Effects of Electron-Electron Interactions on Cyclotron Resonances in Gaseous Plasmas*

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The effect of mutual interactions between electrons on the cyclotron resonance of a gaseous discharge plasma has been studied. From the Boltzmann-Fokker-Planck equation in cylindrical coordinates, a set of simultaneous integro-differential equations is obtained and solved numerically on a digital computer for various values of magnetic field and electron density. The results indicate that the real part of the electrical conductivity of the plasma, and hence its power absorption, are reduced by the electron-electron interactions at the peak of the resonance, and that the width of the resonance is increased. The broadening of the resonance width becomes increasingly pronounced at higher charge concentrations. It is also found that, with the magnetic field equal to zero, the high-frequency conductivity of the plasma is practically unaltered by the electron-electron interactions. Thus the usual expression for the electrical conductivity of a Lorentzian gas cannot be used without discretion for high-density plasmas, where mutual electronic encounters cannot be ignored.

I. INTRODUCTION

WHEN the charge density of a gaseous plasma is high, the effects of the interactions that take place between the charges cannot be ignored. The influence of the electron-ion collisions on the electrical conductivity of a plasma has been studied by previous investigators,[1] and has been measured experimentally by using microwave techniques.[2,3] Little attention was given to the electron-electron encounters, since, due to momentum conservation in the electron system, such interactions were considered to have little effect on the electrical conductivity, which is dependent upon the momentum transfer collision frequency of the electrons. Spitzer and Härm[4] have found, however, that the dc conductivity of a completely ionized gas is considerably reduced when the electron-electron interactions are taken into account. It is, therefore, of interest to ascertain whether the high-frequency conductivity, which is measurable by microwave methods, is also affected to the same extent. In the case where a steady magnetic field is applied perpendicularly to the oscillating electric field, a cyclotron resonance may occur in the system. Since the width of the resonance is determined by the collisional processes in the plasma, it is certainly also of great interest to examine the effect of electron-electron interactions on the shape of the cyclotron resonance. Inasmuch as the former problem is a special case of the latter with the magnetic field equal to zero, our investigation will be formulated at the outset in the form suitable for treating cyclotron resonances.

Bohm and Pines[5,6] have shown that there are two types of electron interactions: the long-range and the short-range. The long-range part of the interactions exhibits an organized phenomenon, and is the chief agent that brings about the collective behavior of the system, e.g., the plasma oscillations. On the other hand, the short-range part of the Coulomb interactions is random in nature and is associated with the thermal fluctuations of the electrons in the plasma. To be complete in our study, we must consider both of these types of electron interactions. Upon writing the Boltzmann equation in the collective coordinates of the electron system obtained by a canonical transformation similar to that employed by Bohm and Pines, it was found, however, that for most gaseous discharge plasmas the effect of the long-range electron interactions on the velocity distribution function of the electrons is relatively unimportant.[7] Thus, a detailed description of the treatment of the long-range interactions will not be attempted here. We shall therefore turn our attention in this paper exclusively to the short-range interactions.

Since the cutoff distance which separates the long- and the short-range forces is generally set to be the Debye length λ_D, or in certain instances, even longer than λ_D, the short-range interactions are actually still quite long compared to the close encounters where large-angle scatterings occur. Since the short-range forces are random in nature, and produce only small-angle deflections on the electron paths, the theory of Brownian motion may be adopted, and the usual Fokker-Planck equation[8] may therefore be used. The Fokker-Planck (F-P) equation for particles interacting through shielded Coulomb force has been studied

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[1] E.g., V. Ginsburg, J. Phys. U.S.S.R. **8**, 253 (1944).
[2] J. M. Anderson and L. Goldstein, Phys. Rev. **100**, 1037 (1955).
[3] A. A. Dougal and L. Goldstein, Phys. Rev. **109**, 615 (1958).
[4] L. Spitzer and R. Härm, Phys. Rev. **89**, 977 (1953).
[5] D. Bohm and D. Pines, Phys. Rev. **82**, 625 (1951).
[6] D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952).
[7] See R. C. Hwa, Ph.D. thesis, University of Illinois, 1957 (unpublished).
[8] S. Chandrasekhar, Revs. Modern Phys. **15**, 1 (1943).

before by previous investigators.[9–12] The work presented here is an extension to the high-frequency case and to the case where a steady magnetic field may be applied so that a cyclotron resonance may occur in the plasma. We shall assume that the plasma is uniform and that the applied oscillating electric field is very weak so that the temperature of the electrons is not altered appreciably even at the peak of the cyclotron resonance. These restrictions are essential, since we are interested in the effect on the resonance shape due to electron-electron interactions, instead of that due to the diamagnetism of the plasma arising from nonuniform spatial distribution, or due to the increase in electron temperature caused by strong external fields.

