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Basic Microwave Properties of Hot Magnetoplasmas*

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The usual conductivity tensor of a uniform plasma in a uniform, static magnetic field (along the z axis) is generalized to include four effects of random motion of plasma electrons. These effects are due to (1) the shape of the radio-frequency electric field strength expressed by $\nabla\times\nabla\times\mathbf{E}\neq0$, (2) the partial spanning of a wavelength by an electron cyclotron orbit, and (3) the possible variation of the radio-frequency electric field lines along the z axis of electron drift. The effects resulting cause diffusion damping, the existence of nonzero (x,z), (y,z), (z,x), and (z,y) elements in the conductivity tensor, large changes of the effective plasma density, and phase changes in the conduction current density. These results are applied to evaluation of the index of refraction of microwave signals propagating normal to the magnetic field. The existence of unusual transmission bands is predicted for very dense, hot plasmas.

1. INTRODUCTION

1.1 Objective

THE purpose of this paper is to present a theoretical study of the conductivity tensor of a high-temperature ionized gas in a uniform, static magnetic field. Such a study is of importance in providing the basis for predictions of microwave transmission and absorption in electric arcs[1] as well as in the atmosphere of the sun[2] and the ionosphere of the earth.[3] The areas of technical applications would include new electron stream[4] and gaseous breakdown[5] devices used to generate and control high-frequency radio waves. A recent important area of application is in connection with microwave diagnostic studies in thermonuclear research.[6]

Although some temperature effects have been included in previous analyses,[7] these have not been as complete as might be desired. The present analysis is also incomplete in that the basis of the calculations is the Boltzmann transport equation. This equation omits certain statistical effects that might be of importance.[8] However, within the limitations imposed by use of the Boltzmann equation and a number of other approximations which will be listed shortly, it is the purpose

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[1] A. Guthrie and B. K. Wakerling, *Characteristics of Electrical Discharges in Magnetic Fields* (McGraw-Hill Book Company, Inc., New York, 1949); W. H. Bostick and M. A. Levine, Phys. Rev. **97**, 13 (1955); H. Gamo, J. Phys. Soc. Japan **8**, 176 (1953); A. A. Thm. and Van Trier, Appl. Sci. Research **B3**, 305 (1953); P. S. Epstein, Revs. Modern Phys. **28**, 3 (1956); Phys. Rev. **87**, 227 (1952). Etter and Goldstein, "Guided Wave Propagation through Gyromagnetic Gaseous Discharge Plasma," University of Illinois Electrical Engineering Research Laboratory Technical Report No. 3 (unpublished); H. Suhl and L. R. Walker, Bell System Tech. J. **33**, 579, 939, 1133 (1954) and "Topics in Guided Wave Propagation Through Gyromagnetic Media," Bell Telephone Laboratory Monograph No. 2322 (unpublished).

[2] F. P. Wild, Australian J. Sci. Research **A4**, 36 (1951).

[3] Appleton and Barnett, The Electrician **94**, 398 (1925); H. B. Keller, "Ionospheric Propagation of Plane Waves," New York University Mathematical Research Group Report EM-31 (unpublished); A. Russek, "Scattering Matrices for Ionospheric Models," New York University Mathematical Research Group Report EM-38 (unpublished).

[4] D. A. Watkins and N. Ryan, J. Appl. Phys. **25**, 1375 (1954).

[5] H. Johnson and K. R. Deremer, Proc. Inst. Radio Engrs. **39**, 908 (1951).

[6] R. F. Post, Revs. Modern Phys. **29**, 338 (1956).

[7] D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952); V. A. Bailey, *ibid.* **78**, 428 (1950); J. A. Roberts, *ibid.*, **76**, 340 (1949). D. Bohm and E. P. Gross, *ibid.* **75**, 1851, 1864 (1949); **79**, 992 (1950). E. P. Gross, *ibid.* **82**, 232 (1951); Bhatnagar, Gross, and Krook, *ibid.* **94**, 511 (1954); W. P. Allis, *Handbuch der Physik* (Springer-Verlag, Berlin, 1957), Vol. 21; L. Spitzer and R. Härm, Phys. Rev. **89**, 977 (1953); Hari K. Sen, *ibid.* **88**, 816 (1952); E. P. Gross, *ibid.* **82**, 232 (1951).

[8] Gasiorowicz, Neuman, and Riddell, Phys. Rev. **101**, 922 (1956). Rudolph C. Hwa and L. Goldstein, "Electron Interactions in Gaseous Discharge Plasmas and their Effect on Cyclotron Resonance," University of Illinois Electrical Engineering Research Laboratory Scientific Report No. 3 (1957) (unpublished).

of this paper to derive the four important basic kinds of temperature effects on the microwave conductivity tensor of magnetoplasmas.

1.2 Conditions and Assumptions of the Problem

Consider a plasma filling a bounded region of space. Let the entire region be immersed in a uniform and constant magnetic field. The plasma will be taken as consisting of an equal number density of free electrons and positive ions as well as some neutral gas molecules. The boundary surface of the region is defined as the limiting surface within which this condition is fulfilled and outside of which it is not fulfilled. The boundary may be made up of solid material or the plasma side of an electron or ion sheath. Within the plasma the charged particles spiral around the magnetic lines of force. Occasionally, close-encounter collisions may occur between electrons and ions and neutral molecules. Near the surface the particles will have distorted orbits either because of collisions with a solid part of the boundary or because of entering the electric fields of a sheath. A boundary layer is defined as a subregion of the plasma containing all the electrons of the plasma whose orbits will intersect the boundary surface before close-encounter collision with other plasma particles. It is the purpose of the present paper to present the results of a theoretical study of some radio-frequency properties of the complementary subregion which is assumed to be simply connected. This is made up of the plasma minus its boundary layer. Initially, there are to be no macroscopic electric fields or currents.

Now it is supposed that this condition becomes slightly disturbed. A small oscillating electromagnetic field is assumed to come into existence within the plasma. Somehow the effects of electromagnetic conditions on the boundary must be characterized in the interior region of the plasma. This can be accomplished by expanding the electric field in terms of a complete set of functions. The boundary conditions then restrict the coefficients in the expansion. The properties of the medium are then represented by the relation between the Fourier time components of the current density, $\mathbf{J}_\omega$, and the electric field strength, $\mathbf{E}_\omega$, for each of these modes. It is the purpose of this paper to determine these relations.

The Boltzmann equation for the statistical distribution of particles in configuration-velocity space can be derived[9] from Liouville's theorem by use of the following assumptions:

The interaction potential between any two particles depends only upon their relative coordinates and momenta. (1.2.1)

Three-body and higher correlations may be neglected. (1.2.2)

The distribution function and the time-dependent forces remain nearly constant during the period required for a "collision." (1.2.3)

The range of the screened Coulomb interaction[10] is such that the period required for an electron-electron or electron-ion "collision" is approximately the characteristic plasma-electron period. Thus, in order to justify the use of the Boltzmann equation for high-frequency oscillations, the main effects of electron-electron and electron-ion collisions should be replaced by a self-consistent electromagnetic field. Then only the statistical fluctuation in these collision times within a significant volume must be kept small compared to the oscillation periods.

It can be shown[11] that this statistical fluctuation for a volume $\lambda\!\!\!^{-3}$ is

$$\langle\delta\omega_p{}^2\rangle_{\mathrm{Av}}=\frac{4\pi\omega_p(e^2/m)^{\frac{1}{2}}}{[\lambda\!\!\!^{-3}(\lambda\!\!\!^{-2}/\lambda_D{}^2+4\pi)]^{\frac{1}{2}}},\tag{1.2.4}$$

where ω_p is the angular plasma frequency given by

$$\omega_p\equiv(4\pi ne^2/m)^{\frac{1}{2}}\tag{1.2.5}$$

in cgs electrostatic units with n, e, and m, respectively, the electron number density, charge ($e=-|e|$), and mass. The Debye length λ_D is given by[10]

$$\lambda_D\equiv(\langle v^2\rangle_{\mathrm{Av}}/3\omega_p{}^2)^{\frac{1}{2}},\tag{1.2.6}$$

where $\langle v^2\rangle_{\mathrm{Av}}$ is the mean square electron speed. Consequently, the assumption that the mean fractional variation be small is

$$\left[\frac{m\omega_p\lambda\!\!\!^{-3}}{e^2}\left(\frac{\lambda\!\!\!^{-2}}{4\pi\lambda_D{}^2}+1\right)\right]^{\frac{1}{2}}\gg1.\tag{1.2.7}$$

For many cases of interest, it is undesirable for the microwave signal to be rapidly attenuated except for special frequencies. Hence the frequency of momentum-transfer collisions between electrons and neutral particles, ν_m, must be kept much smaller than the frequency of the microwave signal. This condition will be assumed:

$$\nu_m\ll\omega.\tag{1.2.8}$$

This means that collisions are to be relatively unimportant and hence that the form of the collision integrals can be very grossly approximated. The frequencies of interest will also be assumed to be so high that the motions of positive ions can be neglected[12]:

$$\omega^2\gg4\pi ne^2/M,\tag{1.2.9}$$

where M is the mass of a positive ion.

[9] H. L. Frisch, J. Chem. Phys. **22**, 1713 (1954); H. Grad, Comm. Pure Appl. Math. **2**, 331 (1994); J. G. Kirkwood, J. Chem. Phys. **14**, 180 (1946).

[10] P. Debye and E. Hückel, Physik. Z. **24**, 185, 305 (1923).

