Calculation of Electron-Electron Scattering

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The matrix elements involved in the calculation of electron-electron and positron-electron scattering are written down explicitly for a complete set of initial and final spin states of the particles. This permits the evaluation of the cross section for any polarization combination desired, with a minimum of labor.

TWO papers have recently appeared in which the polarization-sensitive parts of the electron-electron and the positron-electron scattering cross sections are calculated.[1,2] In both these calculations the spin-sum method is used. This method is not simpler than the direct calculation of matrix elements when all polarizations are summed over[3] and has the feature of becoming more complicated the greater the specialization of initial and final states, as more projection operators must be used. All variables summed over are irretrievably lost.

This note provides a set of matrix elements from which any electron-electron or positron-electron scattering cross section may be calculated, complete with instructions for such calculations.

We deal with an "initial" state containing two electrons described by spinors $u(p_1)$ and $u(p_2)$. $u(p_1)$ is a positive-energy spinor describing the target electron. $u(p_2)$ is of positive energy for the electron-electron scattering problem, where it represents the *incident* electron, and is of negative energy for the positron-electron problem where it represents the *scattered* positron. Similarly $u(p_3)$ is of positive energy and $u(p_4)$ is of positive or negative energy, so that the "final" state is specified by these two spinors. The matrix elements of interest have the form

$$M=\sum_{\mu=1}^{4}\left[(\bar{u}(p_3)\gamma_\mu u(p_1))(\bar{u}(p_4)\gamma_\mu u(p_2))\frac{1}{(p_1-p_3)^2}-(\bar{u}(p_3)\gamma_\mu u(p_2))(\bar{u}(p_4)\gamma_\mu u(p_1))\frac{1}{(p_2-p_3)^2}\right]. \quad (1)$$

M depends on the four spins associated with the momenta $p_1\cdots p_4$.

TABLE I. Values of $4p^2M$ for electron-electron scattering.

3 component of spin of particle 1	3	2	4	$4p^2M$
+	+	+	+	$\frac{E^2+p^2}{m^2}\cot^2(\theta/2)-1$
+	+	+	−	$\frac{iE}{m}[\cot(\theta/2)-\tan(\theta/2)]$
+	−	+	+	$\frac{iE}{m}[\cot(\theta/2)-\tan(\theta/2)]$
+	−	+	−	$-1+\frac{E^2+p^2}{m^2}\tan^2(\theta/2)$
−	+	+	+	$\frac{iE}{m}[\cot(\theta/2)-\tan(\theta/2)]$
−	+	+	−	-2
−	−	+	+	$\frac{E^2+p^2}{m^2}[\cot(\theta/2)+\tan(\theta/2)]^2-2$
−	−	+	−	$\frac{iE}{m}[\cot(\theta/2)-\tan(\theta/2)]$

To calculate these matrix elements it is convenient to use the center-of-momentum reference system and to let the momenta of the incident particles lie along the 3 axis, the momenta of the scattered particles lie in the 23 plane. Then we may write, for the electron-electron problem,

$$\begin{aligned}u(p_1)&=\exp(-\alpha_3\chi/2)u_1,\\ u(p_2)&=\exp(\alpha_3\chi/2)u_2,\\ u(p_3)&=\exp(-i\sigma_1\theta/2)\exp(-\alpha_3\chi/2)u_3,\\ u(p_4)&=\exp(-i\sigma_1\theta/2)\exp(\alpha_3\chi/2)u_4,\end{aligned} \quad (2)$$

where $\cosh\chi=E/m$, E being the energy of a particle in the center-of-momentum system, and θ is the scattering angle in this system. The spinors u_1-u_4 then represent particles at rest. We choose to make them eigenvectors of σ_3 which is diagonal in the commonest representation of the Dirac matrices.[4]

In the positron-electron problem the operators occuring in the expressions for $u(p_2)$ and $u(p_4)$ are interchanged. Thus in the electron-electron problem the spins of u_1, u_2 being + means that the incident electrons

[1] A. Bincer, Phys. Rev. **107**, 1434 (1957).
[2] G. W. Ford and C. J. Mullin, Phys. Rev. **108**, 477 (1957).
[3] See comment by H. A. Kramers, *Quantum Mechanics* (North Holland Publishing Company, Amsterdam, 1957), p. 479.

