Energy Shifts in the Feynman Formalism

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The relation between the S matrix and the energy-level shift is demonstrated in a form which permits the use of the Feynman methods of calculation. It is also shown that vacuum fluctuations and "unlinked clusters" do not contribute to the energy of a physical system.

IT is often convenient to use a time-dependent formalism to compute the energy shift due to a time-independent interaction. This is especially advantageous when both particles and antiparticles are treated, since a symmetrical treatment using the time-dependent formalism of Feynman[1,2] can often simplify such problems. Examples are relativistic theories such as quantum electrodynamics, and some many-body problems in which the particle-hole idea is useful.[3]

The fundamental quantity of this time-dependent formalism is the S matrix, which describes the propagation of the state vector from $t=-\infty$ to $t=+\infty$. The S matrix has been used in the past to compute energy shifts, although often in a modified form. Although the relation between the S matrix and the energy shift has been stated in the literature,[1,2] a rigorous proof was not given until recently.[4] However, this result is not in a form which can be directly applied in the Feynman formalism; it is the purpose of this article to supply this connection.

Our proof shows that unconnected Feynman graphs representing, for instance, vacuum fluctuations in field theory or "unlinked clusters"[3,5] in the many-body problem, make no contribution to the energy of a real system.

We start with an unperturbed state φ, an eigenstate of H_0 with eigenvalue E_0. The energy-level shift which arises when the interaction gH_1 is "turned on" adiabatically can be shown[4] to be

$$\Delta E=\lim_{\alpha\to 0} i\frac{\alpha}{2}g\frac{\partial}{\partial g}\ln\langle\varphi|S_\alpha|\varphi\rangle. \tag{1}$$

Here $S_\alpha=U_\alpha(\infty,-\infty)$ is the adiabatic S matrix defined by

$$U_\alpha(t,-\infty)=1-i\int_{-\infty}^{t}dt'e^{-\alpha|t'|}gH_1(t')U_\alpha(t',-\infty) \tag{2}$$

in the limit $t\to\infty$.

In the Feynman method it is customary to interchange the order of integration and limiting process in (1) and (2) so that the convergence factor $e^{-\alpha|t|}$ becomes unity and does not appear explicitly. We wish to show that one can do this and obtain a simple prescription for computing the level shift.

Let us denote by M_α the contribution to $\langle\varphi|S_\alpha|\varphi\rangle$ arising from all terms represented by only one connected diagram. Then, in terms of M_α,

$$\langle\varphi|S_\alpha|\varphi\rangle=1+M_\alpha+\frac{M_\alpha^2}{2!}+\cdots=e^{M_\alpha}, \tag{3}$$

where we use the fact that n disconnected loops are counted in $n!$ ways while performing the integrations involved in M_α^n. [This argument is due to Feynman.[1] We assume here that the sum of the perturbation series has a meaning, and, consequently, that this series can be reordered. In any case, (3) is formally correct.] Thus $\ln\langle\varphi|S_\alpha|\varphi\rangle=M_\alpha$ and contains only connected diagrams.

Let $M_\alpha=\sum_n M_\alpha^{(n)}$, where $M_\alpha^{(n)}$ is the nth-order contribution to M_α. It contains space-time integrals over various combinations of wave functions and propagators. If each of these is expanded in the energy representation, the resulting expression has the form[6]

$$M_\alpha^{(n)}=g^n\int_{-\infty}^{\infty}dt_1\cdots dt_n dE_1\cdots dE_{n-1}\exp\left(-\alpha\sum_{j=1}^{n}|t_j|\right)\times\exp\left(i\sum_{j=1}^{n}E_jt_j\right)f(E_1\cdots E_{n-1}). \tag{4}$$

Here $f(E_1\cdots E_{n-1})$ contains the Fourier transforms of the appropriate wave functions and propagators. In terms of diagrams, E_j is the algebraic sum of the energies of all lines which meet at the jth vertex (positive if they enter, negative if they leave). The set

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[2] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Press, Inc., Cambridge, 1955).
[3] J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957).
[4] J. Sucher, Phys. Rev. **107**, 1448 (1957).
[5] K. A. Brueckner, Phys. Rev. **100**, 36 (1955).