[11] D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952); J. E. Drummond, Electronic Defence Laboratory Report No. E-14, 1956 (unpublished).

[12] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956); Thomas H. Stix, Phys. Rev. **106**, 1146 (1957); A. B. Bernstein, Phys. Rev. **109**, 10 (1958).

It will be assumed that classical kinematics and mechanics is applicable. This imposes double bounds on the mean kinetic energy of plasma electrons. On the one hand the mean kinetic energy must be much greater than the energy of a plasma quantum, and on the other hand the mean kinetic energy must be much smaller than the electronic rest energy. This is expressed by the following inequalities:

$$\hbar\omega_p \ll \tfrac{1}{2}m\langle v^2\rangle_{\text{Av}} \ll mc^2, \tag{1.2.10}$$

where $\hbar$ is Planck's constant divided by 2π and c is the speed of light. This still leaves a wide range of practical values for the density and energy parameters.

With these restrictions, the equation for the distribution function, $f(\mathbf{r},\mathbf{v},t)$ of electrons in position $(\mathbf{r})$, velocity $(\mathbf{v})$ space at time t is assumed to be

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla} f+(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\omega}_c)\cdot\boldsymbol{\nabla}_{\mathbf{v}} f=-\nu_m(f-f_0). \tag{1.2.11}$$

The left-hand side of the Boltzmann equation (1.2.11) represents the total time rate of change of the distribution function f along an electron trajectory in $(\mathbf{r},\mathbf{v})$ space as defined by Lagrange's system of characteristic equations[13]:

$$d\mathbf{r}/dt=\mathbf{v}, \tag{1.2.12}$$

$$d\mathbf{v}/dt=\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\omega}_c, \tag{1.2.13}$$

where $\boldsymbol{\mathcal{E}}$ is the electric acceleration field strength defined by

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t)\equiv(e/m)\mathbf{E}(\mathbf{r},t), \tag{1.2.14}$$

with $\mathbf{E}$ the electric field strength, and $\boldsymbol{\omega}_c$ the cyclotron frequency given by

$$\boldsymbol{\omega}_c\equiv e\mathbf{H}/mc, \tag{1.2.15}$$

where $\mathbf{H}$ is the magnetic field strength.

The right-hand side of Eq. (1.2.11) is a gross approximation to the collision integrals. It represents a relaxation term to a steady state distribution, f_0. The solution will be assumed to depend analytically upon ν_m.

Maxwell's equations governing $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ will be assumed:

$$\boldsymbol{\nabla}\times\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi}{c}\mathbf{J}, \tag{1.2.16}$$

$$\boldsymbol{\nabla}\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}. \tag{1.2.17}$$

A second connection between the field variables $\mathbf{E}$ and $\mathbf{H}$ and the distribution function f is provided by identifying the first velocity moment of f as the current density $\mathbf{J}(\mathbf{r},t)$ in Eq. (1.2.16)

$$\mathbf{J}\equiv e\int_\infty \mathbf{v}fd^3v. \tag{1.2.18}$$

The quiescent plasma is represented by f_0 which will be assumed to be constant in time, uniform in space, and symmetric in velocity:

$$f_0=f_0(v^2), \tag{1.2.19}$$

such that

$$\int_0^\infty v^2f_0(v^2)v^2dv<\infty, \tag{1.2.20}$$

in order to insure a bounded kinetic energy density.

In all problems of practical interest, the field variables $\mathbf{E}$ and $\mathbf{J}$ are of bounded magnitude and may be assumed to vanish outside a finite interval of time. The assumption may be expressed as follows:

$$\int_{-\infty}^\infty |\mathbf{E}(\mathbf{r},t)|^2dt<\infty, \tag{1.2.21}$$

$$\int_{-\infty}^\infty |\mathbf{J}(\mathbf{r},t)|^2dt<\infty. \tag{1.2.22}$$

Thus the conditions for the existence of Fourier time transforms are realized. The transforms are written as follows:

$$\mathbf{E}_\omega(\mathbf{r})\equiv\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^\infty \mathbf{E}(\mathbf{r},t)e^{j\omega t}dt, \tag{1.2.23}$$

$$\mathbf{J}_\omega(\mathbf{r})\equiv\frac{e}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^\infty e^{j\omega t}dt\int_\infty \mathbf{v}f(\mathbf{r},\mathbf{v},t)d^3v. \tag{1.2.24}$$

The electric field $\mathbf{E}$ will be assumed to be so small that the distribution function, f, will be only slightly perturbed away from f_0, or that the orbit of an individual electron will be only slightly noncircular. The condition that the perturbation be small is

$$\langle(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\omega}_c')\cdot\boldsymbol{\nabla}_{\mathbf{v}}f_0\rangle_{\text{Av}} \gg\langle(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\omega}_c')\cdot\boldsymbol{\nabla}_{\mathbf{v}}(f-f_0)\rangle_{\text{Av}}, \tag{1.2.25}$$

where the angular brackets mean velocity average with any continuous weighting factor and where

$$\mathbf{h}\equiv(mc/e)\boldsymbol{\omega}_c' \tag{1.2.26}$$

is the radio-frequency magnetic field strength.

It will be found that with these assumptions the current density within one small volume element will be determined by the electric field strengths throughout many other volume elements. The question as to the existence of a conductivity tensor is "Is it possible to find a proportionality constant (tensor) between the ω frequency component of the electric field strength at a

[13] Lyman M. Kells, *Elementary Differential Equations* (McGraw-Hill Book Company, Inc., New York, 1947), pp. 223–227.

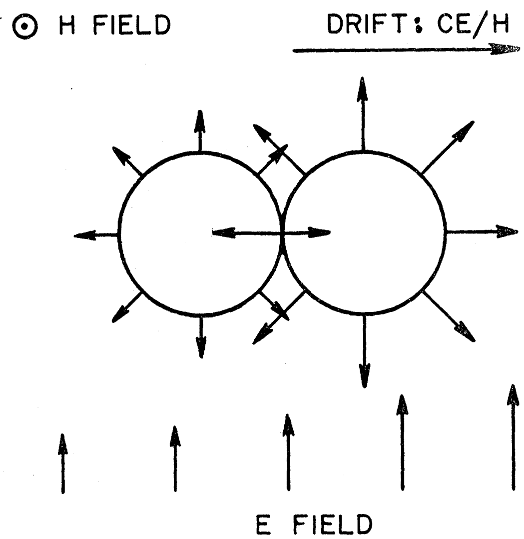

FIG. 1. Drift and expansion of orbits for $\nabla\times\nabla\times\mathbf{E}\neq 0$.

point and the ω frequency component of the current density at the same point?" In general the answer has to be "No." For some possible electric field configurations the transconductance property just noted makes $\mathbf{E}_\omega(\mathbf{r})$ not proportional to the current density, $\mathbf{J}(\mathbf{r})$. If the useful concept of conductivity is to be retained, it must be generalized a little. The question of conductivity may be reformulated as follows: "For what possible electric field configurations, $\mathbf{E}_\omega(\mathbf{r})$ is the current density, $\mathbf{J}_\omega(\mathbf{r})$, proportional to $\mathbf{E}_\omega(\mathbf{r})$ and its second order derivatives only, and what are the proportionality constants in these cases?" This extended concept will be useful only if all the tensors can be given by a few formulas and all possible electric fields can be composed out of the particular fields that satisfy the condition stated above. In what follows, it will be shown that both of these requirements can be met for homogeneous plasmas in a uniform static magnetic field directed along the z axis.

In the next subsection the four basic kinds of temperature effects on the conductivity tensor are examined qualitatively, and in Sec. 2 a quantitative analysis is given. In Sec. 3 various consequences are considered.

1.3 Qualitative Examination of the Problem

The qualitative aspects of the thermal motion of electrons in an oscillating plasma can be seen by superposing typical electron orbits upon some special radio-frequency electric field configurations. This is done in Figs. 1–3.

Figure 1 illustrates the thermal orbits of two electrons (circles) and the over-all orbital drift (boldface arrow) due to the electric field strength (vertical arrows) of frequency small compared to the orbital frequency. The small arrows pointing radially outward from the center of the orbits indicate the expansion of the orbits due to $\oint \mathbf{E}\cdot d\mathbf{s}$ around the orbits. Where the circles touch, the radial expansion of the left-hand orbit augments the current due to the orbital drift and the expansion of the right-hand orbit reduces the current resulting from orbital drift. Because of this cancellation, the $\nabla\times\mathbf{E}$ has in itself no net effect on the current response of the plasma electrons. However, because $c^2\neq\infty$, $\nabla\times\nabla\times\mathbf{E}\neq 0$. Hence there is a spatial dependence of $\nabla\times\mathbf{E}$ so that the cancellation noted above is not complete. This is shown in the figure by the shorter radial arrows on the left-hand orbit compared to those on the right-hand orbit. Thus the character of the electric field *form* due to a finite speed of light has given rise to an extra contribution to the conduction current in a magnetoplasma. This effect has usually been neglected. For the case illustrated it reduces the conduction current making the plasma seem less dense as measured by a probing electromagnetic wave. In the following sections this as well as the other three thermal effects will be computed quantitatively. It will be shown that for high-density, high-temperature plasma these effects can be of major importance to the basic microwave properties.

In Fig. 2 a second thermal effect is illustrated again for an orbit frequency large compared to the oscillation frequency. This shows the canceling effect of electric fields on an electron orbit which spans half a wavelength. This too reduces the apparent density of the plasma. This effect is sometimes included but not always systematically or completely in the conductivity tensor.