[4] We use the normalization $\bar{u}_r\gamma_4u_s=\delta_{rs}$ for our rest spinors. If τ_k are the Pauli matrices ($k=1, 2, 3$), then

$$\gamma_k=\begin{pmatrix}0 & -i\tau_k\\ i\tau_k & 0\end{pmatrix},\quad \gamma_4=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$
$$\alpha_k=\begin{pmatrix}0 & \tau_k\\ \tau_k & 0\end{pmatrix},\quad \sigma_k=\begin{pmatrix}\tau_k & 0\\ 0 & \tau_k\end{pmatrix}.$$

have their spins in the $+3$ direction, and u_3, u_4 being $+$ means that the scattered electrons have their spins along the $+$ direction of an axis rotated through θ from the $+3$ direction. In the positron-electron problem the spins of u_1, u_4 being $+$ means that the incident *electron* has its spin in the $+3$ direction and the incident *positron* has its spin in the -3 direction, and similarly for u_3, u_2 relative to the inclined axis.

The quantities to be calculated now have the form

$$\bar{u}_3 \exp(\alpha_3\chi/2) \exp(i\sigma_1\theta/2)\gamma_\mu \exp(-\alpha_3\chi/2)u_1. \quad (3)$$

The calculations are all made with the spin of the incident particle travelling in the $+3$ direction positive. The matrix elements are invariant under a reversal of all spins, so this is sufficient. Table I contains the results for electron-electron scattering; Table II, those for positron-electron scattering. The spins tabulated in Table II are those for the electron (1 and 3) and for the *positron* (4 and 2), not for the hole which represents the positron.

As an example of the use of these tables, we reproduce Ford and Mullin's cross section for the scattering of a longitudinally polarized electron by an electron whose spin is in the direction defined by polar angles ψ, ϕ.[5] To do this we must express the spinor u_1', describing the target spin, as a superposition of the two spin states represented in Table I. We obtain u_1' from u_+ by two rotations;

$$u_1' = \exp(-i\sigma_3\phi/2) \exp(-i\sigma_1\psi/2)u_+ = au_+ + bu_-, \quad (4)$$

with

$$a = \exp(-i\varphi/2) \cos(\psi/2), \quad b = -i \exp(i\varphi/2) \sin(\psi/2). \quad (5)$$

The cross section is now obtained by taking the linear combinations $\{a\times$element in row $1+b\times$element in row $5\}$, etc., squaring absolutely, adding, and multiplying by the factor

TABLE II. Values of $4p^2M$ for positron-electron scattering.

3 component of spin of particle 1	3	4	2	$4p^2M$
+	+	+	+	$-\cot^2(\theta/2)\left[1+\frac{2p^2}{m^2}\cos^2(\theta/2)\right]$
+	+	+	−	$\frac{iE}{m}\cot(\theta/2)\left[1-\frac{2p^2}{E^2}\sin^2(\theta/2)\right]$
+	−	+	+	$-\frac{iE}{m}\cot(\theta/2)\left[1-\frac{2p^2}{E^2}\sin^2(\theta/2)\right]$
+	−	+	−	$-1+\frac{2p^2}{m^2}\sin^2(\theta/2)$
−	+	+	+	$-\frac{iE}{m}\cot(\theta/2)\left[1-\frac{2p^2}{E^2}\sin^2(\theta/2)\right]$
−	+	+	−	$-\left[\frac{m^2}{E^2}+\frac{2p^2}{E^2}\cos^2(\theta/2)\right]$
−	−	+	+	$-\left(\frac{2p^2}{m^2}+\frac{p^2}{E^2}\right)$
				$-\cot^2(\theta/2)\left[\frac{E^2+p^2}{m^2}-\frac{2p^2}{E^2}\sin^2(\theta/2)\right]$
−	−	+	−	$\frac{iE}{m}\cot(\theta/2)\left[1-\frac{2p^2}{E^2}\sin^2(\theta/2)\right]$

$$\frac{r_0^2(\gamma+1)}{2\gamma^4\beta^4},$$

which contains all necessary factors π, densities of states, etc. r_0 is the classical electron radius. The first part of the procedure leads to

$$\left|a\left[\frac{E^2+p^2}{m^2}\cot^2(\theta/2)-1\right]+b\frac{iE}{m}[\cot(\theta/2)-\tan(\theta/2)]\right|^2+\left|a\frac{iE}{m}[\cot(\theta/2)-\tan(\theta/2)]+b(-2)\right|^2$$
$$+\left|a\frac{iE}{m}[\cot(\theta/2)-\tan(\theta/2)]+b\left[\frac{E^2+p^2}{m^2}[\cot(\theta/2)+\tan(\theta/2)]^2-2\right]\right|^2$$
$$+\left|a\left[-1+\frac{E^2+p^2}{m^2}\tan^2(\theta/2)\right]+b\frac{iE}{m}[\cot(\theta/2)-\tan(\theta/2)]\right|^2$$
$$=\frac{1+\cos\psi}{2\sin^4\theta}\{16\gamma^2-(16\gamma^2+4\gamma-4)\sin^2\theta+2(\gamma^2-1)\sin^4\theta\}+\frac{1-\cos\psi}{2\sin^4\theta}\{16\gamma^2-(12\gamma-4)\sin^2(\theta/2)-4(\gamma-1)\sin^4\theta\}$$
$$+\frac{\sin\psi\cos\varphi}{2\sin^4\theta}\left\{-8(\gamma-1)\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}}\cos\theta\sin^3\theta\right\}. \quad (6)$$

[5] Equations (7) of reference 2. Note that a factor 4 is omitted before $\sin^2\theta$ in the second term of this equation. The factor is correct in the previous equation, Eq. (6).

