The examination of the particular solution of the field equations, given by Case III, shows that its physical interpretation is completely different depending on whether (2.5) or (2.6) are used as equations of motion. Obviously, if a third set of equations of motion were to be selected, we might have a third independent physical interpretation of exactly the same solution. Since the problem of selecting a suitable set of equations is far from solved, there seems little point to attempting to give physical interpretations of the remaining solutions found in this paper.

10. CONCLUSION

We have, in this paper, discovered a set of rigorous solutions of Einstein's latest form of unified field theory. It was hoped that an examination of these solutions would clarify a great deal of ambiguity that arises in the identification of fundamental physical quantities. Instead of this clarification, our solutions have shown that the unified field theory is not complete, in that equations giving the motion of test particles have to be selected.

In regard to equations of motion, we would like to emphasize that (2.5) and (2.6) are put forth only for basis of illustration. Actually it is our opinion that neither set is suitable for a unified field theory. These equations act to separate the actions of the two fields and enable us to label the effects as gravitational or electromagnetic in character. Thus the present theory and these particular equations of motion would have little claim to be a unified theory. Its character would be closer to a theory of two distinct fields which interact with each other.

Finally, to conclude this paper we would like to make a few general remarks about Einstein's present form of theory. Basing his theory on a nonsymmetric tensor introduces too many degrees of freedom in the sense that an infinity of tensor quantities can be defined in terms of such a tensor. For this reason we believe that a mathematical type of approach is doomed to failure and a more physical type of approach should be used. Certainly an intermediate theory based on a symmetric tensor and a generalized electromagnetic potential vector is worth further investigations and might point the way to a clearer understanding of a theory based on a nonsymmetric tensor.

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Effect of a Gravitational Field, Due to a Rotating Body, on the Plane of Polarization of an Electromagnetic Wave

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It is shown that for electromagnetic waves the gravitational field of a rotating body acts as an optically active medium. Thus, the plane of polarization of the wave rotates while it passes through this field. The effect is small. The angle of rotation due to the gravitational field of the sun is about 10^{-12} radian.

I

IT is well known that according to the general theory of relativity, gravitational fields influence electromagnetic ones. To be specific, the rays associated with an electromagnetic wave are not straight but curved if the wave passes through a gravitational field, and the frequency of the wave is changed if the gravitational potentials are different at the points of emission and observation.[1] The two effects are connected. The change in frequency shows that the phase of the wave is affected; but then, in general, so will be the rays, since they are the orthogonal trajectories of the surfaces of constant phase. However, an electromagnetic wave is not completely characterized by the rays and its energy; it also has a state of polarization. Will this polarization be uninfluenced by a gravitational field? This effect, if it exists, must be independent of the two effects mentioned since they must take place even if the field is a scalar one and the latter effect obviously cannot take place in a scalar field. For this reason its order of magnitude may also be quite different. Unfortunately, we shall see it is much less.

In this paper we shall find that if the gravitational field is caused by rotating bodies the *plane* of polarization of the wave can change, in a manner similar to that which would occur if the wave passed through an optically active medium whose optical axis coincides with the axis of rotation of the source. The effect seems to be too small to be observed in practice.

In Sec. II we shall discuss the effect using essentially the equivalence principle. This treatment already gives approximately the size and sign of the effect.

In Sec. III we derive the results using Maxwell's equations in the presence of gravitational fields.

[1] There is no change in the frequency if the gravitational potentials are the same at these two points, even if the wave has passed through a region where the potential was different. If it were otherwise we could violate the law of conservation of energy.

II

The effects of a gravitational field are completely contained in the metric tensor, g_{ik}. If two different observers associate the same metric tensor with a given portion of space-time they will observe the same gravitational field, and the physical phenomena taking place in that portion of space-time will be influenced in the same way by the gravitational field for both observers. (This is essentially the equivalence principle.)