II. F-P EQUATION IN CYLINDRICAL COORDINATES

The Fokker-Planck equation describing the time rate of change of the distribution function $f(\mathbf{v},t)$ due to encounters resulting in small-angle deflections is usually expressed in Cartesian coordinates as

$$\frac{\delta f}{\delta t}=-\sum_i \frac{\partial}{\partial v_i}(f\langle\Delta v_i\rangle)+\tfrac{1}{2}\sum_{ij}\frac{\partial^2}{\partial v_i\partial v_j}(f\langle\Delta v_i\Delta v_j\rangle), \quad (1)$$

where the indices i and j designate the components in the rectangular system. $\langle\Delta v_i\rangle$ is the average change per unit time of the i component of the velocity of the test particle; the average is to be taken over all collisional parameters. In order that the Fokker-Planck method may be used, Δv_i must be very small so that the higher order moments of Δv_i may be neglected. It may be shown[13] that if τ represents a time interval, long compared to the mean period of the fluctuating forces, but short compared to the average time for an appreciable change of the particle momentum, then to the first power of τ, only the first two moments of Δv_i should be retained. The averages $\langle\Delta v_i\rangle$ and $\langle\Delta v_i\Delta v_j\rangle$ are to be calculated as usual on the basis of binary collisions, using the Coulomb force, cut off at an appropriate distance, for charge interactions in plasmas.[11] Hence

$$\begin{aligned}\langle\Delta v_i\rangle&=\int\!\!\int\!\!\int \Delta v_i F(\mathbf{V})g\sigma(\theta_g,g)\,\sin\theta_g d\theta_g d\phi_g d^3V,\\ \langle\Delta v_i\Delta v_j\rangle&=\int\!\!\int\!\!\int \Delta v_i\Delta v_j F(\mathbf{V})g\sigma(\theta_g,g)\,\sin\theta_g d\theta_g d\phi_g d^3V,\end{aligned} \quad (2)$$

where $F(\mathbf{V})$ is the velocity distribution function of the field particles, g is the relative velocity between the colliding particles, $\sigma(\theta_g,g)$ is the differential scattering cross section, and θ_g and ϕ_g are the scattering angles in the center-of-gravity system.

For problems where cyclotron resonances may occur, the F-P equation must be written in cylindrical coordinates. With $v_\perp$, $v_{||}$, and ϕ denoting the radial velocity, the axial velocity, and the azimuthal angle, respectively, the equation becomes

$$\frac{\delta f}{\delta t}=-\sum_k \frac{1}{v_\perp}\frac{\partial}{\partial v_k}(fv_\perp\langle\Delta v_k\rangle)+\tfrac{1}{2}\sum_{kl}\frac{1}{v_\perp}\frac{\partial^2}{\partial v_k\partial v_l}(fv_\perp\langle\Delta v_k\Delta v_l\rangle), \quad (3)$$

where v_k and v_l may each be $v_\perp$, $v_{||}$, or ϕ. To facilitate later calculations, it is advantageous to express the velocity change $\Delta\mathbf{v}$ in a rectangular coordinate system ξ, η, ζ, where ξ is in the direction of $\mathbf{v}$, while η and ζ are in the directions of increasing θ and ϕ, respectively (θ being the polar angle of $\mathbf{v}$). In terms of changes along the ξ, η, ζ axes, *viz.*, Δv_ξ, Δv_η, Δv_ζ, one may show that

$$\begin{aligned}\Delta v_\perp&=\frac{v_\perp}{v}\Delta v_\xi+\frac{v_{||}}{v}\Delta v_\eta+\frac{1}{2v^2}\left(v_\perp+\frac{v_{||}^2}{v_\perp}\right)(\Delta v_\zeta)^2,\\ \Delta\phi&=\frac{\Delta v_\zeta}{v_\perp}-\frac{\Delta v_\xi\Delta v_\zeta}{v_\perp v}-\frac{v_{||}}{v_\perp^2 v}\Delta v_\eta\Delta v_\zeta,\\ \Delta v_{||}&=\frac{v_{||}}{v}\Delta v_\xi-\frac{v_\perp}{v}\Delta v_\eta,\end{aligned} \quad (4)$$

where only the first and second power terms have been retained. Writing γ_α for $\langle\Delta v_\alpha\rangle$ and $\gamma_{\alpha\beta}$ for $\langle\Delta v_\alpha\Delta v_\beta\rangle$ where α and β may each be ξ, η, or ζ, and making use of (4), the right side of Eq. (3) may be expanded into nine terms, that is,