This does indeed lead to Ford and Mullin's result. The quantities γ and β used above refer to the laboratory frame, while Ford and Mullin's refer to the center-of-momentum frame. The connection is

$$\gamma_{\text{lab}} = 2\gamma^2_{\text{c.m.}} - 1.$$

All angles are center-of-momentum angles.

In a similar way the cross section for any combination of incident and scattered polarized electrons or positrons can be constructed. The depolarization of electrons or positrons scattered from polarized electrons may be found, a result not obtainable from the spin-sum calculations published, as can the scattering of transversely polarized electrons. So many possible combinations occur that it is best to await the need for a cross section before carrying out the combination of the parts, rather than make all possible combinations now.

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Integral Representations of Causal Commutators

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An integral representation is found for the matrix element, between given states, of the commutator of two field operators. The representation makes use of the information derivable from the local commutativity of the operators and from the mass spectrum of the fields. The representation was discovered by Jost and Lehmann and proved by them for the case of two fields of equal mass. It is here extended to the case of unequal masses.

The mathematical basis of this work is the fact that every function $f(q)$ of a four-vector q, with a Fourier transform $\tilde{f}(x)$ which vanishes for space like x, has a unique extension which is a solution of the wave equation in six dimensions.

I. STATEMENT OF THE PROBLEM

JOST and Lehmann[1] have introduced an integral representation for matrix elements of causal commutators. Their representation is a powerful tool for investigating the analytic behavior of scattering amplitudes as a function of all relevant momentum and energy variables. In particular, Lehmann[2] was able to obtain a proof of dispersion relations, shorter and simpler than the original proof of Bogolyubov.[3] Unfortunately, the Jost-Lehmann analysis applied only to the commutator of two fields of equal mass. The Lehmann proof of dispersion relations was restricted to the scattering of equal-mass particles, and for this reason remained unpublished. In this paper we extend the integral representation to the case of unequal masses.

A more important task for the future is to explore systematically the analytic consequences of causality, not restricting attention to the special type of information which can be embodied in dispersion relations. Progress in this direction has already been made by Källén and Wightman,[4] using only the causality condition and assuming nothing about particle masses. We hope that the integral representations here established may make it possible to extend the Källén-Wightman analysis so as to include detailed information about the mass spectrum.

We are concerned with two fields $A(x)$ and $B(x)$, obeying the causality principle

$$[A(x), B(x')] = 0 \quad \text{for} \quad (x-x')^2 < 0. \tag{1}$$

Here x and x' are 4-vectors, and the scalar product is defined by

$$x\cdot y = x_0 y_0 - \mathbf{x}\cdot\mathbf{y} = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3. \tag{2}$$

The problem is to determine the analytic form of the function

$$\tilde{f}(x) = i\langle P, \alpha | [A(\tfrac{1}{2}x), B(-\tfrac{1}{2}x)] | Q, \beta\rangle, \tag{3}$$

or of its Fourier transform

$$f(q) = \frac{1}{(2\pi)^4}\int d_4 x \exp(-iq\cdot x)\tilde{f}(x). \tag{4}$$

Here $|P,\alpha\rangle$ and $|Q,\beta\rangle$ are any two states specified by the energy-momentum vectors P and Q and by other quantum numbers α and β.

By Eq. (1), the function $\tilde{f}$ in x space has the property

$$\tilde{f}(x) = 0 \quad \text{for} \quad x^2 < 0. \tag{5}$$

From Eqs. (3) and (4), the function f in q space also has

[1] R. Jost and H. Lehmann, Nuovo cimento **5**, 1598 (1957).

[2] H. Lehmann (private communication). *Note added in proof.*—Lehmann has used the results of this paper to prove that certain scattering amplitudes are analytic functions of the momentum transfer (to be published in Nuovo cimento).

[3] Bogolyubov, Medvedev, and Polivanov, Uspekhi Math. Nauk (to be published).

[4] G. Källén and A. S. Wightman (private communication to W. Pauli). See report by G. Källén in *Proceedings of Seventh Annual Rochester Conference on High-Energy Nuclear Physics, 1957* (Interscience Publishers, New York, 1957, Session IV, p. 17.