We shall make use of this fact to show that according to the general theory of relativity a physical process occurring in the presence of a gravitational field generated by a rotating body has locally the same appearance as if the same process would take place in the gravitational field of the same body without rotation, while the frame of reference of the observer is rotating with a certain angular velocity of much smaller magnitude and opposite sign from that of the body.

The metric tensor of a weak gravitational field generated by a rotating body is given by the following expressions[2]:

$$\begin{aligned} g_{11}=g_{22}=g_{33}&=(1-2\phi/c^2),\\ g_{44}&=(1+2\phi/c^2), \qquad\qquad (1)\\ \gamma=-i(g_{14},g_{24},g_{34})&=-(2G/c^3R^3)\mathbf{L}\times\mathbf{R}. \end{aligned}$$

$\mathbf{R}$ is the radius vector from the center of the mass to the point of observation; $\mathbf{L}$ is the angular momentum of the body; ϕ is the Newtonian gravitational potential of the mass, except for a small addition which is essentially the gravitational potential of the mass associated with the energy of rotation of the body. (Since this contribution is small and will not influence the effect we seek, we may disregard it.) In the derivation of (1) one has assumed that the gravitational field is small and that R is large compared with the linear dimensions of the body.

The diagonal elements of g_{ik} are responsible for the frequency shift and bending of light rays in the presence of a gravitational field.

We shall show that γ causes a rotation of the plane of polarization.

It is interesting to observe that while the diagonal elements depend only on the gravitational potential, the off-diagonal elements, collected in γ, contain the angular momentum of the source. The latter does not appear in the nonrelativistic theory and in this sense the effects due to the vector γ are truly relativistic, stemming entirely from the fact that the gravitational field is described by the metric tensor.

If the body were not rotating, γ would vanish, but the diagonal elements would stay the same (except for the unimportant term mentioned). If we transform now to a frame of reference rotating with angular velocity Ω, the diagonal elements remain essentially unchanged (g_{44} receives a small additional term which can be neglected as long as Ω is small) but γ will become $2\Omega\times\mathbf{R}/c$.[3] Hence at the point $\mathbf{R}$ the metric tensor will have the same value as for the rotating body if the directions of $\mathbf{\Omega}$ and $\mathbf{L}$ are opposite and $GI\omega/c^2R^3=\Omega$, where I is the moment of inertia of the rotating body and ω the magnitude of its angular velocity. If the physical process takes place in a portion of space whose linear dimensions are negligible compared with R, we can then say that the two physical situations, rotating body and fixed frame, fixed body and frame rotating with the above-specified angular velocity, are equivalent.

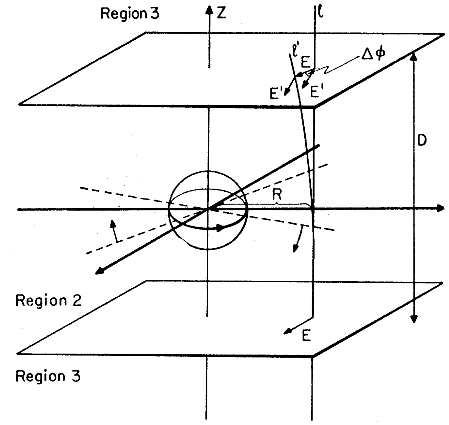

FIG. 1. Rotating mass with equivalent coordinate system and light wave.

Hence, to investigate how the plane of polarization will change, it is sufficient to contemplate the following situation. Let a plane wave travel in the direction of $\mathbf{L}$, passing the body at a distance R in the equatorial plane. We subdivide the path into three sections. In section (3) the influence of the gravitational field cannot be felt; in (2) the wave is under the influence of the gravitational field. (See Fig. 1.) In view of the results in the last section we shall consider, instead of the rotating body (rotating counterclockwise in Fig. 1), a rotating frame (dotted lines in Fig. 1). The two frames should coincide as the light wave enters region (2). Now we pursue the wave as seen from the rotating frame until it reaches frame (3). We now turn back the dotted frame to its original position. The path in this arrangement, l', will be approximately the actual path (neglecting the bending due to the static field, and the fact that the angular velocity will vary as the distance between points along the path and the origin vary). The electric vector $\mathbf{E}'$ of the plane wave will be turned now with an angle relative to the position $\mathbf{E}$ it would occupy along path l, which is the path in the absence of rotation. This angle will be approximately

[2] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Press, Cambridge, 1951), p. 328. (Observe that we use an imaginary time coordinate while the book uses a real one.)

