assumed to be subject to a spin-orbit interaction of strength c, and the particles interact by attractive quadrupole-quadrupole forces of strength G^2. The total intrinsic spin Σ is assumed to vanish. The quantity T_0, which is half of the largest possible value of total orbital angular momentum Λ, is taken equal to 20, a rather large value. This corresponds to the problem of two particles in the harmonic oscillator shell of principal quantum number $N=20$ [see Eq. (3.55)]. However, it also corresponds to a variety of problems involving a larger number of particles in shells of smaller principal quantum number. It is convenient to define a coupling parameter $x=T_0G^2/c$. For $x<1$ (spherical equilibrium shape), the excitation energies, in units of c, are plotted against x. Below $x=0.8$, the spectrum is assumed to be of the vibrational form (3.64). For $x>1$, (nonspherical equilibrium shape) the excitation energies are expressed in units of T_0G^2 and plotted against $1/x$. Above $x=1.25$, the spectrum is assumed to be of the rotational form [see Eqs. (3.71) and (3.72)]. The dotted lines in the transition region represent qualitative interpolations. Each level is characterized by the oscillator quantum number ν and by its orbital angular momentum Λ in the limit $x\rightarrow 0$, and by the quantum numbers T and Λ in the limit $x\rightarrow\infty$. Only states corresponding to values of ν up to 3 are shown. The states which correspond to $\nu=4$ become the following in the limit of $x\rightarrow\infty$: $T=T_0$, $\Lambda=8$; $T=T_0-1$, $\Lambda=4$; $T=T_0-2$, $\Lambda=0$. Also plotted for comparison is the energy spacing e between intrinsic levels as function of x. (See 3.52, 3.70.)

where γ denotes a constant of order unity. In the transition region, the collective energies are small compared to the intrinsic excitation energy; in particular, using Eqs. (3.70–3.72), we find that

$$(E_{\text{Coll}}/e)=\text{const}\times T_0^{-\frac{1}{3}}. \tag{3.75}$$

Of course in this region the collective motion does not separate into rotations and harmonic vibrations. On the other hand, in the strong-coupling case ($x\gg1$) the adiabatic condition holds for the rotational motion, but not for the vibrational motion.

Figure 4 shows a sketch of the energy spectrum as function of the coupling parameter for a rather large value of T_0, namely 20. Note that the energy of the first excited $\Lambda=0$ vibrational state has a minimum in the transition region. This is expected since the minimum of the intrinsic energy curve is flattest in this region (as shown in Fig. 3).

We now treat the intermediate-coupling problem (3.55) by explicit consideration of all the interactions. We expect to find that the calculated energy spectra will essentially agree with the results just calculated by means of the unified model.

In the weak-coupling case ($x\ll1$), it is convenient to use the independent-particle wave functions (characterized by $\lambda+$, $\lambda-$) as a basis set. The spin-orbit interaction is diagonal in this representation. Its matrix elements are given by

$$(H_{\text{s.o.}})_{\lambda+'\lambda-',\lambda+\lambda-}=-\tfrac{1}{2}c[(\lambda+)-(\lambda-)]\delta_{\lambda+',\lambda+}\delta_{\lambda-',\lambda-}. \tag{3.76}$$

The matrix elements of the quadrupole-quadrupole interaction may be obtained with the help of (3.14), (3.16), and (3.18). The diagonal elements are

$$(H_{Q-Q})_{\lambda+\lambda-,\lambda+\lambda-}=\tfrac{1}{4}G^2[-T_0(T_0+2)+\tfrac{1}{2}(\lambda+)^2+\tfrac{1}{2}(\lambda-)^2]. \tag{3.77}$$

This interaction also couples pairs of states with the λ of each particle differing by two units (but with the same total Λ). These off-diagonal elements are given by

$$\begin{aligned}(H_{Q-Q})_{\lambda+'\lambda-',\lambda+\lambda-}=-\tfrac{1}{8}G^2\{&[T_0-(\lambda+)_<]\\ &\times[T_0+(\lambda+)_>][T_0-(\lambda-)_<][T_0+(\lambda-)_>]\}^{\frac{1}{2}}\\ &\times\delta_{\lambda+',(\lambda+)\pm2}\delta_{\lambda-',(\lambda-)\mp2},\end{aligned} \tag{3.78}$$

where $(\lambda+)_<$ means the smaller of $\lambda+$ and $\lambda+'$, and similarly for the other symbols. The quantum numbers $\lambda+$, $\lambda-$ are related to ν and Λ as follows:

$$(\lambda+)+(\lambda-)=\Lambda, \tag{3.79a}$$

$$(\lambda+)-(\lambda-)=2(T_0-\nu). \tag{3.79b}$$

The energy spectrum, up to second order in the x and first order in $1/T_0$, is

$$\frac{E_{\nu\Lambda}-E_{00}}{c}=\left(1-\frac{x}{2}-\frac{x^2}{8}\right)\nu+\frac{1}{T_0}\left[\frac{x}{4}\left(\nu^2+\frac{\Lambda^2}{4}\right)+\frac{x^2}{16}(3\nu^2+2\nu-\tfrac{1}{4}\Lambda^2)\right]. \tag{3.80}$$

In the limit $T_0\rightarrow\infty$, this gives the same vibrational spectrum as calculated previously (at least up to second order in x, but presumably to higher order as well). The levels in a vibrational multiplet are split by an amount proportional to $1/T_0$, and arranged in order of increasing Λ^2.

It is of interest to compare these results with the ordering of levels in even-even nuclei exhibiting vibrational spectra.[25] In particular, consider the so-called "two-phonon states" ($I=0$, 2, 4). The experimental evidence on their level ordering is still very inconclusive.[26] However, it appears that the 2+ state is

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[25] G. Scharff-Goldhaber and J. Weneser, Phys. Rev. **98**, 212 (1955), see also M. Nagasaki and T. Tamura, Progr. Theoret. Phys. Japan **12**, 248 (1954).

[26] C. A. Mallmann (to be published).

frequently below the 4+ state,[26,27] while the 0+ state seems to be higher in energy.

The same effects which split the levels in a multiplet must lead to deviations from the uniform level spacing. Thus the $\nu=2$ states occur at energies slightly larger than twice the energy of the $\nu=1$ state:

$$\frac{E_{20}-E_{00}}{E_{12}-E_{00}}=2+\frac{1}{T_0}(\tfrac{1}{2}x^2), \tag{3.81a}$$

$$\frac{E_{24}-E_{00}}{E_{12}-E_{00}}=2+\frac{1}{T_0}(x+\tfrac{3}{4}x^2). \tag{3.81b}$$

When the coupling parameter exceeds unity, it is convenient to use the strong-coupling wave function (characterized by T, Λ) as a basis set. The quadrupole-quadrupole force is diagonal in this representation:

$$(H_{Q-Q})_{T'\Lambda,\,T\Lambda}=\tfrac{1}{2}G^2[-T(T+1)+\tfrac{1}{4}\Lambda^2]\delta_{T'T}. \tag{3.82}$$

For the present case, the strong-coupling wave functions are symmetric or antisymmetric in the spatial coordinates of the two particles, depending on whether T_0-T is even or odd. Using this fact, it can be readily verified that the matrix elements of the spin-orbit interaction satisfy the relation:

$$(H_{\text{s.o.}})_{T'\Lambda,\,T\Lambda}=-c[1-(-1)^{T'-T}][t_z(+)]_{T'\Lambda,\,T\Lambda}. \tag{3.83}$$