Figure 3 illustrates an effect evidently not previously realized in the literature. This is the production of radio-frequency currents along the axis of the static magnetic field as a result of electric fields directed in the plane transverse to the static magnetic field. At the top of the drawing is shown converging arrows indicating electron bunching due to the transverse electric fields

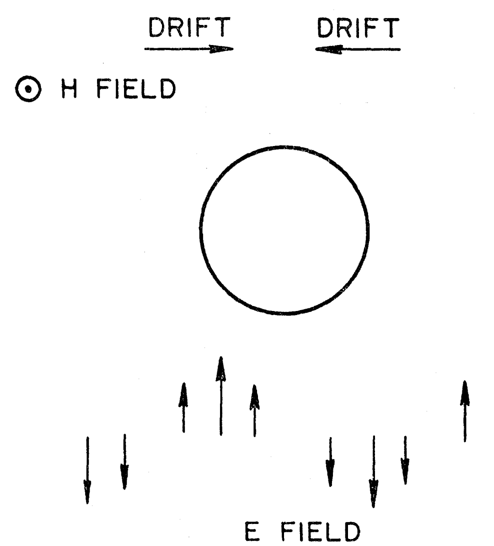

FIG. 2. Canceling drifts of large orbit.

in that plane. At the bottom of the drawing is shown diverging arrows indicating the opposite effect of transverse electric fields 180° out of phase with the top fields. Midway between these planes is shown a plane receiving bunched thermal electrons from the top and fewer than average thermal electrons from the rarefied region below. Thus when these groups of electrons reach the central plane due to their thermal motion, they will produce radio-frequency current pulses which will not completely cancel. Consequently, there must exist (z,x) and (z,y) elements in the conductivity tensor. Furthermore these elements must contain differential operators with respect to z and x or y in order to describe the dependence upon z-axis phase and (x,y)-plane bunching. Of course, like the other effects described above, they must vanish in the limit of zero temperature. These conclusions are made quantitative in the next section. The effects of these new elements of the conductivity tensor make desirable the re-evaluation of propagation theories which have sometimes depended heavily upon these elements being zero.[1]

In addition to the production of an axial component of radio-frequency current density, the thermal drift of electrons along the magnetic field carries them into regions where the local bunching or current is out of phase with that carried by the drifting electrons. This mixing increases entropy and results in an effective damping of the Landau type. This will also be shown in the analysis that follows.

It might be thought that there are other kinetic effects due to the term $(\mathbf{v}\times\boldsymbol{\omega}_c)\cdot\nabla_{\mathbf{v}}f$ in the Boltzmann equation (1.2.11). However, because of the symmetry of the unperturbed distribution (1.2.19) these effects cancel to second order in the perturbing field strength, $\mathbf{E}$, as is shown in the next section.

2. DERIVATION OF CONDUCTIVITY TENSOR

2.1 Plan of Derivation

The purpose of this section is to carry through a rigorous derivation of the current density resulting from the existence of an electric field strength within a plasma under the conditions specified in Sec. 1.2. This general kind of problem has been done many times in various ways and for various specific cases.[14] However,

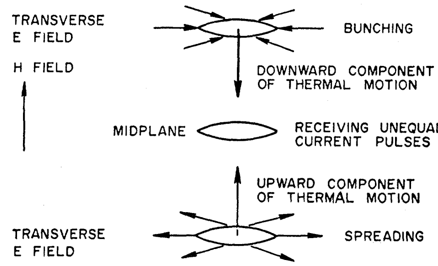

FIG. 3. **H** axis current from transverse **E** fields.

it is believed that it has not been done before in quite the present manner nor for quite as wide a range of important physical parameters. What is required, then, is to remove the unknown distribution function f from the Fourier components of Eq. (W1.2.18) in favor of the Fourier components of the electric field strength $\mathbf{E}_\omega(\mathbf{r})$ and a given unperturbed distribution function f_0. Perhaps the most direct approach would be to solve the Boltzmann equation (1.2.11) and Maxwell's equations (1.2.16) and (1.2.17) simultaneously for f. However, this is in general rather complicated and unnecessary since it is equivalent to computing an infinite number of velocity moments of f whereas only one is required by Eq. (1.2.18). This suggests that the first velocity moment of the Boltzmann equation (1.2.11) be taken as it stands, and that the resulting equation be solved for $\mathbf{J}$. However, this equation involves the second velocity moment. When an equation for this second moment is sought, it is found to involve the third moment and so on. The most frequent solution to this dilemma is to assume what some higher moment is equal to and proceed from there. This is the magnetohydrodynamic approach. Since it is desirable to avoid this uncertainty, a middle course is taken here in which the Boltzmann equation (1.2.11) is formally solved *by itself* for $f(\mathbf{r},\mathbf{v},t)$ in terms of a functional dependence on the perturbation, $\mathbf{E}(\mathbf{r},t)$. This is then substituted into Eq. (1.2.18) and the integration for each Fourier component performed. The result shows that $\mathbf{J}_\omega(\mathbf{r})$ depends linearly and homogeneously upon $\mathbf{E}_\omega(\mathbf{r})$ and $\nabla\times\nabla\times\mathbf{E}_\omega(\mathbf{r})$. But Maxwell's equations (1.2.16) and (1.2.17) can be combined to give a second linear, homogeneous equation for $\mathbf{J}_\omega(\mathbf{r})$ in terms of $\mathbf{E}_\omega(\mathbf{r})$ and $\nabla\times\nabla\times\mathbf{E}_\omega(\mathbf{r})$ from which $\nabla\times\nabla\times\mathbf{E}_\omega(\mathbf{r})$ can be eliminated to yield the desired relation between $\mathbf{J}_\omega(\mathbf{r})$ and $\mathbf{E}_\omega(\mathbf{r})$. This is rigorously done without finding a simultaneous solution of Eqs. (1.2.11) and (1.2.16) to (1.2.18) or guessing at higher moments of Eq. (1.2.11). At the same time it is shown that without further restrictions (such as boundary conditions) on the problem, a dispersion relation does not exist. This is because there were only two homogeneous equations in three unknowns which yield an infinite number of solutions.

In the next subsection the perturbation approximation (1.2.25) is introduced into the Boltzmann equation. In the following subsection the formal solu-

[14] Feodore Berz, Ph.D. thesis, Imperial College, London, 1955 (unpublished); L. Landau, J. Phys. U.S.S.R. **10**, 25 (1946); M. Bayet, J. phys. radium **15**, 258 (1954); H. R. Mimno, Revs. Modern Phys. **9**, 1 (1937); W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill Book Company, Inc., New York, 1950), second edition, pp. 445–446; C. H. M. Turner, Can. J. Phys. **32**, 16 (1954); H. B. Keller, "On the Electromagnetic Field Equations in the Ionosphere," New York University Mathematical Research Group Report EM-57 (unpublished); J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 327; W. P. Allis, *Handbuch der Physik* (Springer Verlag, Berlin, 1957), Vol. 21; R. Jancel and T. Kahan, Nuovo cimento **12**, 575 (1955); J. phys. radium **16**, 136 (1955); "The Physics of the Ionosphere," *Proceedings of the* **1954** *Cambridge Conference* (The Physical Society, London, 1954), pp. 365–384; J. G. Huxley, Proc. Phys. Soc. (London) **B64**, 844 (1951).

tion or integral representation of the Boltzmann equation is obtained. This is used in subsection 2.4 to obtain the current density.

2.2 Linearized Boltzmann Equation

The purpose of this subsection is to simplify the Boltzmann equation by use of the linearization condition (1.2.25) and the symmetry condition (1.2.19). This is facilitated by introducing a small perturbation function, $f_1(\mathbf{r},\mathbf{v},t)$, which is to be the difference between the actual distribution function, $f(\mathbf{r},\mathbf{v},t)$, and the unperturbed function, $f_0(v^2)$:

$$f_1(\mathbf{r},\mathbf{v},t)\equiv f(\mathbf{r},\mathbf{v},t)-f_0(v^2). \tag{2.2.1}$$

Upon substituting this form into the Boltzmann equation (1.2.11), the following Boltzmann type equation for f_1, is obtained:

$$\frac{\partial f_1}{\partial t}+\nu_m f_1+\mathbf{v}\cdot\nabla f_1+(\mathbf{v}\times\boldsymbol{\omega}_{c0})\cdot\nabla_{\mathbf{v}}f_1 = -2\boldsymbol{\varepsilon}\cdot\mathbf{v}\frac{df_0}{dv^2}, \tag{2.2.2}$$

where

$$\boldsymbol{\omega}_{c0}\equiv e\mathbf{H}_0/mc. \tag{2.2.3}$$

The radio-frequency magnetic field disappeared from this equation because the term, $(\mathbf{v}\times\boldsymbol{\omega}_c')\cdot\nabla_{\mathbf{v}}f_0$, in which it appeared is zero by virtue of the symmetry, Eq. (1.2.19), of f_0.

The linearization condition (1.2.25) means that the second order terms, $(\boldsymbol{\varepsilon}+\mathbf{v}\times\boldsymbol{\omega}_c')\cdot\nabla_{\mathbf{v}}f_1$, can always be dropped compared to the first order term $2\boldsymbol{\varepsilon}\cdot\mathbf{v}(df_0/dv^2)$ even at extrema of $f_0(v^2)$ because the velocity averages in (1.2.25) will make such exceptional points unimportant for all except delta function distributions f_0.

This linearization procedure is equivalent to Nelson's[15] where the electrical forces acting on an electron are taken as those that would be experienced by an electron exploring the field along the unperturbed circular orbit, these forces then being used to derive a first order correction to the orbit.