[3] Reference 2, p. 246.

$(D/c)(GI\omega/c^2R^3)$ since in each second the frame turns an angle $GI\omega/c^2R^3$, and it takes D/c seconds to pass through the region. Let us take $D\sim R$ and R to be about the order of the radius of the body. [This is admittedly crude, since Eq. (1) was derived by assuming that R is much larger than the radius of the body.] We then get for our effect $\Delta\varphi = GI\omega/c^3R^2 \sim GM/c^3$ since $I\sim MR^2$. For the sun we take $M\sim 2\times 10^{33}$ g, $\omega\sim 10^{-6}$ sec; then $\Delta\varphi\sim 4\times 10^{-12}$ radian, which is a very small quantity.

III

We shall now show that this effect can also be derived by using Maxwell's equations in the presence of a gravitational field.

If the gravitational field is given by the g_{ik} tensor, and the field by the F_{ik} tensor, the equations which have to be satisfied are

$$\begin{aligned} \partial F_{ik}/\partial x_l + \partial F_{li}/\partial x_k + \partial F_{kl}/x_i &= 0, \\ \partial(g^{\frac{1}{2}}F^{ik})/\partial x_k &= 0, \end{aligned} \qquad (2)$$

if no charges and currents are present; g is the absolute value of the determinant of g_{ik}.

If the gravitational field is strong, it is somewhat complicated to discuss the relation between F_{ik} and the electric and magnetic field. If, however, the gravitational field is weak, as is always the case in practice, we can interpret Eq. (2) easily. This we shall now do. As in (1), the metric tensor is given by

$$g_{ik} = \delta_{ik} + \gamma_{ik} \begin{bmatrix} 1-2\phi & 0 & 0 & i\gamma_1 \\ 0 & 1-2\phi & 0 & i\gamma_2 \\ 0 & 0 & 1-2\phi & i\gamma_3 \\ i\gamma_1 & i\gamma_2 & i\gamma_3 & 1+2\phi \end{bmatrix}. \qquad (3)$$

The trivial difference between (1) and (3) consists of putting $c=1$ in (3) and introducing explicitly γ_{ik}, the small part of g_{ik}.

Also, put

$$F_{ik} = \begin{bmatrix} 0 & B_z & -B_y & -iE_x \\ -B_z & 0 & B_x & -iE_y \\ B_y & -B_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{bmatrix}. \qquad (4)$$

Then the form of Eq. (2) is simply

$$\begin{aligned} &\mathrm{curl}\mathbf{E} + \partial\mathbf{B}/\partial t = 0, \quad \mathrm{div}\mathbf{B} = 0, \\ &\mathrm{curl}\mathbf{H} + \partial\mathbf{D}/\partial t = 0, \quad \mathrm{div}\mathbf{D} = 0, \end{aligned} \qquad (5)$$

with

$$\begin{aligned} &\mathbf{B} = \mathbf{H} + \mathbf{M}, \quad \mathbf{D} = \mathbf{E} + \mathbf{P}, \\ &\mathbf{P} = \alpha\mathbf{E} + (\boldsymbol{\gamma}\times\mathbf{B}), \quad \mathbf{M} = \alpha\mathbf{B} - (\boldsymbol{\gamma}\times\mathbf{E}), \\ &\alpha = -2\phi. \end{aligned} \qquad (6)$$