In particular, the spin-orbit interaction has no diagonal elements. Since the spin-orbit operator is a vector, it can only connect pairs of states with T differing by unity. By using the standard techniques of Racah algebra, it can be shown that

$$\begin{aligned}[t_z(+)]_{T-1\,\Lambda,\,T\Lambda}=&(-1)^T(2T+1)^{\frac{1}{2}}\\ &\times(T1\,\tfrac{1}{2}\Lambda\,0|T1\,T-1\,\tfrac{1}{2}\Lambda)\\ &\times W(\tfrac{1}{2}T_0\,\tfrac{1}{2}T_0\,T\,T-1;1\,\tfrac{1}{2}T_0)(\tfrac{1}{2}T_0||t||\tfrac{1}{2}T_0),\end{aligned} \tag{3.84}$$

where the quantities on the right-hand side denote Clebsch-Gordan coefficient, Racah coefficient, and reduced one-particle matrix elements, respectively.[28] Evaluation of these quantities gives

$$\begin{aligned}(H_{\text{s.o.}})_{T-1\,\Lambda,\,T\Lambda}=&-c[(T_0+1+T)(T_0+1-T)\\ &\times(T^2-\tfrac{1}{4}\Lambda^2)/(4T^2-1)]^{\frac{1}{2}}.\end{aligned} \tag{3.85}$$

Since the spin-orbit coupling has no diagonal elements, there is no first order effect on the energy levels. In second order, the spin-orbit interaction alters both the rotational and vibrational level spacings.

The effective moments of inertia of the rotational bands are given by

$$\frac{\hbar^2}{2\mathfrak{J}}=\frac{G^2}{8}\left\{1+\frac{1}{x^2}\left[1+\frac{10(T_0-T)+1}{2T_0}\right]\right\}, \tag{3.86}$$

to second order in $1/x$. In the limit of infinitely large T_0, this result agrees with the unified model result (3.72). The effect of finite T_0 is to further increase the energy spacings, especially within the higher bands.

Deviations from the Λ^2 law appear only in fourth order of the spin-orbit strength. To study this effect, say for the lowest rotational band, we may start from the general perturbation equation:

$$\begin{aligned}E=E_0+H_{00}'+\sum_{n\neq0}\frac{H_{0n}'H_{n0}'}{E-E_n}\\ +\sum_{n\neq0}\sum_{m\neq0}\frac{H_{0n}'H_{nm}'H_{m0}'}{(E-E_n)(E-E_m)},\end{aligned} \tag{3.87}$$

where the subscript 0 refers to the state in question in absence of perturbation ($H'=0$), and the other subscripts represent the other unperturbed states. In the present case, the spin-orbit perturbation term $H_{\text{s.o.}}$ couples only states whose T differs by 1; thus each state in the ground-state band (T_0) is coupled only to the state of the same Λ in the next band (T_0-1). Denoting the states by their values of T_0-T, we obtain

$$\begin{aligned}E=E_0-\frac{|H_{10}'|^2}{E_1-E_0+[|H_{10}'|^2/(E_1-E_0)]}\\ -\frac{|H_{21}'|^2|H_{10}'|^2}{(E_2-E_0)(E_1-E_0)^2}\end{aligned} \tag{3.88}$$

for the energy up to fourth order in the spin-orbit coupling. We can pick out of (3.88) the term proportional to Λ^4:

$$(\Delta E)_4=-\frac{G^2}{8}\left[\frac{\Lambda^4}{4x^4T_0^2}\left(1+\frac{21}{8T_0}\cdots\right)\right]. \tag{3.89}$$

These terms lead to deviations of the rotational energy spectrum from the Λ^2 law. In the language of the unified model, these deviations are due to rotation-vibration interaction, i.e., an increase of $\mathfrak{J}$ with Λ^2, due to centrifugal forces. It is then easy to show that

$$\frac{E_{T_0\Lambda}-E_{T_00}}{E_{T_02}-E_{T_00}}=\frac{\Lambda^2}{4}\left[1-\frac{\Lambda^2-4}{4x^4T_0^2}\left(1+\frac{21}{8T_0}\cdots\right)\right], \tag{3.90}$$

to fourth order in x^{-1}. If T_0 is large, the deviations from the Λ^2 law become significant only when x is close to unity.

Consider next the effect of the spin-orbit interaction on vibrational energies. In particular, we may regard the excitation energy of the state with $T=T_0-1$ and $\Lambda=0$ as the first excited vibrational state. Using second order perturbation theory, it can be shown that:

$$\begin{aligned}\hbar\omega_{\text{vib}}&=E_{T_0-1,0}-E_{T_0,0}\\ &=G^2T_0\left[1-\frac{1}{2x^2}\left(1+\frac{2}{T_0}\cdots\right)\right],\end{aligned} \tag{3.91}$$

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[27] F. Raz (to be published). See however, L. Wilets and M. Jean, Phys. Rev. **102**, 788 (1956).

[28] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1957).

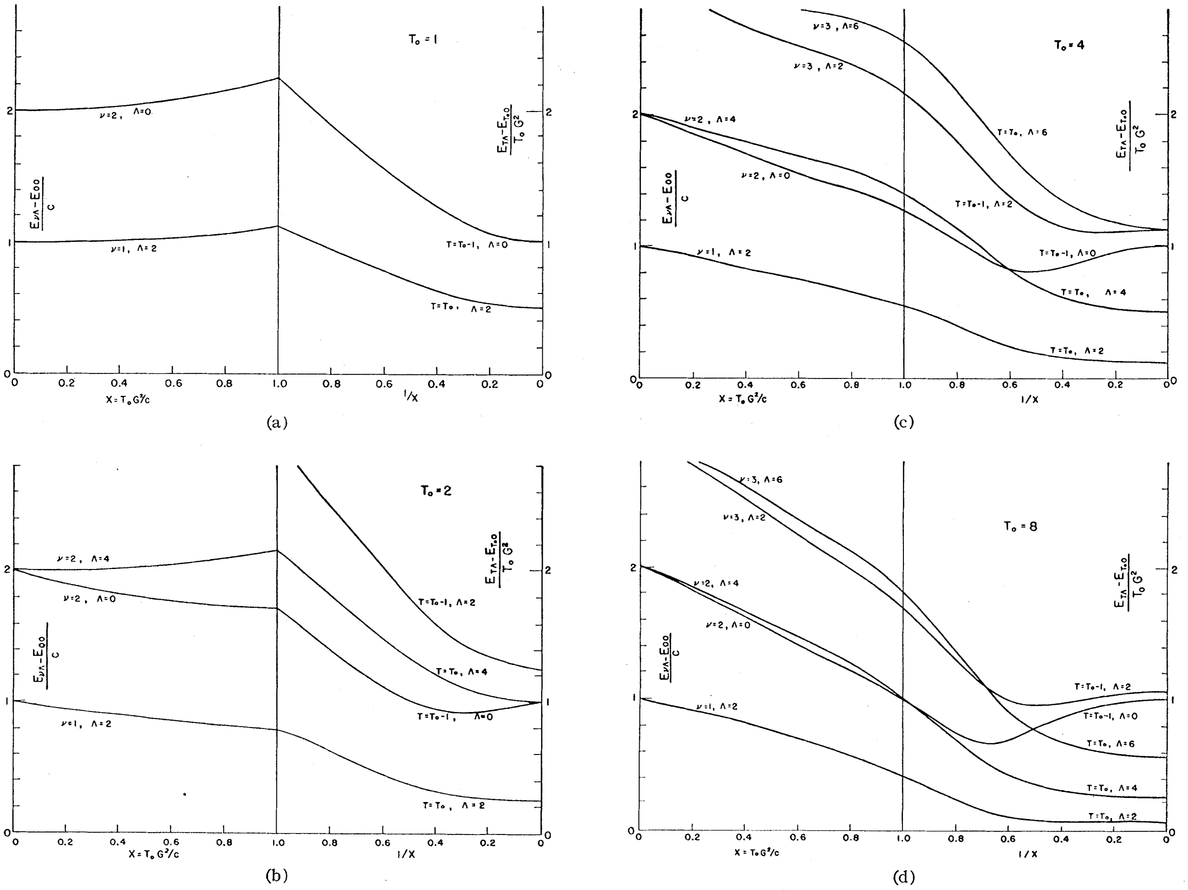

FIG. 5. Energy spectrum for intermediate-coupling problem calculated by diagonalization of energy matrix of (3.55) as function of the coupling parameter, ($\Sigma=0$ states only). The energies are plotted on the same scale as in Fig. 4. For explanation of the symbols, see the caption to Fig. 4. Part (a)—for $T_0=1$. Part (b)—for $T_0=2$, on the lowest states (up to $\nu=3$ are shown). In the limit $x\to\infty$, we obtain the spectrum shown in Fig. 10(b). Part (c)—for $T_0=4$. Part (d)—for $T_0=8$.