$$\delta f/\delta t=(a)+(b)+\cdots+(i), \quad (5)$$

where

$$\begin{aligned}(a)&=-\frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left[f\left(\frac{v_\perp^2}{v}\gamma_\xi+\frac{v_\perp v_{||}}{v}\gamma_\eta+\tfrac{1}{2}\gamma_{\zeta\zeta}\right)\right],\\ (b)&=-\frac{1}{v_\perp}\frac{\partial}{\partial\phi}\left[f\left(\gamma_\zeta-\frac{1}{v}\gamma_{\xi\zeta}-\frac{v_{||}}{vv_\perp}\gamma_{\eta\zeta}\right)\right],\\ (c)&=-\frac{\partial}{\partial v_{||}}\left[f\left(\frac{v_{||}}{v}\gamma_\xi-\frac{v_\perp}{v}\gamma_\eta\right)\right],\\ (d)&=\frac{1}{2v_\perp}\frac{\partial^2}{\partial v_\perp^2}\left[f\left(\frac{v_\perp^3}{v^2}\gamma_{\xi\xi}+\frac{2v_\perp^2 v_{||}}{v^2}\gamma_{\xi\eta}+\frac{v_\perp v_{||}^2}{v^2}\gamma_{\eta\eta}\right)\right],\\ (e)&=\frac{1}{2v_\perp^2}\frac{\partial^2}{\partial\phi^2}[f\gamma_{\zeta\zeta}],\\ (f)&=\frac{1}{2}\frac{\partial^2}{\partial v_{||}^2}\left[f\left(\frac{v_{||}^2}{v^2}\gamma_{\xi\xi}-\frac{2v_\perp v_{||}}{v^2}\gamma_{\xi\eta}+\frac{v_\perp^2}{v^2}\gamma_{\eta\eta}\right)\right],\\ (g)&=\frac{1}{v_\perp}\frac{\partial^2}{\partial v_\perp\partial\phi}\left[f\left(\frac{v_\perp}{v}\gamma_{\xi\zeta}+\frac{v_{||}}{v}\gamma_{\eta\zeta}\right)\right],\end{aligned} \quad (6)$$

[9] Cohen, Spitzer, and Routly, Phys. Rev. **80**, 230 (1950).
[10] W. P. Allis, *Handbuch der Physik* (Springer-Verlag, Berlin, 1956), Vol. 21, p. 429.
[11] Gasiorowicz, Neuman, and Riddell, Phys. Rev. **101**, 922 (1956).
[12] Rosenbluth, MacDonald, and Judd, Phys. Rev. **107**, 1 (1957).
[13] H. C. Brinkman, Physica **23**, 82 (1957).

$$(h)=\frac{\partial^2}{\partial v_{\parallel}\partial\phi}\left[f\left(\frac{v_{\parallel}}{vv_{\perp}}\gamma_{\xi\zeta}-\frac{1}{v}\gamma_{\eta\zeta}\right)\right],$$

$$(i)=\frac{1}{v_{\perp}}\frac{\partial^2}{\partial v_{\perp}\partial v_{\parallel}}\left[f\left(\frac{v_{\perp}{}^2v_{\parallel}}{v^2}\gamma_{\xi\xi}+\frac{v_{\perp}(v_{\parallel}{}^2-v_{\perp}{}^2)}{v^2}\gamma_{\xi\eta}-\frac{v_{\perp}{}^2v_{\parallel}}{v^2}\gamma_{\eta\eta}\right)\right].$$

To evaluate γ_α and $\gamma_{\alpha\beta}$ we shall adopt the procedure used by Allis.[10] $\Delta\mathbf{v}$ is first averaged over the scattering angles in the center-of-gravity system, the result being used to generate the corresponding vector and tensor components in the ξ, η, ζ coordinates. The averages over the velocity distribution function $F(\mathbf{V})$ of the field particles are then performed, yielding γ_α and $\gamma_{\alpha\beta}$ as functions of integrals involving $F(\mathbf{V})$. For plasmas at cyclotron resonance, the test and field particle distribution functions shall be expanded respectively in the forms

$$f(\mathbf{v})=f_0(v)+f_1(v)\sin\theta\cos\phi+f_2(v)\sin\theta\sin\phi, \quad (7)$$

and

$$F(\mathbf{V})=F_0(V)+F_1(V)\sin\Theta\cos(\phi-\Phi)+F_2(V)\sin\Theta\sin(\phi-\Phi). \quad (8)$$

In both cases, the perturbing parts of the distribution functions are very small compared to the isotropic parts, since the applied electric field has been assumed to be very weak. Second and higher order terms are entirely negligible. To the same order of approximation we may assume that $f_0(v)$ and $F_0(V)$ are Maxwellian. In Eq. (8), Θ is the polar angle of $\mathbf{V}$ measured from the z axis, and Φ is the azimuthal angle of $\mathbf{V}$ measured negatively from the $\mathbf{v}-z$ plane (see Fig. 1). Since $\mathbf{v}$ is considered fixed in the integrals of (2), the integrations over $\mathbf{V}$ can be performed straightforwardly, if $F(\mathbf{V})$ is expressed in terms of the spherical angles having $\mathbf{v}$ as the polar axis. Thus, introducing

$$\begin{aligned}\mathfrak{F}_1(V,\phi)&\equiv F_1(V)\cos\phi+F_2(V)\sin\phi,\\ \mathfrak{F}_2(V,\phi)&\equiv F_1(V)\sin\phi-F_2(V)\cos\phi,\end{aligned} \quad (9)$$