2.3 Integral Form of Linearized Boltzmann Equation

The purpose of this section is to obtain an integral representation of the solution of the linearized Boltzmann equation (2.2.2). This can be done formally by means of the Lagrangian system of characteristic equations or by appealing to the definition of the Boltzmann operator, the left-hand side of Eq. (2.2.2), as indicated under Eq. (1.2.11). In either case the result can be put into the form[16]

$$f_1=-2\frac{df_0}{dv^2}\int_0^\infty e^{-\nu_m s}[\boldsymbol{\varepsilon}\cdot\mathbf{v}]'ds, \tag{2.3.1}$$

where the prime denotes that the path of integration is such that t is to be replaced by $t'\equiv t-s$, $\mathbf{v}$ by $\mathbf{v}'\equiv R(s)\mathbf{v}$, and $\mathbf{r}$ by $\mathbf{r}'\equiv\mathbf{r}-\int_0^\infty R(\sigma)\mathbf{v}d\sigma$ where $R(s)$ stands for a rotation operator which rotates a vector operand through an angle $\omega_c s$. The path of integration defined above is such that

$$d\mathbf{r}'/ds=-\mathbf{v}', \tag{2.3.2}$$

$$d\mathbf{v}'/ds=\boldsymbol{\omega}_{c0}\times\mathbf{v}', \tag{2.3.3}$$

$$\mathbf{r}'(0)=\mathbf{r}, \tag{2.3.4}$$

$$\mathbf{v}'(0)=\mathbf{v}. \tag{2.3.5}$$

Thus the dummy variable s may be thought of as time measured backward along the unperturbed trajectory. The damping factor $e^{-\nu_m s}$ and the infinite extent of the integration have removed any initial transient terms and properly weighted the past influences of the electric field on the orbit. (See appendix for a derivation.) It is, of course, evident that any accelerations, $\boldsymbol{\varepsilon}(t-s)$, produced in the past would only perturb the present distribution function if $f_0(v'^2)$ actually varied with v'^2 in the neighborhood of $v'^2=v^2$; hence the dependence of f_1, on the gradient of $f_0(v^2)$ in Eq. (2.3.1.). Thus Eq. (2.3.1) has a form that can be intuitively understood as well as the original Boltzmann equation itself.

2.4 Current Density

The purpose of this subsection is to calculate the conduction current density from Eqs. (1.2.18) and (2.3.1). Substituting Eq. (2.3.1) into Eq. (1.2.18), and taking Fourier transforms on time (conjugate ω), one finds that the Fourier components of the current density are given by

$$\mathbf{J}_\omega(\mathbf{r})=-2e\int_0^\infty e^{(i\omega-\nu_m)s}\mathcal{R}(-\tfrac{1}{2}s)ds \int_\infty\left[\mathcal{R}(-\tfrac{1}{2}s)\boldsymbol{\varepsilon}_\omega\left(\mathbf{r}-\frac{2}{\omega_c}\xi\sin(\tfrac{1}{2}\omega_c s)-\mathbf{v}_{||}s\right)\right] \cdot(\xi+\mathbf{v}_{||})(\xi+\mathbf{v}_{||})f_0'(\xi^2+v_{||}^2)d^2\xi dv_{||}, \tag{2.4.1}$$

with

$$\xi=\mathcal{R}(\tfrac{1}{2}s)\mathbf{v}_\perp=\mathcal{R}(-\tfrac{1}{2}s)\mathbf{v}_\perp', \tag{2.4.2}$$

where $\mathbf{v}_\perp$ and $\mathbf{v}_{||}$ are, respectively, the projections of $\mathbf{v}$ onto the plane transverse to the static magnetic field and along the static magnetic field. It can be seen from (2.4.1) that in general $\mathbf{J}_\omega(\mathbf{r})$ depends functionally on $\mathbf{E}_\omega(\mathbf{r}')$ for all $\mathbf{r}'$. Thus $\mathbf{J}_\omega(\mathbf{r})$ depends generally upon $\mathbf{E}_\omega(\mathbf{r})$ and all its spatial derivatives. In addition, $\mathbf{J}_\omega(\mathbf{r})$ must fit into Maxwell's equations, (1.2.16) and (1.2.17). Furthermore, particular forms for the electric field strength, $\mathbf{E}_\omega(\mathbf{r})$, are to be sought such that

$$\mathbf{J}_\omega(\mathbf{r})=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}\cdot\mathbf{E}_\omega(\mathbf{r}). \tag{2.4.3}$$

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[15] Donald Nelson, Ph.D. thesis, Oregon State College, 1954 (unpublished).

[16] For mathematical details see Drummond, reference 11, or differentiate Eq. (2.3.1) to reobtain Eq. (2.2.2).

Setting Eq. (2.4.3) into Eq. (2.4.1), one obtains the following homogeneous integral equation for $\mathbf{E}_\omega(\mathbf{r})$:

$$\boldsymbol{\sigma}\cdot\mathbf{E}_\omega(\mathbf{r})=\int_\infty \mathbf{K}(\mathbf{r}-\boldsymbol{\varrho})\cdot\mathbf{E}_\omega(\boldsymbol{\varrho})d^3\rho, \tag{2.4.4}$$

where $\mathbf{K}(\mathbf{r}-\boldsymbol{\varrho})$ is a symmetric difference kernel obtained by changing the integration variable in Eq. (2.4.1) and then integrating on s. It is not necessary to write $\mathbf{K}$ out explicitly, but only to note its form as just stated. Since it is symmetric, its eigenfunctions should form a complete orthogonal set; and since it is a difference kernel, these eigenfunctions should be relatively easy to find.[17] For one-dimensional integral equations with difference kernels, the eigenfunctions are e^{ikx}. If in addition, the kernel is symmetric, the eigenvalues will be doubly degenerate, either sign of k yielding the same eigenvalue. These results can be readily extended to the present case by the use of Weber's integrals.[18] In this case the eigenfunctions satisfy the following Helmholtz equations:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\mathbf{E}_\omega(\mathbf{r})+k_\perp{}^2\mathbf{E}_\omega(\mathbf{r})=0, \tag{2.4.5}$$

$$\frac{\partial^2}{\partial z^2}\mathbf{E}_\omega(\mathbf{r})+\mathbf{k}_{||}{}^2\mathbf{E}_\omega(\mathbf{r})=0. \tag{2.4.6}$$

Using these equations, the integrations indicated in Eq. (2.4.1) can be carried out explicitly, with results of the following form:

$$\boldsymbol{\sigma}\cdot\mathbf{E}_\omega(\mathbf{r})=\text{const}_1\,\mathbf{E}_\omega(\mathbf{r})+\text{const}_2\,\boldsymbol{\omega}_c\times\mathbf{E}_\omega(\mathbf{r})+\text{const}_3\,\nabla\times\nabla\times\mathbf{E}_\omega(\mathbf{r}). \tag{2.4.7}$$

Between this and Maxwell's equations, the $\nabla\times\nabla\times\mathbf{E}_\omega(\mathbf{r})$ term can be eliminated to give a single, self-consistent relationship between $\mathbf{E}_\omega(\mathbf{r})$ and $\mathbf{J}_\omega(\mathbf{r})$. This determines every element of the conductivity tensor in terms of $k_\perp$, $k_{||}$, and ω as well as the parameters of the plasma:

$$\boldsymbol{\sigma}(\omega)=\begin{bmatrix} C_{||}/C & -C_\perp/C & \sigma_{xz}\\ C_\perp/C & C_{||}/C & \sigma_{yz}\\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}\end{bmatrix}, \tag{2.4.8}$$

with

$$C(\omega)=1+\frac{16\pi^2 i\omega e^2}{mc^2\omega_c{}^2}\int_0^\infty e^{(i\omega-\nu_m)s}\bar{F}_c[1-\cos(\omega_c s)]ds, \tag{2.4.9}$$

$$C_{||}(\omega)=\frac{-4\pi e^2}{m}\int_0^\infty\left[\frac{\bar{F}_c(s)}{\omega_c{}^2}\left(\frac{\omega^2-k_{||}{}^2c^2}{c^2}-\frac{k_\perp{}^2}{4}\frac{\omega^2-k_{||}{}^2c^2}{c^2}\cos(\omega_c s)+\frac{k_\perp{}^2}{4}\cos(2\omega_c s)\right)-f_c(s)\cos(\omega_c s)\right]e^{(i\omega-\nu_m)s}ds, \tag{2.4.10}$$

$$C_\perp(\omega)=\frac{-4\pi e^2}{m}\int_0^\infty e^{(i\omega-\nu_m)s}\left[\frac{\bar{F}_c(s)}{\omega_c{}^2}\frac{k_\perp{}^2}{2}[\sin(\omega_c s)-\tfrac{1}{2}\sin(2\omega_c s)]+\bar{f}_c(s)\sin(\omega_c s)\right]ds, \tag{2.4.11}$$

$$\sigma_{zz}=\frac{-2\pi e^2\int_0^\infty e^{(i\omega-\nu_m)s}\left[\left(\frac{\omega^2}{c^2}-k^2\right)s^2\bar{F}_c(s)-2\bar{f}_c(s)\right]ds}{m\left[1+\frac{8\pi^2 i\omega e^2}{mc^2}\int_0^\infty e^{(i\omega-\nu_m)s}s^2\bar{F}_c(s)ds\right]}, \tag{2.4.12}$$

$$\sigma_{zx}=\frac{+2\pi e^2\int_0^\infty e^{(i\omega-\nu_m)s}s[1-\cos(\omega_c s)]\bar{F}_c(s)ds}{m\omega_c\left[1+\frac{8\pi^2 i\omega e^2}{m\omega_c c^2}\int_0^\infty e^{(i\omega-\nu_m)s}s\sin(\omega_c s)\bar{F}_c(s)ds\right]}\frac{\partial}{\partial y\partial z}, \tag{2.4.13}$$

$$\sigma_{zy}=\frac{-2\pi e^2\int_0^\infty e^{(i\omega-\nu_m)s}s[1-\cos(\omega_c s)]\bar{F}_c(s)ds}{m\omega_c\left[1+\frac{8\pi^2 i\omega e^2}{m\omega_c c^2}\int_0^\infty e^{(i\omega-\nu)s}s\sin(\omega_c s)\bar{F}_c(s)ds\right]}\frac{\partial^2}{\partial x\partial z}, \tag{2.4.14}$$

$$\sigma_{zx}=-\sigma_{xz}, \tag{2.4.15}$$

$$\sigma_{zy}=-\sigma_{yz}, \tag{2.4.16}$$

[17] P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), pp. 907 ff; W. V. Lovitt, *Linear Integral Equations* (Dover Publications, New York, 1950), pp. 116 ff.