to second order in $1/x$ and first order in $1/T_0$. In the limit of large T_0, this agrees with (3.71) (at least up to second order in $1/x$, but presumably to all orders). It might be thought that the spin-orbit interaction should increase the vibrational energies, since it has no diagonal elements and interacting levels tend to repel. This argument is indeed valid for the case $T_0=1$, in which case there are only two interacting levels with $\Lambda=0$ (i.e., $T=0$, 1). However, when T_0 is larger, the situation is different. True, the energy of the ground state is pushed down, energetically, but the coupling to the lowest state in the first excited band. This coupling by itself, would push the T_0-1 state up by the same amount. However, the T_0-1 state is also pushed *down* by its coupling to the T_0-2 state. Explicit calculations show that the latter coupling is considerably stronger than the coupling to the ground state, in fact the T_0-1 state is actually lowered even *more* than the ground state. Consequently, the relative energy difference between the two states, i.e., the vibrational energy, is decreased by the spin-orbit interaction.

The energy spectra in the intermediate coupling region were also obtained by explicit diagonalizations[29] of the energy matrices for $T_0=1$, 2, 4, and 8, each for several values of x. The $T\Lambda$ representation was used even in the region of weak coupling. The resulting energy spectra of the lowest states as function of the coupling parameter are plotted in Fig. 5 for $T_0=1$, 2, 4, and 8, respectively. For $T_0=1$ there are only three states, not enough to exhibit any appreciable collective effects. As T_0 increases, more states are added, and the spectrum acquires form expected on the basis of the unified model. Detailed calculations were not made for values of T_0 larger than 8, but there seems little doubt that the energy spectra will look very similar to the one sketched in Fig. 6 for $T_0=20$. As expected [see (3.74)], the transition between weak coupling and strong coupling occurs for values of x slightly larger than unity. As T_0 increases, the transition occurs more suddenly. This is also illustrated by plots of the ratios

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[29] These diagonalizations were performed on the Johnniac at RAND Corporation. The author is indebted to the RAND Corporation for making time available for the matrix diagonalizations.

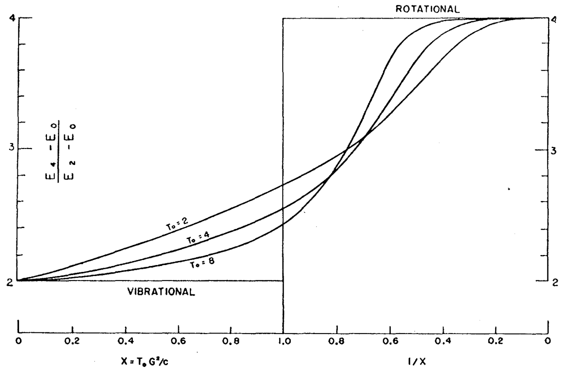

FIG. 6. Ratio $(E_4-E_0)/(E_2-E_0)$ of excitation energies plotted as function of x for various values of T_0. The subscripts 0, 2, 4 denote the lowest states with $\Lambda=0$, 2, 4, respectively.

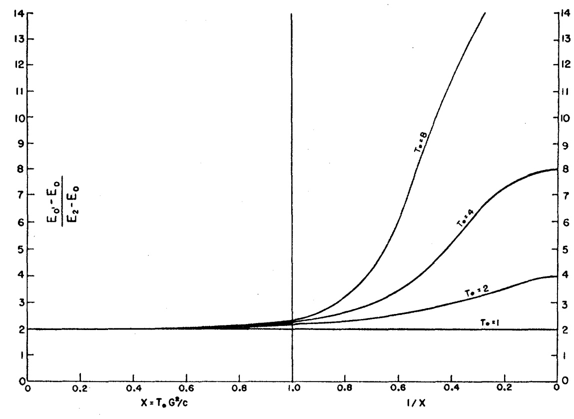

FIG. 7. Ratio $(E_{0'}-E_0)/(E_2-E_0)$ of excitation energies plotted as function of x for various values of T_0. The subscript 0′ denotes the first excited (vibrational) state with $\Lambda=0$. The other subscripts have the same meaning as in Fig. 6.

of excitation energies shown in Figs. 6 and 7, respectively.

4. ELECTRIC QUADRUPOLE TRANSITION PROBABILITIES ACCORDING TO THE TWO-DIMENSIONAL MODEL

An important characteristic of nuclear spectra is the frequent enhancement of electric quadrupole moments and transition probabilities for electric quadrupole radiation far above the values associated with single protons.[4,5] In the present case, the quadrupole operator is given by

$$Q=\sum_i r_i^2\cos 2\varphi_i. \tag{4.1}$$

Each particle is assumed to have unit charge. For the sake of convenience, we will express the quadrupole operator in units of the quantity $(\hbar/2m\omega_0)$ and restrict ourselves to transitions involving no change in the principal quantum number N of any particle. Using (3.8, 3.17, and 3.19) we then find

$$Q=\sum_i f_i(i)=\sum_i t_x(i)=T_x. \tag{4.2}$$

Note that this operator only connects pairs of states whose orbital angular momenta Λ differ by two units. Thus it has no diagonal matrix elements; i.e., the density distribution, averaged over time, is independent of direction (although the instantaneous density distribution may not be). In this respect, the two-dimensional problem is simpler than the three-dimensional one, since in the latter, each state usually has a static quadrupole moment. However, even in the two-dimensional problem, certain quadrupole matrix elements exhibit pronounced collective behavior.

Let us first consider the strong-coupling limit (only quadrupole-quadrupole interactions present). For this case, the spectrum separates into a series of rotational bands, each of which is characterized by a definite value of the quantum number T. Since the operator T_x only connects states with the same value of T, it follows that quadrupole transitions can only occur between states of the same rotational band. In particular, the matrix element between the ground state $(T=T_0, \Lambda=0)$ and the first excited state $(T=T_0, \Lambda=2)$ for an even-n configuration is given by:

$$Q_{T_02,T_00}=\tfrac{1}{2}[T_0(T_0+1)]^{\frac{1}{2}}. \tag{4.3}$$

The corresponding single-particle value of the matrix element is $\frac{1}{2}T_0^{\frac{1}{2}}$ as will be shown later. Thus if T_0 is large, the quadrupole matrix elements are larger than the single-particle values.

The matrix elements may also be calculated by expressing the wave functions in the form (3.40). As an illustration, suppose we have n particles, each in the lowest intrinsic state of the $N=1$ shell. It is easily shown that the overlap integral is given by

$$N_{LL}(\theta,\theta')=\cos^n(\theta-\theta'). \tag{4.4}$$

For $N>1$, the overlap integral is expected to fall off even faster as function of $\theta-\theta'$. Thus, if the number of particles is sufficiently large, the overlap integral behaves essentially like a δ-function of the argument $(\theta-\theta')$.[20] If this δ-function assumption is valid, the wave function can be written in the form given by Bohr and Mottelson[6]:

$$\psi_{L\Lambda}=\chi_L(r,\varphi')(2\pi)^{-\frac{1}{2}}e^{i\Lambda\theta} \tag{4.5}$$

for the evaluation of matrix elements. Let us use this form to calculate the matrix elements of the quadrupole operator. Now each of the χ corresponds to a unique value of T, and, the quadrupole operator connects only intrinsic states of the same T. Furthermore the total wave function has the same T as the intrinsic wave function which generates it. It follows that all matrix elements between states of different T must vanish. This result was already derived above.