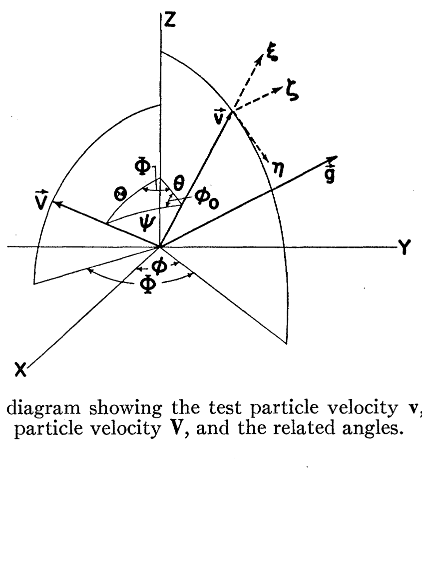

FIG. 1. A diagram showing the test particle velocity $\mathbf{v}$, the field particle velocity $\mathbf{V}$, and the related angles.

we have, after some trigonometric manipulations,

$$F(\mathbf{V})=F_0+\mathfrak{F}_1(\cos\psi\sin\theta-\sin\psi\cos\theta\cos\phi_0)+\mathfrak{F}_2\sin\psi\sin\phi_0, \quad (10)$$

where ψ is the angle between $\mathbf{v}$ and $\mathbf{V}$, and ϕ_0 is the angle between $\mathbf{v}-\mathbf{V}$ and $\mathbf{v}-z$ planes. The integrations, although tedious, are essentially similar to those of Allis.[10] Using his notation

$$\begin{aligned}K&=4e^4/m^2, \quad (Z=1),\\ L&=\ln[1+(3kT\lambda_c/2e^2)^2]^{\frac{1}{2}},\end{aligned} \quad (11)$$

where m is the electron mass, e its numerical charge, k the Boltzmann constant, T the temperature, and λ_c the cutoff distance of the screened Coulomb field,[14] and writing the indefinite integrals in the form

$$\begin{aligned}I_j{}^i(v)&=\frac{4\pi}{v^j}\int_0^v \mathfrak{F}_iV^{j+2}dV,\\ J_j{}^i(v)&=\frac{4\pi}{v^j}\int_v^\infty \mathfrak{F}_iV^{j+2}dV,\end{aligned} \quad (12)$$

one may show that

$$\begin{aligned}\gamma_\xi&=-\frac{\pi KL}{3\mu v^2}[3I_0{}^0+\sin\theta(2I_1{}^1-J_{-2}{}^1)],\\ \gamma_\eta&=\frac{\pi KL}{3\mu v^2}\cos\theta(I_1{}^1+J_{-2}{}^1),\\ \gamma_\zeta&=-\frac{\pi KL}{3\mu v^2}(I_1{}^2+J_{-2}{}^2)\end{aligned} \quad (13)$$

$$\begin{aligned}\gamma_{\xi\xi}&=\frac{2\pi KL}{15v}[5(I_2{}^0+J_{-1}{}^0)+3\sin\theta(I_3{}^1+J_{-2}{}^1)],\\ \gamma_{\eta\eta}&=\gamma_{\zeta\zeta}=\frac{\pi KL}{15v}[5(3I_0{}^0-I_2{}^0+2J_{-1}{}^0)+\sin\theta(5I_1{}^1-3I_3{}^1+2J_{-2}{}^1)],\\ \gamma_{\xi\eta}&=\frac{\pi KL}{15v}\cos\theta(5I_1{}^1-3I_3{}^1+2J_{-2}{}^1),\\ \gamma_{\xi\zeta}&=-\frac{\pi KL}{15v}(5I_1{}^2-3I_3{}^2+2J_{-2}{}^2),\\ \gamma_{\eta\zeta}&=0,\end{aligned} \quad (14)$$

where $\mu=M/(m+M)$. $\gamma_{\eta\zeta}$ and $\gamma_{\zeta\eta}$ are the only vanishing elements. In the above expressions, L, being a logarithmic term, has been treated as a constant. Since the Fokker-Planck method is valid only when there are many particles in the interaction sphere of radius λ_c,

[14] λ_c is ordinarily set to be the Debye length λ_D. But in certain cases, e.g., $T\simeq300°$K, $n\simeq10^{12}$ cm^{-3}, λ_D is less than the interionic separation, which suggests that setting λ_D to be the cutoff distance would be an overestimation of the effectiveness of screening. Thus λ_c may be somewhat larger than λ_D when the latter approaches interionic distance.

the magnitude of L should not be too small. The tensor terms in (14) actually also contain terms which do not depend on L, but they are neglected not only because they are small, but also because their inclusion would prevent the resultant F-P equation from satisfying the momentum conservation law. It is expected that they must be cancelled by the higher order terms of the F-P equation.