[18] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, New York, 1952), pp. 450–453.

where $\bar{f}_c(s)$ and $\bar{F}_c(s)$ are combined Fourier-Hankel transforms:

$$\bar{f}_c(s)\equiv\int_0^\infty\int_0^\infty f_0(v^2)v_\perp J_0\left(2\frac{k_\perp v_\perp}{\omega_c}\sin(\tfrac{1}{2}\omega_c s)\right)dv_\perp\cos(k_{||}v_{||}s)dv_{||}, \tag{2.4.17}$$

$$\bar{F}_c(s)\equiv\int_0^\infty\int_0^\infty \frac{-\omega_c f_0(v^2)}{k_\perp\sin(\omega_c s/2)}v_\perp^2 J_1\left(2\frac{k_\perp v_\perp}{\omega_c}\sin(\tfrac{1}{2}\omega_c s)\right)dv_\perp\cos(k_{||}v_{||}s)dv_{||}. \tag{2.4.18}$$

In the limit $\omega_c\to 0$, the off-diagonal elements approach zero and the diagonal elements approach a common value, σ:

$$\sigma=\frac{\dfrac{2\pi e^2}{m}\displaystyle\int_0^\infty e^{(i\omega-\nu_m)s}\left[2\mathfrak{f}(s)-s^2\left(\frac{\omega^2}{c^2}-k^2\right)\mathfrak{F}(s)\right]ds}{1+\dfrac{8\pi^2 i\omega e^2}{mc^2}\displaystyle\int_0^\infty e^{(i\omega-\nu_m)s}s^2\mathfrak{F}(s)ds}, \tag{2.4.19}$$

where

$$\mathfrak{F}(s)\equiv\frac{2}{(ks)^2}\int_0^\infty\left[v\cos(kvs)-\frac{1}{ks}\sin(kvs)\right]vf_0(v^2)dv, \tag{2.4.20}$$

$$\mathfrak{f}(s)\equiv\frac{1}{ks}\int_0^\infty v\sin(kvs)f_0(v^2)dv. \tag{2.4.21}$$

This result could also have been derived directly for a plasma without magnetic field by use of the same methods developed here. Of course k^2 is the sum of $k_\perp^2$ and $k_{||}^2$. The eigenfunctions belonging with the eigenvalue σ given by Eq. (2.4.19) are solutions of a single three-dimensional Helmholtz equation with parameter k^2. The eigenvalue (2.4.19) is infinitely degenerate because the ratios of k_x and k_y to k_z are completely unspecified in Eq. (2.4.19). Likewise the eigenvalue $\boldsymbol{\sigma}$ given by Eq. (2.4.8) is infinitely degenerate because the ratio of k_x to k_y is completely unspecified in it and in Eq. (2.4.5).

Any realizable electric field can be expanded throughout all space in terms of solutions of Eqs. (2.4.5) and (2.4.6) for various $k_\perp^2$'s and $k_{||}^2$'s. No particular boundary conditions on $\mathbf{E}_\omega(\mathbf{r})$ have to be specified to make this statement true in general. Whatever boundary conditions may apply to the actual function being expanded will determine the issue; whether periodic or radiation or other boundary conditions. Often only a single term in the expansion will be required. Two such cases such are illustrated in the following section.

It can be seen from Eqs. (2.4.8) to (2.4.11) and (2.4.17) to (2.4.19) that in the limit of an unperturbed distribution, f_0, representing very low kinetic energies, the elements $C_{||}/C$ and $C_\perp/C$ approach the standard low-temperature forms[7]:

$$\sigma_{zz}\xrightarrow[\langle v^2\rangle_{\mathrm{Av}}\to 0]{}\sigma\xrightarrow[\langle v^2\rangle_{\mathrm{Av}}\to 0]{}\frac{i\omega_p^2}{4\pi(\omega+i\nu_m)}, \tag{2.4.22}$$

$$\sigma_{xx}\xrightarrow[\langle v^2\rangle_{\mathrm{Av}}\to 0]{}\frac{in_0e^2}{m}\frac{\omega+i\nu_m}{(\omega+i\nu_m)^2-\omega_c^2}, \tag{2.4.23}$$

$$\sigma_{xy}\xrightarrow[\langle v^2\rangle_{\mathrm{Av}}\to 0]{}\frac{n_0e^2}{m}\frac{\omega_c}{(\omega+i\nu_m)^2-\omega_c^2}, \tag{2.4.24}$$

where n_0 is the number density of electrons in the unperturbed distribution,

$$n_0\equiv\int_\infty f_0(v^2)d^3v. \tag{2.4.25}$$

3. APPLICATIONS

3.1 Special Cases

As an illustration of the applications of the foregoing results, the conductivity tensor will be specialized by restricting the unperturbed distribution function to be Maxwellian and it will then be applied to the study of propagation of microwave signals normal to the magnetic field. Only first order temperature effects will be retained. These assumptions are summarized below.

$$f_0=n_0(m/2\pi KT)^{\frac{3}{2}}\exp(-mv^2/2KT), \tag{3.1.1}$$

$$k_{||}=0, \tag{3.1.2}$$

$$k_\perp^2KT/m\omega_c^2\lesssim 0.1, \tag{3.1.3}$$

$$KT/mc^2\lesssim 0.01, \tag{3.1.4}$$

$$\frac{k_\perp^2KT}{m\omega_c^2}>\frac{\nu_m}{\omega}>\left(\frac{k_\perp^2KT}{m\omega_c^2}\right)^2, \tag{3.1.5}$$

where K is Boltzmann's constant.

As shown elsewhere,[19] Eqs. (3.1.3) and (3.1.5) permit

[19] See Drummond, reference 11.

the retention of only first order terms in $k_\perp^2KT/m\omega_c^2$ without significant loss of physical content. After the conductivity tensor is obtained for these conditions, the indices of refraction will be plotted using this result but dropping the collision frequency, ν_m. This would be expected only to remove the "rounding-off" effects of the collisions on resonances.

3.2 Specialized Conductivity Tensor

With the help of Eq. (3.1.1), the integrations indicated in Eqs. (2.4.17) and (2.4.18) can be carried out explicitly. The results are

$$\bar{f}_c(s)=\frac{n_0}{4\pi}\exp\left[-\frac{KT}{2m}\left(\frac{4k_\perp^2}{\omega_c^2}\sin^2(\tfrac{1}{2}\omega_c s)+k_{||}^2s^2\right)\right], \quad (3.2.1)$$

$$\bar{F}_c(s)=\frac{-n_0}{2\pi}\frac{KT}{m}\exp\left[-\frac{KT}{2m}\left(\frac{4k_\perp^2}{\omega_c^2}\sin^2(\tfrac{1}{2}\omega_c s)+k_{||}^2s^2\right)\right]. \quad (3.2.2)$$

Now upon applying conditions (3.1.2)–(3.1.5), the elements of the conductivity tensor become

$$C(\omega)\equiv\mathfrak{D}(\omega_c)=1-\frac{2\omega\omega_p^2KT}{mc^2(\omega+i\nu_m)[(\omega+i\nu_m)^2-\omega_c^2]}, \quad (3.2.3)$$

$$\sigma_{xx}=\sigma_{yy}=\frac{(i\omega-\nu_m)\omega_p^2\left\{1+\dfrac{3k_\perp^2KT}{m[(\omega+i\nu_m)^2-4\omega_c^2]}\right\}}{4\pi\mathfrak{D}(\omega_c)[(\omega+i\nu_m)^2-\omega_c^2]}, \quad (3.2.4)$$

$$\sigma_{xy}=-\sigma_{yx}=\frac{-\omega_c\omega_p^2\left\{1+\dfrac{6k_\perp^2KT}{m[(\omega+i\nu_m)^2-4\omega_c^2]}\right\}}{4\pi\mathfrak{D}(\omega_c)[(\omega+i\nu_m)^2-\omega_c^2]}, \quad (3.2.5)$$

$$\sigma_{xz}=-\sigma_{zx}=0, \quad (3.2.6)$$

$$\sigma_{yz}=-\sigma_{zy}=0, \quad (3.2.7)$$

$$\sigma_{zz}=\frac{\omega_p^2i\left\{1+\dfrac{KT}{m}\left[\dfrac{3k_\perp^2}{\omega^2}-\dfrac{2}{c^2}+\dfrac{k_\perp^2\omega_c^2/\omega^2}{(\omega+i\nu_m)^2-\omega_c^2}\right]\right\}}{4\pi(\omega+i\nu_m)\mathfrak{D}(0)}. \quad (3.2.8)$$

Waves traveling normal to the magnetic field break up into two independent plane polarized waves traveling with different velocities. The two succeeding subsections treat the indices of refraction for these two polarizations.