To evaluate the matrix elements between states in the same rotational band, we express the quadrupole operator in the body reference frame as follows:

$$Q=T_x=T_x'\cos 2\theta. \tag{4.6}$$

We define an intrinsic quadrupole moment by

$$Q_0 = T_x'|_{LL}. \tag{4.7}$$

This is the instantaneous quadrupole moment of the density distribution in the body frame of reference.

The quadrupole matrix element is a product of Q_0 (which equals T_0 for the lowest intrinsic state) and an integral over the θ. Thus we obtain

$$Q_{T_0 2, T_0 0} = \tfrac{1}{2} T_0. \tag{4.8}$$

This approaches the exact result (4.3) if $T_0 \gg 1$.

Next we study the weak coupling limit (i.e., only spin-orbit coupling present), using the two-dimensional model of Sec. 3. In this case we have an independent-particle spectrum. Since the quadrupole operator can act only on one particle at a time, it is evident that transitions are allowed only between states differing with respect to the wave function of a single particle. The orbital angular momentum λ of this particle must change by two units. It can be seen from (3.79) that this is equivalent to the selection rule $\Delta\nu = 1$. Thus, for example, the transition between the $\nu = 0$, $\Lambda = 0$ ground state and the $\nu = 3$, $\Lambda = 2$ excited state is forbidden, since it would require a change in the λ of both particles. The matrix element for the transition between ground state and first excited state ($\nu = 1$, $\Lambda = 2$) is given by

$$Q_{12, 00} = \tfrac{1}{2} T_0^{\frac{1}{2}}. \tag{4.9}$$

In the language of the unified model, the collective wave functions are the eigenfunctions of a two-dimensional isotropic oscillator problem (3.53). The quadrupole operator is evidently proportional to the collective variable α_1, and thus it permits only "one-phonon" transitions,[6] in agreement with the above result.

Now suppose that we turn on the two-body interactions. This has the effect of increasing the amplitude $\Delta\beta$ of quadrupole vibration [Eq. (3.65)] but without significantly changing the form of the energy spectrum (provided the quantity T is large, and the coupling strength x is less than unity). The quadrupole matrix elements between states with $\Delta\nu = 1$ are therefore expected to increase with x. In fact explicit calculations show that they are proportional to $1 - \frac{1}{4}x$ up to first order in x. On the other hand, transitions involving $\Delta\nu = 1$ remain forbidden. In any case, all ratios of transition probabilities, like ratios of excitation energies, are essentially independent of x, provided $x < 1$ and $T_0 \gg 1$.

Finally, consider the case of nonspherical equilibrium shapes. The quadrupole matrix elements between states within a rotational band will increase with x, since they are proportional to $\beta_{\text{Eq.}}$ [Eq. (3.68)]. For example, it can be shown that in the limit of large T_0 we have

$$Q_{T_0 \Lambda, T_0\, \Lambda \pm 2} = \tfrac{1}{2} T_0 (1 - \tfrac{1}{2} x^{-2}) \tag{4.10}$$

up to terms of order x^{-2}. On the other hand, the transition strengths between states in different rotational bands (which can be regarded as vibrational transitions[30]) depend on the oscillations of β about its equilibrium value. Thus, according to (3.73) we would expect that vibrational transitions are weak compared to rotational transitions, at least if T_0 is large. To take a specific example, it can be verified that

$$Q_{T_0 - 1\, 2, T_0 0} = \tfrac{1}{2} T_0^{\frac{1}{2}} x^{-1}, \tag{4.11}$$

up to first order in x^{-1} and assuming $T_0 \gg 1$. In the strong-coupling limit, $x \to \infty$, the vibrational transitions cannot occur at all, as was already discussed at the beginning of this section.

5. NUCLEAR 1*p* SHELL

The most striking evidence for collective behavior in nuclei is found in heavy nuclei far removed from magic numbers. However, the connection between the shell model and unified model is perhaps most simply illustrated near the other end of the nuclear periodic table, in the 1*p* shell. Consider a system of n interacting nucleons in 1*p* orbits. We assume at the outset that all configurations mixing with other states, i.e., virtual 1*p*−2*p* excitations, etc., can be neglected. This is expected to be a fairly good approximation since the 1*p* shell is well separated from other shells.[31] Suppose that there are two kinds of interactions, (a) one-body spin-orbit interactions, and (b) two-body central, charge-independent, and velocity-independent interactions, i.e., linear combination of the four well-known types: Wigner, Heisenberg, Majorana, and Bartlett interactions. The nuclear coupling scheme in the 1*p*-shell is determined by the competition between these two kinds of interactions.[2,32] If the spin-orbit forces dominate, the particles move essentially independently and the j of each nucleon remains a good quantum number, i.e., we have jj coupling. If the two-body interactions dominate, the motions of the particles will be correlated, but the total orbital angular momentum L and total spin S will be constants of the motion, i.e., we have LS coupling. In actual 1*p* nuclei we have an intermediate situation. At the beginning of the shell (A up to 9) the coupling scheme is fairly close to the LS limit, but for heavier nuclei, the strength of the effective spin-orbit coupling is larger,[33] and jj coupling provides a better approximation. Of course, for any detailed fits to the experimental data, it is necessary to treat both spin-orbit and two-body interactions.[2]

Let us first consider the LS coupling scheme, and restrict ourselves to states which are completely symmetric with respect to the spatial coordinates of all nucleons. These completely space-symmetric states will be lowest in energy, since the mutual interactions are predominantly attractive. The energy levels of

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[30] They can also be regarded as particle transitions involving a change of the intrinsic structure.

[31] Such configuration mixing must, however, be taken into account for some applications. See reference 2.

[32] D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953).

[33] D. Kurath, Phys. Rev. **101**, 216 (1956).

such states of the p^n configuration can be expressed in terms of the energy levels of the space-symmetric two-particle states.[17] Two p nucleons may form a space-symmetric state of orbital angular momentum 0 or 2. We can always fit the energies of these two states by expressing the interaction between the nucleons in the form

$$v_{ik}=a+b\mathbf{l}_i\cdot\mathbf{l}_k, \tag{5.1}$$

where a and b are constants, which depend on the details of the effective two-body interaction. The interaction Hamiltonian for the p^n configuration may then be written as $\sum\sum_{i<k} v_{ik}$, i.e., excluding self interaction terms, and the energy levels are given by

$$E(p^n)_L=\tfrac{1}{2}n(n-1)a+[\tfrac{1}{2}L(L+1)-n]b. \tag{5.2}$$

For attractive interactions, the constant b is positive, so that a characteristic rotational spectrum is obtained. We may express (5.2) in the form

$$E_L-E_{L_0}=(\hbar^2/2\mathfrak{J})[L(L+1)-L_0(L_0+1)], \tag{5.3}$$

where the subscript 0 refers to the ground state and the effective moment of inertia is given by

$$(\hbar^2/2\mathfrak{J})=\tfrac{1}{2}b, \tag{5.4}$$

independent of the number of particles. Because of symmetry requirements, the allowed values of L are restricted to the following: n, $n-2,\cdots$ (1 or 0), i.e. all even or odd values up to n occur according to whether n is even or odd. Figure 8 shows the level schemes (completely space-symmetric states only) for p^2, p^3, and p^4 configurations in LS coupling.