III. REDUCTION OF THE F-P EQUATION

Substituting the Eqs. (13) and (14) into (5) and (6), one obtains a long integro-differential equation for the Fokker-Planck equation. It may, however, be simplified considerably. The ϕ derivatives can first be removed by noting that (9) gives

$$\partial\mathfrak{F}_2/\partial\phi=\mathfrak{F}_1,\quad \partial\mathfrak{F}_1/\partial\phi=-\mathfrak{F}_2.$$

Hence, in view of (12), we have

$$\frac{\partial\gamma_\zeta}{\partial\phi}=-\frac{v}{v_{||}}\gamma_\eta;\quad \frac{\partial\gamma_{\xi\zeta}}{\partial\phi}=-\frac{v}{v_{||}}\gamma_{\xi\eta};$$

$$\frac{\partial^2}{\partial\phi^2}(f\gamma_{\zeta\zeta})=-f\gamma_{\zeta\zeta}.$$

Since $\gamma_{\eta\zeta}=0$, the use of the above relations in (6) removes all differentiations with respect to ϕ; furthermore, the equations become dependent upon five γ terms only, *viz.*, γ_ξ, γ_η, $\gamma_{\xi\xi}$, $\gamma_{\eta\eta}$ and $\gamma_{\xi\eta}$. These terms will be collected in groups and treated separately as such. In the course of reduction we shall encounter products of $f(\mathbf{v})$ and $F(\mathbf{V})$ throughout all the equations in (6). Inasmuch as $f_0(v)F_0(V)$ has no contribution to the electrical conductivity of the plasma (because of its isotropy in velocity), we shall consider only the first order terms. If we write Eq. (7) in the form

$$f(\mathbf{v})=f_0(v)+\frac{v_\perp}{v}\chi(v,\phi),\tag{15}$$

where

$$\chi(v,\phi)=f_1(v)\cos\phi+f_2(v)\sin\phi,\tag{16}$$

we see from (12) and (13) that

$$\frac{f\gamma_\xi}{v_\perp}=-\frac{\pi KL}{3\mu v^3}[3I_0{}^0\chi+f_0(2I_1{}^1-J_{-2}{}^1)],$$

which is a function of v only,[15] independent of $v_\perp$ or $v_{||}$. The other γ terms may be treated in like manner. Thus, if we define five functions

$$P(v)\equiv v^3f\gamma_\xi/v_\perp;\quad Q(v)\equiv f\gamma_\eta/v_{||},$$

$$R(v)\equiv v^2\frac{d}{dv}(v^2f\gamma_{\xi\xi}/v_\perp),\tag{17}$$

$$S(v)\equiv v^3f\gamma_{\eta\eta}/v_\perp;\quad T(v)\equiv v^3f\gamma_{\xi\eta}/v_{||},$$

which are all dependent on v only, we find, after some manipulations, that Eq. (5) may be expressed as

$$\frac{\delta f}{\delta t}=-\frac{v_\perp}{v^4}\{vP'-2v^3Q-\tfrac{1}{2}R'+S'+2T'\},\tag{18}$$

where the primes indicate differentiations with respect to v.

For electron-electron collisions the field particles are the same as the test particles, so $F_0=f_0$, $F_1=f_1$, and $F_2=f_2$. Comparing Eqs. (9) and (16), we further find that $\mathfrak{F}_1=\chi$. The distribution f_0, being Maxwellian at temperature T, varies as $\exp(-\beta v^2)$, where $\beta=m/2kT$. Using the relations

$$\frac{\partial}{\partial v}(v^mI_j{}^i)=(m-j)v^{m-1}I_j{}^i+4\pi v^{m+2}\begin{Bmatrix}f_0\\ \chi\end{Bmatrix};\quad \begin{Bmatrix}i=0\\ i=1\end{Bmatrix}$$

$$\frac{\partial}{\partial v}(v^mJ_j{}^i)=(m-j)v^{m-1}J_j{}^i-4\pi v^{m+2}\begin{Bmatrix}f_0\\ \chi\end{Bmatrix};\quad \begin{Bmatrix}i=0\\ i=1\end{Bmatrix}$$

and making the appropriate substitutions and differentiations in (18), one obtains finally

$$\begin{aligned}\left.\frac{\delta f}{\delta t}\right|_{ee}=\frac{\pi KLv_\perp}{15v^4}\{&5v^2\chi''(I_2{}^0+J_{-1}{}^0)\\ &+5v\chi'(3I_0{}^0-I_2{}^0+2J_{-1}{}^0)\\ &-5\chi(3I_0{}^0-I_2{}^0+2J_{-1}{}^0-24\pi v^3f_0)\\ &-10\beta v^2f_0(I_1{}^1+J_{-2}{}^1)+12\beta^2v^4f_0(I_3{}^1+J_{-2}{}^1)\}.\end{aligned}\tag{19}$$

This is the rate of change of f due to collisions between electrons.

A check on Eq. (19) can be made if we let the mass of the field particles approach infinity, and see whether (19) leads to the corresponding result for electron-ion collisions. Since the temperature of the ions is essentially the same as that of the electrons, the velocity distribution function of the field particles can be approximated by a delta function at the origin of the velocity space. Thus, from Eq. (12), we see that only $I_0{}^0$ equals N, the number density of the field particles (ions); all other integrals yield zeros. Recognizing this fact and remembering that $\mu=M/(m+M)\simeq1$ for electron-ion collisions, one may simplify Eqs. (13) and (14) considerably, and obtains

$$\left.\frac{\delta f}{\delta t}\right|_{ei}=-\frac{N\pi KLv_\perp}{v^4}\chi=-\nu_{ei}\frac{v_\perp\chi}{v},\tag{20}$$

where $\nu_{ei}=N\pi KL/v^3$ is the momentum transfer collision frequency between electrons and ions.[1] Equation (20) is, indeed, the correct expression of $\delta f/\delta t$ for electron-ion collisions.