[6] As an example, the second-order contribution to the electromagnetic self-energy of a charged particle has the form

$$M_\alpha^{(2)}=g^2\int_{-\infty}^{\infty}dt_1dt_2dq^0dk^0\exp\left(-\alpha\sum_{j=1}^{2}|t_j|\right)\times\exp[i(p_1^0-q^0-k^0)(t_1-t_2)]h(q^0,k^0).$$

The spatial integrals and the integrals over the three-momenta are included in $h(q^0,k^0)$; p_1^0 is the initial (and final) energy of the charged particle; q^0 and k^0 are the energies of the intermediate particle and photon. If we change integration variables to $E_1=p_1^0-q^0-k^0$ and k^0, and let $E_2=-E_1$, then

$$M_\alpha^{(2)}=g^2\int_{-\infty}^{\infty}dt_1dt_2dE_1\exp\left(-\alpha\sum_{j=1}^{2}|t_j|\right)\exp\left(i\sum_{j=1}^{2}E_jt_j\right)f(E_1).$$

The k^0-integral, which depends on the dynamics of the electromagnetic interaction and is not related to the time integrals, has been absorbed into $f(E_1)=\int dk^0h(E_1,k^0)$.

E_j is related by $\sum_{j=1}^n E_j=0$ arising from over-all energy conservation.

Using (1), and the fact that $g(\partial/\partial g)M_\alpha=\sum_n nM_\alpha^{(n)}$, we find that

$$\Delta E=\lim_{\alpha\to 0} i\frac{\alpha}{2}g\frac{\partial}{\partial g}M_\alpha=\sum_n \lim_{\alpha\to 0} i\frac{n\alpha}{2}M_\alpha^{(n)}\equiv\sum_n \Delta E^{(n)}. \quad (5)$$

Therefore

$$\Delta E^{(n)}=\lim_{\alpha\to 0} i\frac{n\alpha}{2}M_\alpha^{(n)}=ig^n\int_{-\infty}^{\infty}dE_1\cdots dE_{n-1} \times\mathcal{O}(E_1\cdots E_n)f(E_1\cdots E_{n-1}), \quad (6)$$

where

$$\mathcal{O}(E_1\cdots E_n)=\lim_{\alpha\to 0}\frac{n\alpha}{2}\int_{-\infty}^{\infty}dt_1\cdots dt_n \times\exp\left(-\alpha\sum_{j=1}^{n}|t_j|\right)\exp\left(i\sum_{j=1}^{n}E_jt_j\right). \quad (7)$$

To evaluate $\mathcal{O}(E_1\cdots E_n)$ it is simplest to perform all the integrals involved in $\Delta E^{(n)}$, and then examine the result. We assume that the orders of integration may be freely interchanged, although one must still go to the α-limit *after* the time-integration.

If we first perform the integrations over $t_1\cdots t_{n-1}$ in (7), we find

$$\mathcal{O}(E_1\cdots E_n) =\lim_{\alpha\to 0}\frac{n\alpha}{2}\int_{-\infty}^{\infty}dt_n\prod_{j=1}^{n-1}\left(\frac{2\alpha}{E_j^2+\alpha^2}e^{-iE_jt_n}\right)e^{-\alpha|t_n|}. \quad (8)$$

We do not integrate over t_n at this stage so as to permit each of the $(n-1)$ energy integrals to be performed independently.