If self-interaction terms are included, each energy level is displaced by a constant, but there is no change in the relative spacing, e.g., the moment of inertia.

To be more specific, let us suppose that the two-body interactions are spin-independent and of the form:

$$v_{ik}=-G^2P_2(\cos\varphi_{ik}). \tag{5.5}$$

The energies of the two-particle S and D states may then be readily evaluated by standard shell-model techniques. It is found that

$$E(p^2)_0=-\tfrac{2}{5}G^2, \tag{5.6a}$$

$$E(p^2)_2=-(1/25)G^2. \tag{5.6b}$$

The spacing between these two levels gives the moment of inertia:

$$\hbar^2/2\mathfrak{J}=(3/50)G^2. \tag{5.7}$$

In order to interpret the above energy spectra on the basis of the rotational model, we proceed by essentially the same method as in Sec. 3. The particles are regarded as moving independently, but in a variable nonspherical potential.[34] For the present case, we assume that the surfaces of constant potential are spheroids.

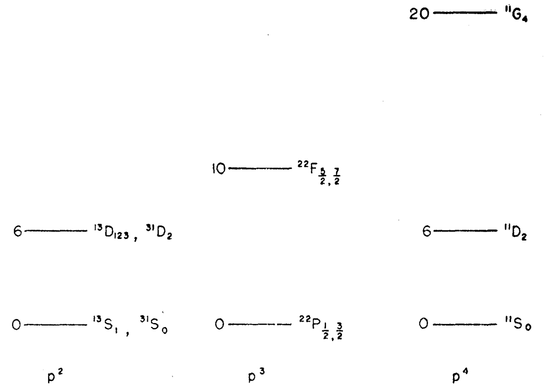

FIG. 8. Level schemes for p^2, p^3, and p^4 configurations in LS coupling. Only states which are completely symmetric in the spatial coordinates of all nucleons are shown. Since the total nuclear wave function must be antisymmetric with respect to interchange of any two particles, it must also be antisymmetric with respect to spin and isotopic spin variables. Such states cannot be constructed for more than 4 particles in the p shell. Each level is characterized by $^{2T+1,2S+1}L_I$ and its excitation energy in units of $\hbar^2/2\mathfrak{J}$. Note that while all states shown follow the rotational $L(L+1)$ law, only the states with $S=0$ also follow an $I(I+1)$ law.

The well is characterized by a deformation α which is proportional to the fractional elongation of the symmetry (3) axis, and by three Euler angles collectively denoted by θ, which specify the orientation of the system with respect to a fixed reference frame. The part of the intrinsic Hamiltonian which describes the nonspherical part of the potential is

$$h=-G\alpha\sum_i P_2(\cos\varphi_i')+\tfrac{1}{2}\alpha^2 \tag{5.8}$$

analogous to (3.28), where φ_i' denotes the angular coordinate of particle i with respect to the 3-axis. The single-particle states have components of orbital angular momentum (denoted by Λ_i) along the symmetry axis equal to 0 or ±1. (Note that the quantity Λ now has a different meaning than in Sec. 3.) The intrinsic single-particle energies are given by

$$E_{\Lambda_i=0}=-\tfrac{2}{5}G\alpha, \tag{5.9a}$$

$$E_{\Lambda_i=\pm1}=\tfrac{1}{5}G\alpha, \tag{5.9b}$$

to first order in the deformation α.

Clearly it is energetically most favorable to put all the particles into the $\Lambda_i=0$ orbit. Then the total component of orbital angular momentum Λ along the symmetry axis vanishes. The spatial part of the resulting intrinsic function, being a product of identical single-particle functions, is of course symmetric in the spatial coordinates of all the particles. The equilibrium deformation, i.e., the value of α which minimum intrinsic energy, is given by

$$\alpha_{\mathrm{Eq}}=G\sum_i[P_2(\cos\varphi_i')]_{LL}=\tfrac{2}{5}nG, \tag{5.10}$$

where the subscripts LL denote the diagonal element with respect to lowest intrinsic state χ_L. The spacing

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[34] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd 29, No. 16 (1955).

between adjacent intrinsic levels is given by

$$e=(3/5)G\alpha_{\mathrm{Eq}}=(6/25)nG^2. \tag{5.11}$$

The uniform spacing of intrinsic levels is a consequence of the fact that there are only two separate single-particle levels. Also it is readily shown that the collective rotation operators connect only states separated by e in energy. Finally the intrinsic energy is independent of the orientation angles. Consequently, we may obtain an exact solution of this particular many-body problem (including self interaction terms):

$$H=-\tfrac{1}{2}\sum_i\sum_k G^2P_2(\cos\varphi_{ik}), \tag{5.12}$$

by the method of Sec. 2. The procedure is very similar to that followed for the derivation of (3.38).

In the present case, there are two collective degrees of freedom; *viz.*, rotations of the system about the 1 and 2 axes, the directions perpendicular to the symmetry axis. Rotation about the symmetry axis does not constitute collective motion, since the intrinsic structure is entirely unaffected by such a rotation. The differential equation for the collective motion (2.40) is of the form:

$$-\frac{e^2}{2}\sum_{\perp=1,2}\frac{1}{\gamma_\perp}\left(\frac{\partial}{\partial\theta_\perp}\right)^2\Phi+(E_L-e)\Phi=E\Phi, \tag{5.13}$$

where

$$\gamma_\perp=3\alpha_{\mathrm{eq}}{}^2=(12/25)n^2G^2. \tag{5.14}$$

The eigensolutions of (5.13) are the well-known D functions.[28] Thus the eigenfunctions of H are of the form

$$\psi_{L\Lambda M}(\mathbf{r})=\int\chi_L(\mathbf{r},\theta)D_{M\Lambda}{}^L(\theta)d\theta, \tag{5.15}$$

where $d\theta$ denotes integration over all orientation angles. The effective moment of inertia is given by

$$(\hbar^2/2\mathfrak{J})=(e^2/2\gamma_\perp), \tag{5.16}$$

which equals the result (5.7) obtained previously by explicit treatment of the interactions.

Next consider the calculation of electric quadrupole moments and transition probabilities. According to the remarks in Sec. 2, the calculation of physical quantities may be greatly simplified by not integrating the wave function over the collective variables, and instead regarding the latter as extra variables. The resulting error is small, provided the number of particles involved is large. For the present case, this procedure implies that we write the wave functions in the well-known form,

$$\psi_{\mathrm{rot}}(\mathbf{r},\theta)=\chi_L(\mathbf{r},\theta)D_{M\Lambda}{}^L(\theta). \tag{5.17}$$

According to this version of the rotational model, the quadrupole moment of any state in the lowest rotational band is given by

$$Q_{\mathrm{rot}}=\frac{3\Lambda^2-L(L+1)}{(L+1)(2L+3)}Q_0=-\frac{L}{2L+3}Q_0, \tag{5.18}$$

where Q_0, the intrinsic quadrupole moment (its instantaneous value in the body system) is

$$Q_0=\sum_i(3z_i^2-r_i^2)_{LL}=(4n/5)\langle r^2\rangle, \tag{5.19}$$

and $\langle r^2\rangle$ denotes the average value of r^2 for any p-shell nucleon. The coefficient of Q_0 in (5.18) is the well-known projection factor.[6] Explicit calculations show that the "exact" value of the quadrupole moment [calculated using (5.15)] is related to the approximate value (5.18) as follows:

$$\frac{Q_{L,L}}{(Q_{\mathrm{rot}})_{L,L}}=\frac{2n+3}{2n}. \tag{5.20}$$

As n becomes larger, the rotational model comes closer to reproducing the exact result.