3.3 Polarization Normal to Magnetic Field

For polarization normal to the magnetic field, the square of the propagation constant is[20] (in Gaussian units):

$$k_\perp^2=\frac{\omega}{c^2}\left(\frac{\omega^2+8\pi i\omega\sigma_{xx}-16\pi^2\sigma_{xx}-16\pi^2\sigma_{xy}^2}{\omega+4\pi i\sigma_{xx}}\right). \quad (3.3.1)$$

Substituting σ_{xx} and σ_{xy} from Eqs. (3.2.4) and (3.2.5) into Eq. (3.3.1) yields for the index of refraction:

$$n^2\equiv\left(\frac{k_\perp\omega^{-1}}{c^{-1}}\right)^2=\frac{-b\pm(b^2-4ac)^{\frac{1}{2}}}{2a}, \quad (3.3.2)$$

where

$$a\equiv\frac{-3v'^2X(1+3v'^2X)}{1-4\omega_c'^2}, \quad (3.3.3)$$

$$b\equiv1-X+\frac{6Xv'^2}{1-4\omega_c'^2}(1-X+2X\omega_c'^2), \quad (3.3.4)$$

$$c\equiv-1+2X-X^2+\omega_c'^2X^2, \quad (3.3.5)$$

$$X\equiv\frac{\omega_p'^2}{1-\omega_c'^2-2v'^2\omega_p'^2}, \quad (3.3.6)$$

$$\omega_p'\equiv\omega_p/\omega, \quad (3.3.7)$$

$$\omega_c'\equiv\omega_c/\omega, \quad (3.3.8)$$

$$v'\equiv(1/c)(KT/m)^{\frac{1}{2}}. \quad (3.3.9)$$

Equation (3.3.2) is an explicit expression for the square of the index of refraction in terms of the three independent variables $\omega_p'^2$, ω_c', and v'^2. It is readily tabulated on a digital computer. The sign of the radical must be chosen to be the same as the sign of b in order to insure convergence to the proper limit as $T\rightarrow0$. Graphical representations of the results are given in Figs. 4 and 5.

Figure 4 shows the index of refraction for a 500-volt plasma. A region of abnormal transmission is seen to occur for $|\omega_c/\omega|$ between $\frac{1}{2}$ and 1. Within this region the curves for various densities hug a single curve. The individual curves jump back to the extinguishing region at various frequencies determined by both density and temperature of the electron distribution. Notice that in the region just to the right of $|\omega_c/\omega|=1$, the indices have both real and imaginary parts. This is a region of power exchange between the plasma and the radiation field.

Figure 5 shows the same thing for 5000-volt plasmas. The smaller vertical scale here shows that the index of refraction in the abnormal transmission band depends almost entirely upon electron temperature. Again, however, the frequency of crossover between transmission and extinction bands depends upon both electron temperature and density.

In the next subsection the same things are done for polarization parallel to the static magnetic field.

[20] Lyman Mower, "Propagation of Plane Wave in an Electrically Anisotropic Medium," Sylvania Report MPl-1, 1956 (unpublished).

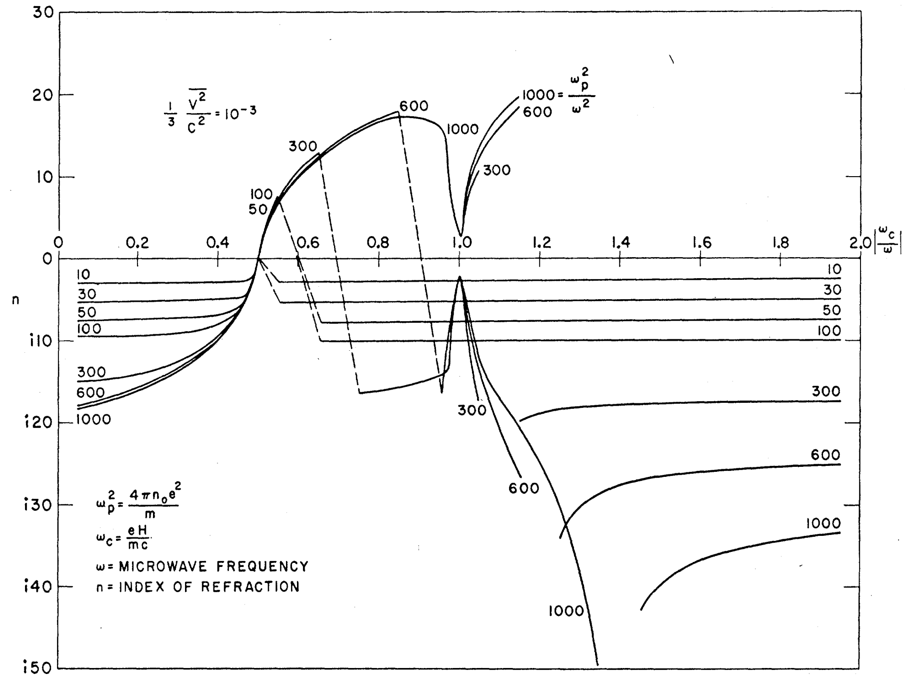

Fig. 4. Index of refraction for microwave signal propagating across magnetic field and polarized normal to the magnetic field in a plasma.

3.4 Polarization Parallel to Magnetic Field

For polarization parallel to the magnetic field, the square of the propagation constant is[20] (in Gaussian units):

$$k_\perp^2 = \frac{\omega^2}{c^2}[1 + 4\pi i\omega\sigma_{zz}]. \quad (3.4.1)$$

Substituting σ_{zz} from Eq. (3.2.8) into Eq. (3.4.1) yields the following equation for the index of refraction:

$$n^2 = (1-\omega_p'^2) \Big/ \left[1 + v'^2\omega_p'^2\left(1 + \frac{\omega_c'^2}{1-\omega_c'^2}\right)\right]. \quad (3.4.2)$$

Graphical representation of the results are given in Figs. 6 and 7.

Figure 6 shows the index of refraction for a 500-volt plasma. A region of abnormal transmission is seen to occur for $|\omega_c/\omega|$ slightly greater than 1. The index of refraction within this band as well as the width of the band depend upon both electron density and temperature.

Figure 7 shows the index of refraction for 5000-volt plasmas. The values of the indices within this band are smaller and the widths of the band are greater than in the corresponding case for the 500-volt plasmas.

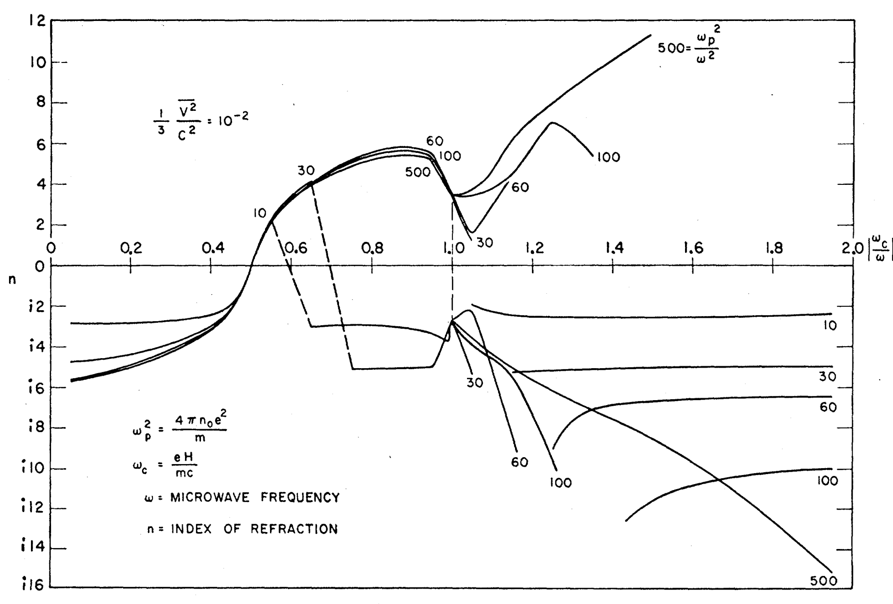

Fig. 5. Index of refraction for microwave signal propagating across magnetic field and polarized normal to the magnetic field in a plasma.