The jth energy integral is

$$\int_{-\infty}^{\infty}dE_j\frac{2\alpha}{E_j^2+\alpha^2}e^{-iE_jt_n}f(E_1\cdots E_j\cdots E_{n-1}).$$

This integral can be performed by extending E_j into the complex plane and transforming to a contour integral. For $t_n>0$ we close the contour in the lower half-plane. In this case there will be contributions from the simple pole at $E_j=-i\alpha$ and from the singularities of $f(E_1\cdots E_j\cdots E_{n-1})$. We shall ignore the latter since they lead to smaller inverse powers of α, and give no contribution in the limit $\alpha\to 0$. Then if we include the results for both positive and negative t_n, this integral is

$$2\pi e^{-\alpha|t_n|}f\left(E_1\cdots,-i\alpha\frac{t_n}{|t_n|},\cdots E_{n-1}\right).$$

Combining all such energy integrals, we find

$$\begin{aligned}\Delta E^{(n)}&=(2\pi)^{n-1}ig^n\lim_{\alpha\to 0}\frac{n\alpha}{2}\int_{-\infty}^{\infty}dt_ne^{-n\alpha|t_n|} \times f\left(-i\alpha\frac{t_n}{|t_n|}\cdots-i\alpha\frac{t_n}{|t_n|}\right)\\&=(2\pi)^{n-1}ig^n\lim_{\alpha\to 0}\tfrac{1}{2}[f(i\alpha,\cdots,i\alpha)+f(-i\alpha,\cdots,-i\alpha)]\\&=(2\pi)^{n-1}ig^nf(0,\cdots,0).\end{aligned} \quad (9)$$

This result shows that

$$\mathcal{O}(E_1\cdots E_n)=(2\pi)^{n-1}\prod_{j=1}^{n-1}\delta(E_j). \quad (10)$$

On the other hand, suppose we let $\alpha\to 0$ *before* we perform the time integrals. The resulting matrix element is

$$M_0^{(n)}=g^n\int_{-\infty}^{\infty}dE_1\cdots dE_{n-1}\mathcal{P}(E_1\cdots E_n)f(E_1\cdots E_{n-1}), \quad (11)$$

where

$$\mathcal{P}(E_1\cdots E_n)=\int_{-\infty}^{\infty}dt_1\cdots dt_n\exp\left(i\sum_{j=1}^{n}E_jt_j\right) =(2\pi)^n\prod_{j=1}^{n}\delta(E_j)=(2\pi)^n\delta(0)\prod_{j=1}^{n-1}\delta(E_j), \quad (12)$$

since $\sum_{j=1}^n E_j=0$. [The expression $\delta(0)$ is not well-defined, but this equation can serve to give it meaning. Alternatively, we could let $\sum_{j=1}^n E_j=\epsilon_{fi}$, the energy difference between the initial and final states, so that $\delta(0)$ is replaced by $\delta(\epsilon_{fi})$, and then let $\epsilon_{fi}\to 0$]. As a result,

$$\mathcal{P}(E_1\cdots E_n)=2\pi\delta(0)\mathcal{O}(E_1\cdots E_n), \quad (13)$$

and

$$M_0^{(n)}=-2\pi i\delta(0)\Delta E^{(n)}. \quad (14)$$

If we sum the contributions of all orders, we find

$$M_0=\ln\langle\varphi|S_0|\varphi\rangle=-2\pi i\delta(0)\Delta E, \quad (15)$$

or

$$\langle\varphi|S_0|\varphi\rangle=e^{-2\pi i\delta(0)\Delta E}. \quad (16)$$

This result was anticipated in a heuristic argument of Feynman.[1] S_0 is the operator which is considered in the Feynman formulation; it is clearly more convenient, for purposes of calculation, than S_α. If we expand the exponential in (16), we arrive at the following prescription: Compute, using the energy representation for all time-dependent functions, those terms in $\langle\varphi|S_0|\varphi\rangle$ which contain only *one* energy-conserving δ-function (i.e., the connected diagrams).[7] The sum of these terms is $-2\pi i\delta(0)\Delta E$.

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[7] These include terms corresponding to the self-energy of the vacuum as well as those yielding the energy shift of the system under consideration. These occur additively, and the vacuum-fluctuation terms can be separated from the real, physically-interesting, energy shifts.