These quadrupole moments increase essentially linearly with the number of particles. The quadrupole matrix elements are also nearly proportional to the number of particles and they obey the relation

$$\frac{Q_{L+2,L}}{(Q_{\mathrm{rot}})_{L+2,L}}=\left[\frac{(n-L)(n+L+3)}{n^2}\right]^{\frac{1}{2}}. \tag{5.21}$$

Again, in the limit of many particles, transition probabilities between states of small L are given correctly by the rotational model, if the number of particles is sufficiently large. Incidentally, since Q is a symmetric operator, it can only connect a state of complete spatial symmetry to another of the same kind, i.e., no $E2$ transitions can occur between the lowest rotational band and any other. This is similar to the selection rule on the quantum number T discussed in Sec. 4.

While all the above spectra follow the rotational law, they are truly collective, i.e., rotational bands extending to large values of L, and quadrupole matrix elements large compared to the single-particle values, only if n is large. However, this cannot occur in the p shell, since the Pauli principle limits n to a maximum value of 4. In particular, at most two protons can act coherently in producing an electric quadrupole moment (neglecting configuration mixing to orbits other than $1p$, which can further increase the quadrupole moment). It is for this reason that collective effects have not been strikingly evident in the nuclear $1p$ shell.[35]

While the nuclei at the beginning of the p shell are rather close to LS coupling, the effect of the spin-orbit interaction is appreciable. In first order, the spin-orbit interaction

$$-c\sum_i\mathbf{l}_i\cdot\mathbf{s}_i \tag{5.22}$$

has the same effect on the energies of the completely space-symmetric levels as the interaction

$$-(c/n)\mathbf{L}\cdot\mathbf{S}. \tag{5.23}$$

Thus there is no effect on $S=0$ states. For n odd and

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[35] See however, R. A. Ferrell and V. M. Visscher, Phys. Rev. **104**, 475 (1956), and D. Kurath, Phys. Rev. **106**, 975 (1957).

$S=\frac{1}{2}$ the energy levels up to first order in the spin-orbit interaction, are given by:

$$E_{LI}-E_0=\frac{3}{50}G^2[L(L+1)-2]+\frac{c}{2n}\times\begin{Bmatrix}-L\\L+1\end{Bmatrix}$$

$$\text{for}\quad I=L\begin{Bmatrix}+\\-\end{Bmatrix}\tfrac{1}{2},\qquad \begin{matrix}(5.24\text{a})\\(5.24\text{b})\end{matrix}$$

where E_0 is the energy of the ground state ($L=1$) in absence of spin-orbit coupling.

The L value corresponding to each I is given by

$$L=I-\tfrac{1}{2}(-1)^{I+\frac{1}{2}}. \qquad (5.25)$$

Substitution of this result into (5.24) makes it possible to express the energy spectrum as function of I. We obtain

$$E_{LI}-E_0=(\hbar^2/2\mathfrak{J})[I(I+1)+a(-1)^{I+\frac{1}{2}}(I+\tfrac{1}{2})]-(\hbar^2/2\mathfrak{J})[(11/4)+a], \qquad (5.26)$$

where the moment of inertia is given by (5.7) and the quantity a is the following:

$$a=-1-(25/3)(c/nG^2). \qquad (5.27)$$

Apart from the last term, which is independent of I, the spectrum is precisely that of a rotational spectrum based on an intrinsic state with $K=\frac{1}{2}$ and with a decoupling parameter a.[3] The calculated energy spectrum of the completely space-symmetric states of the p^3 configuration as function of the spin-orbit strength, and the decoupling parameter is shown in Fig. 9. According to the rotational model, the lowest intrinsic state of the p^3 configuration has no net orbital angular momentum along the symmetry axis. The component K of the total angular momentum equals $\frac{1}{2}$, since it is due entirely to the spin of the last odd nucleon. Thus the level spectrum is expected to be of the form (5.26). The decoupling parameter is given by[3]

$$a=\textstyle\sum_j(-1)^{j-\frac{1}{2}}|c_j|^2(j+\tfrac{1}{2})=|c_{\frac{1}{2}}|^2-2|c_{\frac{3}{2}}|^2, \qquad (5.28)$$

where c_j denotes the probability amplitude for finding the last particle with total angular momentum j. To first order in the spin-orbit coupling strength, this reduces to the value (5.27) obtained previously.

We have seen that the rotational model yields both the correct moment of inertia and decoupling parameter, at least in the $1p$ shell near the LS limit. In second order, the spin-orbit interaction leads to a decrease of the moment of inertia, i.e., an increase of the energy spacing between states, but without altering the characteristic rotational energy ratios. However, in third and higher order, the spin-orbit interaction tends to destroy the rotational structure. This occurs in the second half of the $1p$ shell where the spin-orbit interaction is much stronger than in the first part of the shell.

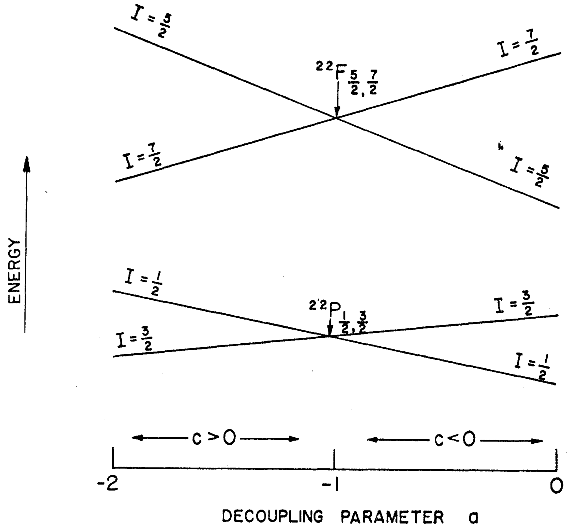

FIG. 9. First-order effect of spin-orbit coupling on the energy levels of the p^3 configuration. Only states of complete spatial symmetry are shown. The energies are plotted as a function of the decoupling parameter a which is related to the strength of the spin-orbit coupling by (5.27). The value $a=0$ corresponds to a pure rotational spectrum in absence of decoupling effects. For $a=-1$, we have the LS coupling limit. The effect of the nuclear spin-orbit coupling is to make a more negative. Thus for Li^7 we have $a\sim-1.2$.

6. CONCLUDING REMARKS

It appears from the treatment of the previous examples that the shell-model approach (the explicit treatment of the effective two-body interactions) and the unified-model approach (replacement of the interactions by a variable potential) are rather complementary. The shell-model method is, in principle, exact, if the Hamiltonian is known, while the unified model gives, in general, only approximate answers. On the other hand, the unified model provides considerably more insight into the physics of the nuclear coupling scheme than does the shell model, and also it is often easier to apply.

Thus, for example, consider the sudden transition between vibrational and rotational spectra for the case discussed in Sec. 3. This feature, which is found experimentally,[25,36] can be reproduced by the shell model, but is difficult to understand physically in terms of the mutual interactions without reference to the unified model. Also, even if the exact form of the effective two-body interactions is known, the calculation of energy levels and wave functions may be extremely difficult in practice unless the number of particles outside of (or missing from) closed shells is very small. For example, to study the nuclei of mass number ~220 it would be necessary to consider the system of about 10 interacting particles outside the Pb^{208} core, assuming that core excitation can be neglected. Even if the resulting huge energy matrices could be diagonalized,

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[36] G. Scharff-Goldhaber, Phys. Rev. **103**, 837 (1956).

we would still not understand the physics of the nuclear coupling scheme. Experimentally, a few odd-A nuclei in this region exhibit striking close-lying doublets,[37] and other interesting features as well, which may suggest that we have a number of nearly uncoupled modes of collective oscillations. One has hopes that the unified model will provide some insight into these phenomena.