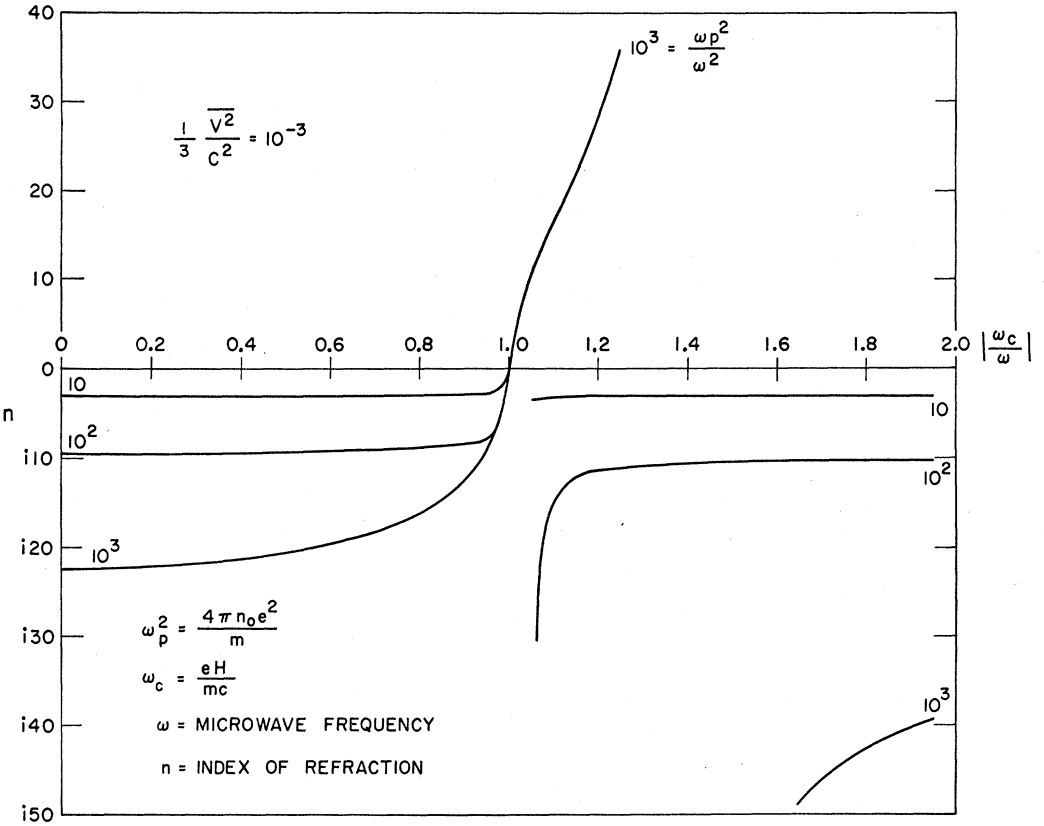

FIG. 6. Index of refraction for microwave signal propagating at right angles to the magnetic field and polarized parallel to the magnetic field.

4. CONCLUSIONS

4.1 Basic Electromagnetic Effects

The purpose of this subsection is to point out the significance of terms in the conductivity tensor which are found to depend upon c the speed of light. The factor $\langle v^2\rangle_{\mathrm{Av}}/c^2$ occurs in three places in the conductivity tensor. First it appears in under the integral sign in $C_{\parallel}(\omega)$, Eq. (2.4.10). Usually this integration is such that $\langle v^2\rangle_{\mathrm{Av}}/c^2$ is subtracted from terms of order $k_{\perp}^2\langle v_{\perp}^2\rangle_{\mathrm{Av}}/\omega^2$ and 1. Thus it may usually be neglected for the present nonrelativistic treatment, Eq. (1.2.10).

Second, $\langle v^2\rangle_{\mathrm{Av}}/c^2$ appears in the denominators of the transverse elements of the conductivity tensor. There, as shown by Eq. (2.4.9) for $C(\omega)$, it is multiplied by the average electron number density, and added to a term not containing n. Consequently, for high-density plasmas, this term can become very important even though $\langle v^2\rangle_{\mathrm{Av}}/c^2\ll 1$. It can completely change the microwave properties of a plasma from predominantly nondissipative to dissipative at certain frequencies; and at other frequencies, it can change a plasma from a nonpropagating to a propagating medium for microwaves. This is illustrated by the special cases worked out in Sec. 3. However, the basic effects can be seen by noting the way in which this $\langle v^2\rangle_{\mathrm{Av}}/c^2$ term enters into the conductivity tensor. It can be shown from Eqs. (2.4.9) and (2.4.8) that if $\omega_c^2\gg\omega^2$ and $k_{\parallel}^2\langle v_{\parallel}^2\rangle_{\mathrm{Av}}/\omega_c^2\ll 1$, then the term reduces the response current of the plasma and hence the apparent density of the plasma as measured by its effect on a microwave signal. This is what was predicted on the basis of the qualitative argument in Sec. 1.3. The present result goes beyond this, however, to show that the opposite happens if $\omega^2\gg\omega_c^2$ provided $k_{\parallel}^2\langle v_{\parallel}^2\rangle_{\mathrm{Av}}/\omega_c^2\ll 1$. In addition, certain power-transfer resonances are found to occur for those frequencies for which the real part of $C(\omega)=0$.[16]

Third, $\langle v^2\rangle_{\mathrm{Av}}/c^2$ appears in the denominators of Eqs. (2.4.13) and (2.4.14) for σ_{xz} and σ_{yz}. Here again it is multiplied by the density n and added to a term not containing n. Thus it can very significantly modify the effects of these newly derived nonzero elements of the conductivity tensor for high-density plasmas.

4.2 Orbital Effects

The ratio of orbit circumference, $(2\pi/\omega_c)(v_{\perp}^2)^{\frac{1}{2}}$, to transverse wave length, $2\pi/k_{\perp}$, appears in two functions, $\bar{f}_c$ and $\bar{F}_c$, as given by Eqs. (2.4.17) and (2.4.18). These two functions enter in every element of the conductivity tensor. Their principle effect is to modify the effective plasma density. For instance if $\omega_c^2\gg\omega^2$ and $k_{\parallel}^2v_{\parallel}^2/\omega_c^2$

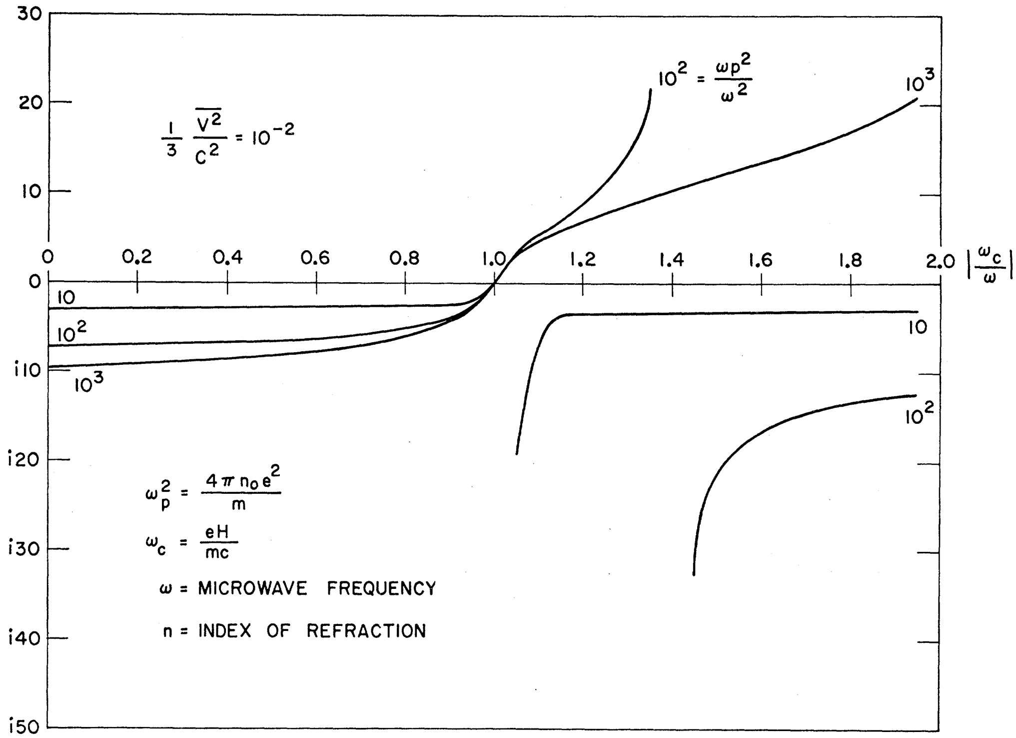

FIG. 7. Index of refraction for microwave signal propagating at right angles to the magnetic field and polarized parallel to the magnetic field.

$\ll 1$, the elements σ_{xx}, σ_{xy}, σ_{yy}, and σ_{yx} will all become smaller to first order terms in $k_\perp^2\langle v_\perp^2\rangle_{\text{Av}}/\omega_c^2$. This is as predicted on the basis of a qualitative argument in Sec. 1.3. The present result goes beyond this, however, to show that the opposite happens if $\omega^2 \gg \omega_c^2$ and $k_{\parallel}'^2\langle v_{\parallel}^2\rangle_{\text{Av}}/\omega_c^2 \ll 1$. In addition, certain minor resonances are predicted.[16]

4.3 Longitudinal Current Density

It was shown in Sec. 2.5 that if the charge density ρ_ω has the same dependence upon x and y as does the z component of radio-frequency magnetic field strength, h_z, then nonzero (z,x) and (z,y) components of the conductivity tensor exist as differential operators. The components vanish as $\langle v^2\rangle_{\text{Av}} \to 0$. Thus for finite temperatures and a z-dependent radio-frequency electric field, transverse radio-frequency field components produce a longitudinal radio-frequency current density just as shown in the qualitative argument in Sec. 1.3. The conductivity elements σ_{xz}, σ_{yz}, and σ_{xy} are determined by reciprocal relations such as (2.4.15) and (2.4.16) which become the Onsager or Callen-Greene relations when f_0 is Maxwellian.[21]

[21] L. Onsager, Phys. Rev. **37**, 405 (1931); **38**, 2265 (1932). H. B. Callen and R. F. Greene, *ibid.* **86**, 702 (1952); **85**, 1378 (1952); Thomas A. Kaplan, *ibid.* **102**, 1447 (1956); S. R. De Groot, *Thermodynamics of Irreversible Processes* (Interscience Publishers, Inc., New York, 1952).