Since collisions between electrons should entail no net change of momentum in the electron system, we

[15] The dependence on ϕ may be ignored temporarily.

must have

$$\int \frac{\delta f}{\delta t}\bigg|_{ee} v_k d^3v = 0, \tag{21}$$

where v_k is the electron velocity in an arbitrary direction k. Using Eq. (19), one finds that the above relation is indeed satisfied. This provides another check on Eq. (19).

IV. BOLTZMANN EQUATION

The Boltzmann equation for the distribution function of the electrons in a uniform, partially ionized gas is given by

$$\frac{\partial f}{\partial t} + \mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{v}} = \frac{\delta f}{\delta t}, \tag{22}$$

where

$$\frac{\delta f}{\delta t} = \frac{\delta f}{\delta t}\bigg|_{em} + \frac{\delta f}{\delta t}\bigg|_{ei} + \frac{\delta f}{\delta t}\bigg|_{ee},$$

with the subscripts *em*, *ei*, *ee* indicating the effects of electron-molecule, electron-ion, and electron-electron collisions, respectively. $\partial/\partial\mathbf{v}$ is the gradient operator in the velocity space. Assuming that a steady magnetic field $\mathbf{H}$ is in the z direction and that an oscillatory electric field $\mathbf{E}$ is in the y direction oscillating at a frequency ω, we have

$$\mathbf{a} = -\mathbf{i}\omega_H v_\perp \sin\phi - \mathbf{j}\left(\frac{eE}{m}e^{i\omega t} - \omega_H v_\perp \cos\phi\right),$$

where the cyclotron frequency $\omega_H = eH/mc$. The appropriate expansion for $f(\mathbf{v})$ is

$$f(\mathbf{v}) = f_0(v) + \frac{v_\perp}{v}\chi(v)e^{i\omega t}, \tag{23}$$

where $\chi(v)$ is as shown in Eq. (16). The collisional terms for electron-ion and electron-electron have already been obtained, and are given by Eqs. (20) and (19), respectively. For electron-molecule collisions, we have[16]

$$\frac{\delta f}{\delta t}\bigg|_{em} = -\frac{v_\perp}{l}\chi, \tag{24}$$

where l is the mean free path of the electrons associated with the momentum transfer collisions with the molecules. Substituting these into Eq. (22) and separating according to $\sin\phi$ and $\cos\phi$ result in two simultaneous integro-differential equations, from which we can solve for f_1 and f_2. By changing the independent variable and some parameters into dimensionless quantities, the equations can be put into a form more readily solvable numerically. Thus, if we define

$$x \equiv \beta^{\frac{1}{2}} v; \quad r \equiv \omega_H/\omega,$$

$$s \equiv 1/\omega\beta^{\frac{1}{2}} l \simeq \bar{\nu}_{em}/\omega.$$

$$t \equiv n\pi K L \beta^{\frac{3}{2}}/2\omega \simeq \bar{\nu}_{ei}/\omega,$$

$$H_j{}^i(x) \equiv \int_0^x f_i x^j dx; \quad G_j{}^i(x) \equiv \int_x^\infty f_i x^j dx,$$

$$\Phi(x) \equiv 2\pi^{-\frac{1}{2}} \int_0^x \exp(-x^2)dx, \text{ error function,}$$

in which $\bar{\nu}_{em}$ and $\bar{\nu}_{ei}$ are the effective electron-molecule and electron-ion collision frequencies, we obtain

$$\begin{aligned} a(x)f_1''(x)+b(x)f_1'(x)+c(x)f_1(x)+d_1(x) &= \frac{r}{t}f_2(x), \\ a(x)f_2''(x)+b(x)f_2'(x)+c(x)f_2(x)+d_2(x) &= -\frac{r}{t}f_1(x)+\frac{h(x)}{t}, \end{aligned} \tag{25}$$

where

$$a(x) = (\Phi - x\Phi')/x^3,$$

$$b(x) = [(2x^2-1)\Phi + x\Phi']/x^4,$$

$$c(x) = -\frac{2}{x^3}\left[1+\left(1-\frac{1}{2x^2}\right)\Phi + \left(\frac{1}{2x}-4x^3\right)\Phi'\right] - \left(\frac{s}{t}x + i\frac{1}{t}\right),$$

$$d_i(x) = \frac{8\Phi'}{15x^2}[-5H_3{}^i + 6H_5{}^i + (-5+6x^2)x^3 G_0{}^i],$$

$$h(x) = \frac{ne\beta^2 E}{\pi m\omega} x\Phi'.$$

This is the set of equations we have to solve in order to determine the velocity distribution function of the electrons at cyclotron resonance, taking into account the electron-molecule, electron-ion, and electron-electron collisions. From the distribution function we can determine the electrical conductivity, and, consequently, the shape of resonance absorption. It is of interest to note that Eq. (25) reduces to Eq. (8) of Spitzer and Härm[4] if we let the magnetic field, the signal frequency, and the electron-molecule collision frequency approach zero.