It may be of interest in this connection to mention the model developed by Elliott.[18] The central potential is assumed to be of the harmonic oscillator form and the two-body interactions are quadrupole-quadrupole interactions:

$$v_{ik} = -D r_i^2 r_k^2 P_2(\cos\varphi_{ik}), \tag{6.1}$$

with the extra condition that all matrix elements between pairs of states differing with respect to the principal quantum number of any nucleon vanish. This is the three-dimensional version of the model discussed in Sec. 3. Elliott has investigated the resulting coupling scheme in detail and finds that it is equivalent to a simple version of the Nilsson rotating anisotropic harmonic oscillator model.[34] The nonspherical intrinsic Hamiltonian corresponding to the interaction (6.1) is

$$h = -D\alpha r_i^2 P_2(\cos\varphi_i') + \tfrac{1}{2}\alpha^2. \tag{6.2}$$

The intrinsic spectrum is quite similar in some respects to that for LS coupling in the p shell (see Sec. 5). In particular, the intrinsic levels are equally spaced, in energy, a special property of the harmonic oscillator potential. Thus as before, we obtain rotational spectra. However, while in the p shell only at most 4 nucleons can act coherently, in the present case a larger number of nucleons can contribute to the deformation because of the larger number of available orbits in each shell (of principal quantum number ≥ 2). Thus the quadrupole-quadrupole force can generate large collective effects, e.g., well developed rotational spectra going up to large values of the angular momentum and large quadrupole matrix elements. These features would also be obtained in the p shell if an arbitrarily large number of nucleons could be put into each orbit. Thus some of the results of the previous section, but with n not necessarily ≤ 4, are applicable, to the present case. For example, Eqs. (5.20) and (5.21) still hold for the lowest rotational band, provided we define n to be the maximum value of L in this band, regardless of the actual number of particles involved.

If we add one-body $\mathbf{l}\cdot\mathbf{s}$ and $\mathbf{l}\cdot\mathbf{l}$ terms to the above Hamiltonian, we obtain essentially the Nilsson model[34] in its most general form. This model, has had considerable success in explaining many properties of deformed odd-A nuclei; e.g. ground-state spins, gyromagnetic ratios, decoupling parameters, selection rules, all of which depend mainly on the motion of the last odd unpaired nucleon.[38] However, in order to understand other features, such as the pairing energy, or the magnitude of moments of inertia, it is necessary to take into account additional two-body interactions involving multipole orders higher than 2, the so-called "residual" interactions.[6,9,†]

Returning to the Elliott model, the $S=0$ states of maximum spatial symmetry form a single rotational band with angular momentum $L=0, 2, 4 \cdots$ i.e., $\Lambda=0$. The $S=0$ states of next lower spatial symmetry form two bands, one with $L=0, 2, 4 \cdots (\Lambda=0)$ and the other with $L=2, 3, 4 \cdots (\Lambda=2)$. States of the same L in the two bands are degenerate. It is tempting to suppose that these bands are related to the so-called β and γ vibrational bands, which are believed to occur in heavy nuclei. In the one nucleus, Pu^{238}, where both bands have been identified, they are nearly degenerate.[39] However, to understand these features in more detail, it is probably necessary to take the residual interactions into account.

Finally, we wish to mention the possibility of simulating the system of interacting particles by a two-nucleon model, i.e., to replace the many-body problem by an effective two-body problem. For the simplified two-dimensional model discussed in Sec. 3, it was possible to do this exactly, because of the particular form of the assumed interactions. In actual nuclei, the situation is much more complicated. However, while any two-nucleon model can at best give approximate results, it might provide some valuable insight into the coupling schemes in heavy nuclei.

In closing, it is hoped that the approach outlined in this paper may be helpful in the development of a unified low-energy nuclear theory.

ACKNOWLEDGMENTS

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APPENDIX A. ROTATIONAL COUPLING BETWEEN CLOSE-LYING INTRINSIC STATES

Let us suppose that the intrinsic spectrum is somewhat more complicated than the kind considered in Sec. 2, in particular consider the case illustrated in part (b) of Fig. 1.

We shall restrict ourselves to a single collective angle variable (rotations in two dimensions). This treatment can be readily generalized to take into account several collective variables. The intrinsic spectrum is assumed to consist of groups of nearly degenerate levels. The energy interval e between the lowest two groups is taken to be large compared to the spacing within each

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[37] R. Pilger, University of California Radiation Laboratory Report UCRL-3877, July, 1957 (unpublished).

[38] B. R. Mottelson and S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd (to be published); K. Gottfried, Phys. Rev. **103**, 1017 (1956).

[39] I. Perlman and J. O. Rasmussen, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 42.

group. Suppose that the collective rotation operator connects pairs of states in the same group, as well as states separated by e. The former gives rise to a rotational coupling between the intrinsic states in each group, in particular the group of lowest energy.

In view of the rotational coupling, we have, corresponding to (2.26), the following equation:

$$\frac{\partial h}{\partial\theta}\chi_\nu=-e\left[\frac{\partial\chi_\nu}{\partial\theta}-\sum_{\nu'}\left(\frac{\partial}{\partial\theta}\right)_{\nu'\nu}\chi_{\nu'}\right],\tag{A.1}$$

where ν and ν' refer to any intrinsic state in the lowest group and the sum is taken over all such states. We have neglected the contribution of states within the same group to the quantity $[\partial h/\partial\theta]\chi_\nu$, since the relevant energy differences will be small compared to e.

By the same techniques as were used in Sec. 2, we obtain

$$H\chi_\nu=(E_L-\tfrac{1}{2}e)\chi_\nu-A_\theta\left[\frac{\partial^2\chi_L}{\partial\theta^2}-2\sum_{\nu'}\left(\frac{\partial}{\partial\theta}\right)_{\nu'\nu}\frac{\partial\chi_{\nu'}}{\partial\theta}\right.$$
$$\left.+\sum_{\nu'}\left(\frac{\partial^2}{\partial\theta^2}\right)_{\nu'\nu}\chi_{\nu'}\right]\tag{A.2}$$

corresponding to (2.37). The quantity A_θ is given by

$$A_\theta=e^2/2\gamma_\theta,\tag{A.3}$$

and γ_θ is defined by (2.32). The total wave functions are of the form (2.46), i.e.,

$$\psi=\int\sum\nolimits_\nu\chi_\nu\Phi_{\nu\Lambda}d\theta,\tag{A.4}$$

where the sum ν extends over all intrinsic states in the lowest group, and Λ refers to the state of collective motion. From (2.45) and (2.47) it follows that the collective wave functions obey the set of coupled differential equations:

$$H\Phi_{\nu\Lambda}=(E_L-\tfrac{1}{2}e)\Phi_{\nu\Lambda}-A_\theta\left[\frac{\partial^2}{\partial\theta^2}\Phi_{\nu\Lambda}-2\sum_{\nu'}\left(\frac{\partial}{\partial\theta}\right)_{\nu'\nu}\frac{\partial\Phi_{\nu'\Lambda}}{\partial\theta}\right.$$
$$\left.+\sum_{\nu'}\left(\frac{\partial^2}{\partial\theta^2}\right)_{\nu'\nu}\Phi_{\nu'\Lambda}\right].\tag{A.5}$$

The above equations are essentially of the form used by Kerman[40]:

$$H=h-A_\theta[\mathbf{I}-\mathbf{j}]^2,\tag{A.6}$$

where h is the intrinsic Hamiltonian, $\mathbf{j}$ denotes the intrinsic angular momentum operator (the latter assumed to connect only close-lying intrinsic states) and $\mathbf{I}$ is the total angular momentum operator which acts on the collective wave function.