4.4 Diffusion Damping

The Fourier series expansion of $J_0[2(k_\perp v_\perp/\omega_c) \times\sin(\frac{1}{2}\omega_c s)]$ on s in $\bar{f}_c$ and $\bar{F}_c$ given in Eqs. (2.4.17) and (2.4.18) contains all the integral multiples of the cyclotron frequency, ω_c: $e^{in\omega_c s}J_n^2(k_\perp v_\perp/\omega_c)$. Each of these spectral components is symmetrically split by the factor $\cos(k_{\parallel}v_{\parallel}c) = \frac{1}{2}[\exp(ik_{\parallel}v_{\parallel}s)+\exp(-ik_{\parallel}v_{\parallel}{}^s)]$. The relative intensity of the resulting spectrum is determined by the weighing factor $f_0(v_\perp^2+v_{\parallel}^2)dv_{\parallel}$ in Eqs. (2.4.17) and Eq. (2.4.18). Thus the integration over $v_{\parallel}$ in these equations has the effect of smearing the spectral lines symmetrically about the cyclotron multiples. This is also the effect of the momentum transfer collisions on the resonance lines. Thus for some cases the Landau term $k_{\parallel}v_{\parallel}$ may be simply regarded as increasing the effective frequency of phase-disrupting collisions as indicated in the qualitative argument in Sec. 1.3. The width of the resulting lines will be small compared to the cyclotron frequency if $k_{\parallel}^2\langle v_{\parallel}^2\rangle_{\text{Av}}/\omega_c^2 \ll 1$.

4.5 Prospectus

There are two basic limitations beside the use of classical mechanics to the work reported here. First, it is limited to small perturbations away from a homogeneous isotropic steady state; and second, statistical correlations have been eliminated by using the Boltzmann equation. Correcting the first deficiency appears to be difficult for general cases. Some of the ground work for correcting the latter has already been done[8]; but needs extension.

However, in spite of its limitations, the present theory is believed to extend the work on plasma properties in two important respects; first, the usual nonzero elements of the conductivity tensor are modified in a way that can have significant effects on microwave propagation in high-density, high-temperature plasmas; and second the existence of four new nonzero elements has been shown for some cases. The nature of these new elements has not been fully investigated, but might be effectively treated by Twersky's general methods.[22]

The modifications brought about by the new form of the usual elements of the conductivity tensor was the subject of Sec. 3 on transmission characteristics of hot plasmas. More sensitive dependence of the index of refraction on electron density and temperature is to be expected for right-circularly polarized waves propagating along the magnetic axis.

The present analysis could probably be extended to cases for which the unperturbed steady state is specified by a distribution function of the form $f_0(v^2,v_{||})$. The kernel of the integral equation for the electric field strength would still be a difference kernel (as well as being symmetrical on its variables normal to the magnetic axis). When the unperturbed state is taken as having gradients and currents, two difficulties arise; the solutions to the characteristic equations become rather involved, and the kernel of the integral equation is no longer either symmetrical nor a difference kernel (though it may have both of these properties in its dependence on variables refering to the magnetic axis).

In summary, the present analysis has shown that a high-temperature plasma is characterized by not one but many different conductivity tensors corresponding to certain eigenfunctions of the electromagnetic field configuration, and representing the coupling and transconductance properties of the plasma. An expansion of the eigenvalues has been used to calculate the indices of refraction for two cases of interest. Further work is in progress to extend these calculations. In addition, it would be desirable to have experimental tests of the theory.

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APPENDIX

The purpose of this Appendix is to derive Eq. (2.3.1) from Eq. (2.2.2). There are several ways in which this can be done. The method chosen is believed to be unique.

The general solution[13] of Eq. (2.2.2) is a differentiable but otherwise arbitrary function connecting the constants of integration of Eqs. (2.3.2) and (2.3.3) and

$$\frac{df_1}{dt}=-2\boldsymbol{\mathcal{E}}(\nu,t)\cdot\mathbf{v}\frac{df_0}{dv^2}-\nu_m f_1. \tag{A.1}$$

Solving this relationship formally for $f_1(\mathbf{r},\mathbf{v},t)$ gives

$$\begin{aligned}f_1(\mathbf{r},\mathbf{v},t)&\\ =-\int_0^\infty\bigg\{&2\boldsymbol{\mathcal{E}}\bigg(\mathbf{r}-\frac{2}{\omega_c}\boldsymbol{\xi}\sin(\tfrac{1}{2}\omega_c s)-\mathbf{v}_{||}s,\,t-s\bigg)\\ &\cdot[\mathbf{v}_{||}+\mathcal{R}(s)\mathbf{v}_\perp]\frac{df_0}{dv^2}\\ &-\nu f_1\bigg(\mathbf{r}-\frac{2}{\omega_c}\boldsymbol{\xi}\sin(\tfrac{1}{2}\omega_c s)-\mathbf{v}_{||}s,\,t-s\bigg)\bigg\}ds\\ &+\Phi\bigg(\mathbf{r}-\frac{\boldsymbol{\omega}_c\times\mathbf{v}}{v^2}-\frac{\omega_c t}{\omega_c^2}\boldsymbol{\omega}_c\cdot\mathbf{v},\,\mathcal{R}(-t)\mathbf{v}\bigg),\end{aligned} \tag{A.2}$$

where Φ is a differentiable but otherwise arbitrary function of its indicated arguments. This is not yet a solution, but rather an integral equation for $f_1(\mathbf{r},\mathbf{v},t)$. It can be solved by summing the Liouville-Neumann series corresponding to it. This series is obtained by successive substitutions of the solution $f_1(\mathbf{r},\mathbf{v},t)$ given by the equation itself in place of f_1 appearing in the integrand of Eq. (A.2). In the present case this gives

$$f_1(\mathbf{r},\mathbf{v},t)=\lim_{y\to\infty}\sum_{n=0}^{\infty}I_y{}^n(\mathcal{G}_y+\Phi), \tag{A.3}$$

where

$$\mathcal{G}_y\equiv-2\frac{df_0}{dv^2}\int_0^y\boldsymbol{\mathcal{E}}\bigg(\mathbf{r}-\frac{2}{\omega_c}\boldsymbol{\xi}\sin(\tfrac{1}{2}\omega_c s)-\mathbf{v}_{||}s,\,t-s\bigg)\cdot[\mathbf{v}_{||}+\mathcal{R}(s)\mathbf{v}_\perp]ds, \tag{A.4}$$

[22] V. Twersky, private communication in which he studied electromagnetic waves in a medium where $\mathbf{J}=\boldsymbol{\sigma}\cdot\mathbf{E}+\boldsymbol{\tau}\cdot\mathbf{H}$.

and I_y^n is the nth iterate of the operator I_y defined by

$$I_y\psi(\mathbf{r},\mathbf{v},t)\equiv -\nu_m\int_0^y \psi\left(\mathbf{r}-\frac{2}{\omega_c}\boldsymbol{\xi}\sin(\tfrac{1}{2}\omega_c s)\right.$$
$$\left.-\mathbf{v}_{||}s,\ \mathcal{R}(s)\mathbf{v},t-s\right)ds. \quad \text{(A.5)}$$

When I_y is applied to Φ, the particular combination of arguments in Φ make it independent of s during the integration. Hence

$$I_y^n\Phi=(-y\nu_m)^n\Phi. \quad \text{(A.6)}$$

Thus for $\nu_m y<1$, the infinite series in $I_y^n\Phi$ can be summed. The result is

$$\sum_{n=0} I_y^n\Phi=\frac{\Phi}{1+y\nu_m}, \quad \text{(A.7)}$$

which is an analytic function of ν_m. Since this explicit expression for the sum is analytic everywhere except for $\nu_m=-1/y$, it is the proper analytic continuation of the sum for $y\nu_m\geq 1$. Since the solution (A.3) is to be an analytic function of ν_m as stated in subsection 1.2, the expression (A.7) may be taken as the sum for all real positive values of $\nu_m y$. In particular it applies in the limit $y\rightarrow\infty$ where it gives the value zero provided only that $\nu_m\neq 0$.

The remaining sum may be evaluated by noting the following property of the operator I_∞:

$$I_\infty e^{\nu_m t}\psi(\mathbf{r},\mathbf{v},t)=e^{\nu_m t}\sum_{n=1}^{\infty} I_\infty^n\psi \quad \text{if} \quad \psi(\mathbf{r},\mathbf{v},-\infty)=0. \quad \text{(A.8)}$$

This expansion is easily obtained by means of repeated integration by parts. Applying this to Eq. (A.3) and, noting that $\mathcal{G}_m=(2/\nu_m)(df_0/dv^2)I_\infty\boldsymbol{\mathcal{E}}\cdot\mathbf{v}$ gives the explicit solution (2.3.1).

The form of the solution (2.3.1) is useful in itself. It shows how physically realizable plasmas may be approximated: f_0 need not be independent of time, but must change by only a small percentage during one mean free time, $1/\nu_m$. Similar considerations applied to the velocity moments of f_0, Eq. (2.4.1), show that it need not be independent of position either, but only vary by a small percentage in one mean free path along the z axis and across one mean orbit diameter in the transverse plane.

Equation (2.3.1) also provides an explicit first order distribution function which could be used directly as a basis for the calculation of second order perturbation functions.