Hence the effect of the weak gravitational field manifests itself as if a material medium were present with an electric and magnetic polarization. Equation (6) gives the relation between the polarization and the field strength. At first sight Eq. (6) does not appear to be of the correct form to produce optical rotation. For it is well known that in an optically active medium the polarization depends on the field strength and its *derivatives*. Thus it seems that if the constitutive equations are as given by Eq. (6), the gravitational field will not act as an optically active medium; hence, there will be no rotation of the plane of polarization. However, we must realize that ordinarily we consider the optical constants to be actually constants along the ray, while in our case it is essential that $\boldsymbol{\gamma}$ is zero at a large distance from the rotating body, while finite in its neighborhood. Indeed, we shall see that Eq. (6) defines an optically active anisotropic medium, if $\boldsymbol{\gamma}$ is not a constant.

We expect that the effect will be the largest if the direction of propagation of the wave coincides with the axis of rotation of the body. Let us then try to find the following solutions of (5) and (6). A wave packet is proceeding in the $+z$ direction which is also the axis of rotation of the body. Its lateral extent in the x, y plane should be limited and much smaller than R which is the distance in the equatorial plane between the path of the wave packet and the center of the rotating mass. These solutions are to be accurate to first order in $\boldsymbol{\gamma}$ but not more.

The effects of the gravitational field manifest themselves through α and $\boldsymbol{\gamma}$. If only α were present we would obtain the bending of light rays, but no change in the polarization. This, then, does not interest us. Hence we shall put $\alpha=0$, and keep only $\boldsymbol{\gamma}$. In this way we can investigate the effect of $\boldsymbol{\gamma}$ unadulterated by other effects.

Putting, then, (6) with $\alpha=0$ into (5), we obtain

$$\begin{aligned} &\mathrm{curl}\mathbf{D} - (\mathbf{B}\cdot\mathrm{grad})\boldsymbol{\gamma} + (\boldsymbol{\gamma}\cdot\mathrm{grad})\mathbf{B} + \partial\mathbf{B}/\partial t = 0, \\ &\mathrm{curl}\mathbf{B} + (\mathbf{D}\cdot\mathrm{grad})\boldsymbol{\gamma} - (\boldsymbol{\gamma}\cdot\mathrm{grad})\mathbf{D} - \partial\mathbf{D}/\partial t = 0, \\ &\mathrm{div}\mathbf{D} = \mathrm{div}\mathbf{B} = 0. \end{aligned} \qquad (7)$$

Let us assume that $\mathbf{D} = (D_1, D_2, 0)$, $(\mathbf{B} = B_1, B_2, 0)$. Then the gradient operator will correspond to differentiation with respect to x and y. ($\boldsymbol{\gamma}$ has no z component either.) What about the x, y derivatives of $\boldsymbol{\gamma}$, $\mathbf{B}$, and $\mathbf{D}$? It is easy to verify that, barring points on and near the equatorial plane, $\partial\gamma_1/\partial y \gg \partial\gamma_1/\partial x$ and $\partial\gamma_2/\partial x \gg \partial\gamma_2/\partial y$. Hence we shall neglect $\partial\gamma_1/\partial x$ and $\partial\gamma_2/\partial y$. The remaining derivatives are $\partial\gamma_1/\partial y = a/r^3 \equiv \sigma$, $\partial\gamma_2/\partial x = -\sigma$; $a = 2G/c^3$, if we take $\boldsymbol{\gamma} = (a/r^3)(y, -x, 0)$, which corresponds to a rotation along the positive z axis.