Let us now suppose that there are only two intrinsic states, denoted by L and M, in the lowest group and that the operator $\partial/\partial\theta$ couples these states to each other, but has no diagonal matrix elements. The nonvanishing matrix elements can be written as follows:

$$i\left(\frac{\partial}{\partial\theta}\right)_{ML}=-i\left(\frac{\partial}{\partial\theta}\right)_{LM}=\lambda e^{i\varphi},\tag{A.7}$$

where λ and φ are real. We also assume that each of the collective wave functions can be expressed in the form

$$\Phi_{\nu\Lambda}=\phi_{\nu\Lambda}e^{i\Lambda\theta},\tag{A.8}$$

where the $\phi_{\nu\Lambda}$ are constants. The collective equations then become

$$[E_L-\tfrac{1}{2}e+A_\theta(\lambda^2+\Lambda^2)]\phi_{L\Lambda}$$
$$+2A_\theta\lambda\Lambda e^{-i\varphi}\phi_{M\Lambda}=E\phi_{L\Lambda},\tag{A.9a}$$
$$2A_\theta\lambda\Lambda e^{i\varphi}\phi_{L\Lambda}$$
$$+[E_M-\tfrac{1}{2}e+A_\theta(\lambda^2+\Lambda^2)]\phi_{M\Lambda}=E\phi_{M\Lambda}.\tag{A.9b}$$

Thus we obtain two coupled rotational bands. In the absence of any coupling, the energy in each band depends quadratically on Λ^2. In general, the two energy eigenvalues of (A.9) are given by

$$E=\tfrac{1}{2}(E_L+E_M-e)+A_\theta(\lambda^2+\Lambda^2)$$
$$\pm[\tfrac{1}{4}(E_M-E_L)^2+4A_\theta{}^2\lambda^2\Lambda^2]^{\frac{1}{2}}.\tag{A.10}$$

In case the states L and M are degenerate, the result is

$$E=E_L-\tfrac{1}{2}e+A_\theta(\lambda^2+\Lambda^2\pm2\lambda\Lambda).\tag{A.11}$$

The last term in (A.11) is essentially of the same form as the well-known decoupling term which occurs for rotational bands in some odd-A nuclei[3] (with $K=\frac{1}{2}$). In the latter, however, the symmetry requirements on the wave function only permit one value of the sign for each I (which corresponds to Λ in our case).

Now, consider the other limit, i.e., coupling energy small compared to the energy difference

$$\epsilon=E_M-E_L.\tag{A.12}$$

The energies may then be expanded as power series in Λ. The result for the lower state of given Λ, is

$$E=E_L-\tfrac{1}{2}e+A_\theta\lambda^2+(1-4A_\theta\lambda^2\epsilon^{-1}$$
$$+16A_\theta{}^3\lambda^4\Lambda^2\epsilon^{-3})A_\theta\Lambda^2\tag{A.13}$$

in second order, the coupling merely increases the moment of inertia in the lower of the two rotational bands.[40,41] We find that the effective moment of inertia for this band is given by

$$\Im=\frac{\hbar^2}{2A_\theta}\left[1+\frac{4A_\theta\lambda^2}{\epsilon}\right],\tag{A.14}$$

up to terms of order λ^2. This agrees with the value calculated by time-dependent perturbation theory[22,23]

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[40] A. K. Kerman, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd **30**, No. 15 (1956).

[41] G. Lüders, Z. Naturforsch. **11a**, 617 (1956).

(the so-called "cranking model"). The first term,

$$\mathfrak{J}_0 = \hbar^2/2A_0, \tag{A.15}$$

is just the value which we obtain when there is no rotational coupling between close-lying states, while the second term,

$$\Delta\mathfrak{J} = 2\hbar^2\lambda^2/\epsilon, \tag{A.16}$$

represents the extra contribution of the close-lying states to $\mathfrak{J}$. Altogether the rotational kinetic energy is given as follows:

$$T = T_0\left[1 - \frac{\Delta\mathfrak{J}}{\mathfrak{J}_0} + \left(\frac{\Delta\mathfrak{J}}{\mathfrak{J}_0}\right)^2\left(\frac{T_0}{\epsilon}\right)\cdots\right], \tag{A.17}$$

where

$$T_0 = \hbar^2\Lambda^2/2\mathfrak{J}_0. \tag{A.18}$$

Perturbations from the simple Λ^2 law thus appear only in higher order, and they are of opposite sign from the perturbations caused by rotation-vibration interaction. Note that if the energy difference ϵ between the intrinsic states is large compared to the rotational kinetic energy, but small compared to e, then there will be relatively little departure from the Λ^2 law, even if the effective value of $\mathfrak{J}$ is increased considerably by the coupling.

APPENDIX B. EFFECT OF THE SELF-INTERACTION TERMS

It may be of interest to consider more precisely the role of the self-interaction terms which occur if the Hamiltonian is expressed in the form (2.5). If these are not taken into account, we do not, in general, obtain the simple spectra described in Sec. 3. Thus, suppose we have 2 particles in the $N=2$ shell of a two-dimensional harmonic oscillator potential and consider only a mutual quadrupole-quadrupole interaction (3.3) between them. The resulting energy level spectrum is shown in Fig. 10(a). The spectrum is not quite of the rotational form. For example, the second excited state ($\Lambda=4$) occurs at an energy (relative to the ground state) of about 3.5 times that of the first excited state ($\Lambda=2$), a somewhat smaller ratio than the value 4 characteristic of a rotational spectrum in two dimensions.

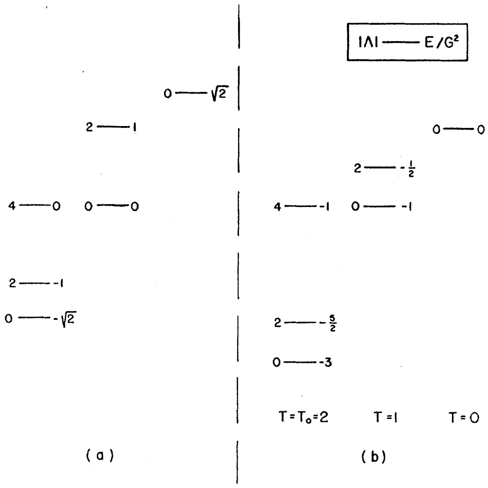

FIG. 10. Energy level schemes for a system of two particles in the $N=2$ shell interacting via attractive quadrupole-quadrupole forces. In part (a), only the interactions between the particles is considered, while in part (b) the self-interaction terms are also included. Each state is labeled by the value of $|\Lambda|$ and its energy, in units of the interaction strength G^2, is also indicated. In case (b) each group forms a rotational band and its value of T is also given. [In case (a), T is not a good quantum number.]

The self-interaction term for each particle is given quite generally by

$$H_{\text{self}}(i) = -\tfrac{1}{2}\sum_\nu c_\nu |g_\nu(i)|^2. \tag{B.1}$$

We may use (B.1) for the present problem if the related quantities f are given by (3.9), i.e., if matrix elements between states of different N vanish automatically. On the other hand, if the f are given by (3.8), the vanishing of the matrix elements between states of different N does not occur automatically, and must be added as an extra condition. This can be accomplished by expressing the matrix elements of the self-interaction for each particle in the form

$$[H_{\text{self}}(i)]_{KL} = -\tfrac{1}{2}\sum_\nu \sum_M c_\nu (g_\nu{}^*)_{KM}(g_\nu)_{ML} \tag{B.2}$$

between single-particle states K and L. If Eq. (3.9) is used for the f's, the sum extends only over single-particle states in the same shell as K and L. In this way we find that the self-interaction matrix is diagonal, at least for the present problem. The diagonal element for $N=2$, $\lambda=0$ is $-G^2$, and for $N=2$, $\lambda=\pm 2$ it is $-\frac{1}{2}G^2$. Adding this self-energy of each particle to the matrix of the mutual interaction and diagonalizing, we obtain the rotational energy spectrum shown in Fig. 10(b).