V. RESULTS

It is obvious that to solve the equations in (25) analytically would be extremely difficult, if at all

[16] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, New York, 1953).

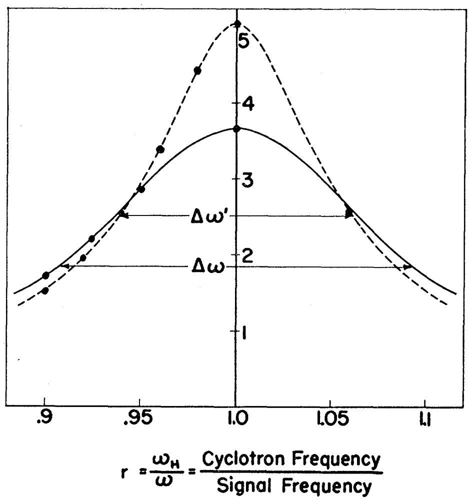

FIG. 2. Cyclotron resonances at $t=0.1$ ($n \simeq 3\times10^{11}$ cm^{-3}) and $s=\bar{\nu}_{em}/\omega=0.01$. The dashed curve is for the case where electron-electron interactions are not considered. The solid curve is obtained when they are taken into account.

possible. To solve them numerically by the usual integration method would also be unfeasible since the coefficients in the equations diverge at both ends; the evaluation of f_1'' and f_2'' would involve subtractions of very large but nearly equal quantities, thus drastically impairing the accuracy of the results in the initial stages of the numerical integration and leading to propagation and accumulation of errors in successive steps. In view of this, the numerical method of finite differences was chosen.

Since the coefficient $c(x)$ is a complex quantity, the two equations in (25) may be separated into four equations relating the real and imaginary parts of f_1 and f_2. Thirty-five nodal points along the axis of the independent variable x were used so that, with four difference equations for each point, there was a total of 140 linear algebraic simultaneous equations to be solved for each problem. The range of values of x was from 0 to 3. For $x>3$ we assumed that the solutions take on asymptotic variations. The solution of the simultaneous equations was performed on the ILLIAC, the University of Illinois digital computer.

The parameter r is the ratio of cyclotron frequency to signal frequency, and was varied from 0.9 to 1.1. For the case of no magnetic field, r was set equal to zero. The parameter s is roughly the ratio of electron-molecule collision frequency to signal frequency, and was given a value of 0.01. The parameter t is approximately the ratio of electron-ion collision frequency to signal frequency; for a known electron temperature T, it gives a measure of the charge concentration n of the plasma. Since the cutoff distance λ_c of the short-range forces is a rather loosely defined quantity, we may let L have an average value of 7 for $T=300°$K and n varying between 10^{10} to 10^{12} cm^{-3}. This approximation is of the same order as that used when L was considered constant in the integrals that led to the expressions in (13) and (14). Thus, the electron density may be calculated from the parameter t using the approximate formula $n \simeq 300\omega t$. It should be noted that the Fokker-Planck method may not apply when n exceeds 10^{12} cm^{-3} (for $T=300°$K), since the screening of the Coulomb field would be so effective that the interaction sphere no longer contains many particles.

From the solutions of the ILLIAC, we can readily calculate the real and imaginary parts of the electrical conductivity tensor, which has the form

$$\sigma_{ij}=\left\|\begin{matrix}\sigma_{11} & \sigma_{12} & 0\\ \sigma_{21} & \sigma_{22} & 0\\ 0 & 0 & \sigma_{33}\end{matrix}\right\|,$$

where

$$\sigma_{11}=\sigma_{22}=-\frac{4\pi ne}{3\beta^2E}\int_0^\infty f_2x^3dx,$$

$$\sigma_{12}=-\sigma_{21}=-\frac{4\pi ne}{3\beta^2E}\int_0^\infty f_1x^3dx,$$

and σ_{33} is the high-frequency conductivity of the plasma in the absence of any magnetic field, and is therefore the limit of σ_{11} or σ_{22} as H approaches zero. Thus, by solving Eqs. (25) for various values of r and t, we obtained resonance curves at different charge densities, taking into account all types of electron encounters, *viz.*, *e-m*, *e-i*, and *e-e*. We also excluded the consideration of the electron-electron encounters by putting equal to zero in (25) the coefficients $a(x)$, $b(x)$, $d_i(x)$, and

$$-\frac{2}{x^3}\left[\left(1-\frac{1}{2x^2}\right)\Phi+\left(\frac{1}{2x}-4x^3\right)\Phi'\right]$$

of $c(x)$. A comparison of the results so obtained for the two cases should reveal the effect of electron-electron interactions on the electrical conductivity of the plasma at cyclotron resonance.