If a true plane wave would propagate in the z direction the x and y derivatives of (D_1, D_2) and (B_1, B_2) would be zero. If the wave has a finite extension this cannot be strictly true. However, if we confine our interest to the central part of the wave packet this will be true to a good degree of approximation. Actually at the end of the calculation we can verify that with this solution $(\boldsymbol{\gamma}\cdot\mathrm{grad})\mathbf{B}$, etc., are small compared with the other terms in the equation. Taking into account these approximations, we have satisfied (approximately) the divergence equations; the other two equations give

$$-\partial D_2/\partial z-\sigma B_2+\dot{B}_1=0,\quad -\partial B_2/\partial z+\sigma D_2-\dot{D}_1=0,$$
$$\partial D_1/\partial z+\sigma B_1+\dot{B}_2=0,\quad \partial B_1/\partial z-\sigma D_1-\dot{D}_2=0,\quad (8)$$

where the dot indicates partial differentiation with respect to t. Let us introduce now $D_1+iD_2=D_+e^{-i\omega t}$, $D_1-iD_2=D_-e^{-i\omega t}$, and the similarly constructed B_-, B_+. We obtain

$$dD_-/dz+(\sigma+\omega)B_-=0,\quad dD_+/dz+(\sigma-\omega)B_+=0,$$
$$dB_-/dz-(\sigma+\omega)D_-=0,\quad dB_+/dz-(\sigma-\omega)D_+=0.\quad (9)$$

If σ were independent of z the solutions would be of the form $D_-=d_-e^{i\omega z}$, where d_- is a constant (z has the coefficient ω since $c=1$). If σ is not a constant but a slowly varying function of z, we can attempt a solution by writing $D_-=d_-e^{i\omega z+il(z)}$, where the derivatives of $l(z)$ are small. If we do this, we get the equations

$$\begin{aligned} &i(\omega+dl/dz)d_-+(\sigma+\omega)b_-=0,\\ &-(\sigma+\omega)d_-+i(\omega+dl/dz)b_-=0,\\ &i(\omega+dl/dz)d_++(\sigma-\omega)b_+=0,\\ &-(\sigma-\omega)d_++i(\omega+dl/dz)b_+=0. \end{aligned}\quad (10)$$

[We have two pairs of homogeneous equations, a pair for each of the sets (b_+, d_+) and (b_-, d_-). For a solution to exist, the determinant of the coefficients in each pair must vanish. This results in

$$dl_-/dz+\sigma=0\quad \text{for the minus pair,}$$
$$dl_+/dz-\sigma=0\quad \text{for the plus pair,}\quad (11)$$

where we have neglected $(dl/dz)^2$ and σ^2. Hence

$$l_+=\int_{-\infty}^{z}\sigma dz,\quad l_-=-\int_{-\infty}^{z}\sigma dz=-l_+.\quad (12)$$

We can now easily verify that with this l, $(\boldsymbol{\gamma}\cdot\text{grad})\mathbf{D}$, etc., are small compared with the other terms in the equation.]

The expressions (12) solve our problem. We see that the two circularly polarized components of the plane wave, the $+$, $-$ vectors, travel with different velocities, and this fact gives rise as usual to a rotation of the plane of polarization. Since l would be zero if γ is a constant, the effect depends on γ being a function of x and y, just as we have said in the discussion of Eq. (6).

We obtain, then,

$$\begin{aligned} D_+(z)&=de^{i(\omega z-\omega t)}\cos l_+(z),\\ D_-(z)&=de^{i(\omega z-\omega t)}\sin l_+(z). \end{aligned}\quad (13)$$

From this we see that if $l_+(z)$ is positive, the D vector is rotating from the x axis toward the y axis as the wave is proceeding in the z direction. This is a rotation along the positive z axis. Since we have taken $\boldsymbol{\gamma}$ to correspond to a rotation of the body along the positive z direction, we see that the plane of polarization rotates in the same sense as the body does.

Moreover, the total angle of rotation is given by $\Delta\varphi=2GI/c^3R^2$ if the ray passes from minus infinity to plus infinity along a path which is parallel to the axis of rotation and intersects the equatorial plane at a distance R from the center of the rotating body.

Both the sense and the magnitude are the same as the results obtained in Sec. II by more qualitative considerations.