A typical set of resonance curves is shown in Fig. 2, where the real part of σ_{11} (or σ_{22}) is plotted *versus* r for

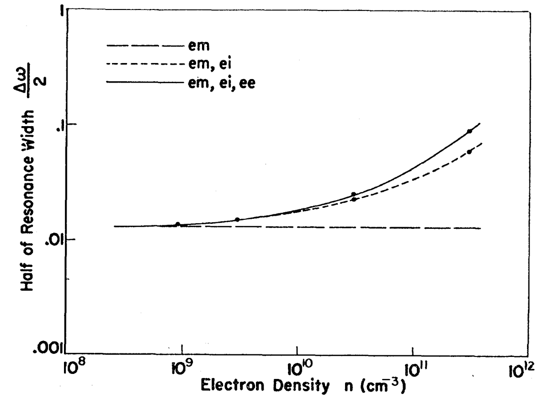

FIG. 3. Effects of charge interactions on cyclotron resonance widths.

both with and without e-e interactions. The value of t is 0.1, which, for $\omega=10^{10}$ sec^{-1}, corresponds to $n\simeq 3\times 10^{11}$ cm^{-3}. It is evident that the effects of the electron-electron interactions are to reduce the absorption of the microwave power by the plasma at the peak of the cyclotron resonance and to increase its half-width. This suggests that the electron-electron interactions increase the scatterings of the electrons.

Resonance curves such as those shown in Fig. 2 were obtained for various values of t. In Fig. 3 is plotted $(\Delta\omega)/2$ against the electron densities, where $\Delta\omega$ designates the half-widths of the resonance curves. The ordinate of the plot corresponds to the effective collision frequency of the electrons in a Lorentzian gas, since, in such a gas, the cyclotron resonance width is very nearly twice the electron collision frequency. Figure 3 clearly depicts the importance of the effect of the electron-ion encounters on the collision frequency of the electrons, and therefore also on the resonance width, for charge densities greater than 10^{10} cm^{-3}, when the electron-molecule collision frequency is of the order of 10^8 sec^{-1}. This corresponds to plasmas in helium gas discharge at 1 mm Hg pressure with a degree of ionization of only 10^{-6} or higher. It is also evident from Fig. 3 that the broadening of the cyclotron resonance width by electron-electron interactions, just as it is with the electron-ion encounters, becomes more pronounced at higher charge densities.

In the case of no magnetic field, i.e., $r=0$, the perturbing distribution function f_1 vanishes, and f_2 is found to be relatively independent of the electron-electron interactions. The real part of the electrical conductivity at the highest charge density investigated ($n\simeq 3\times 10^{11}$ cm^{-3}, $T=300$°K) is increased by $e-e$ interactions by no more than 7%. The imaginary part is practically unaffected. This therefore validates the use of microwave techniques to obtain the correct scattering cross section for electron-molecule or electron-ion collisions.

VI. CONCLUSION

It was found in this investigation that the electron-electron interactions do not appreciably affect the high-frequency electrical conductivity of a plasma when there is no magnetic field. But they do have an effect when a magnetic field is applied, especially if a cyclotron resonance occurs. The interactions reduce the resonance peak and broaden the width, thus effectively increase the electron scatterings. The reduction of the absorption at the peak of the cyclotron resonance is consistent with the result of Spitzer and Härm,[4] who found that the electron-electron interactions reduce the dc conductivity of a plasma. Some insight into the role played by the electron-electron interactions in gaseous plasmas may be gained if we examine the expressions for the electrical conductivity of the plasma under various conditions. For a Lorentzian gas, under cyclotron resonance conditions, the conductivity (σ_{11} or σ_{22}) may be shown to vary as

$$\frac{\nu+i\omega}{(\nu+i\omega)^2+\omega_H^2}, \tag{26}$$

where ν is the effective electron collision frequency, ω the angular frequency of the applied electric field, and ω_H the cyclotron frequency. The above expression would not be valid if electron-electron encounters are also taken into consideration, because the term ν cannot adequately account for the interactions that take place between electrons. This may be seen by noting that Eq. (19) cannot be put in the form of Eqs. (20) or (24), from which the effective momentum transfer collision frequencies of the electrons with ions and molecules, respectively, may be derived. In fact, it is those derivative and integral terms in (19) that distinguish the nature of mutual electronic encounters from that of encounters between electrons and foreign particles.

Results of the numerical computations indicate that the electron-electron interactions reduce the real part of the electrical conductivity when the plasma is at cyclotron resonance, which is in line with the dc case of Spitzer and Härm. Putting first $\omega=\omega_H$, and then $\omega=\omega_H=0$ in (26), we find that the two cases actually belong to the same category, namely, $\sigma\sim 1/\nu$. Thus, a reduction in σ by electron interactions implies an effective increase in ν; or, more appropriately, an effective increase in scattering by such interactions results in a reduction in conductivity under those conditions. In the case of no magnetic field, the high-frequency conductivity behaves quite differently, however. Putting $\omega_H=0$ and assuming $\nu\ll\omega$ in (26), we get $\sigma\sim\nu$. One would then expect that the enhancement of scattering due to electron interactions should increase the conductivity in this case as much as it reduces the latter in the previous cases. But the results indicate that the increase is insignificant. In view of Eq. (21), one may infer that this should not be totally unexpected, but rather should be considered as a natural consequence of the mutual electronic interactions for which no net change of momentum results. Thus, the expression (26) for electrical conductivity of a plasma cannot be used with much reliability when the charge density is high, this being the anomaly of a non-Lorentzian gas.

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